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Introduction to Number Theory

Problem Set 2 (due Jan 22, 2008)

2.1. a) Find all integer solutions to $1357x + 2468y = 1$.

b) Find all integer solutions to $105x + 121y = 13$.

c) We know that the equation $ax + by = c$ has a solution if and only $\gcd(a, b) \mid c$. Let $a_1, \dots, a_k \in \mathbb{Z}$, not all 0. State and prove a similar necessary and sufficient criterion for the solvability of $a_1x_1 + \dots + a_kx_k = c$.

2.2. Let $a, b \in \mathbb{N}$ be coprime, and let $c \in \mathbb{Z}$.

a) Assume $c = ab - a - b$. Find all solutions of $ax + by = c$, and show that there is no solution with $x \geq 0$ and $y \geq 0$.

b) Assume $c > ab - a - b$. Show that there is a solution of $ax + by = c$ with $x, y \geq 0$.

Hint: Why can you assume $x \leq b - 1$? What does this imply for y ?

2.3. a) Let $n = \prod_{j=1}^k p_j^{e_j}$ be the prime power factorization of $n \in \mathbb{N}$. Find a formula in terms of e_1, \dots, e_k for the number of positive divisors of n .

b) Let

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

be the binomial coefficient. Show that $7 \nmid \binom{82}{12}$, but $7 \mid \binom{82}{36}$. *Hint:* count how often the factor 7 appears in the numerator and the denominator.

2.4. Let p be a prime. If $p^\alpha \mid n$ (for some $\alpha \geq 0$), we define $\|n\|_p := 1/p^\alpha$, so for example $\|n\|_p = 1$ if and only if $p \nmid n$. Let $n, m \in \mathbb{N}$.

a) Show $\|nm\|_p = \|m\|_p \|n\|_p$.

b) Show $\|n + m\|_p \leq \max(\|m\|_p, \|n\|_p)$.

c) If $\|m\|_p < 1$, then $\|1 + m\|_p = 1$.