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Introduction to Number Theory

Problem Set 1 (due Jan 15, 2008)

1.1. a) How many pairs of positive consecutive integers $(n, n+1)$ are there, such that both n and $n+1$ are prime?

b) How many triplets $(n, n+2, n+4)$ of positive integers are there, such that all three entries are simulaneously prime?

c) Show that all but one prime can be written as a difference of two squares of positive integers. Which prime p cannot be written as $p = n^2 - m^2$? Show that if $p = n^2 - m^2$ is a difference of two squares, then n and m are unique.

d) Prove by induction that $6 \mid n^3 + 11n$ for every $n \in \mathbb{N}$.

1.2. A prime p is called *Mersenne prime* if it is of the form $2^k - 1$ for some $k \in \mathbb{N}$.

a) Find the first 5 Mersenne primes. *Hint:* The fifth one is close to 10000, so this problem may require the use of a pocket calculator. Try to be as efficient as possible to check if a number of the form $2^k - 1$ is prime or not.

b) Let $n, m \in \mathbb{N}$. Find a closed formula for $(x^n - 1)\sum_{j=0}^{m-1} x^{jn}$.

c) Show that $2^k - 1$ is prime only if k is prime. Does the converse hold?

d) A positive integer n is called *perfect*, if it is twice the sum of all its positive divisors (including 1 and n), for example $2 \cdot 6 = 1 + 2 + 3 + 6$, so 6 is perfect. Find all perfect numbers $1 \leq n \leq 50$.

e) Let $p = 2^k - 1$ be a Mersenne prime, and let $n = 2^{k-1}p$. Write down the positive divisors of n and show that n is perfect.

Remark: It can be shown that all even perfect numbers are of the form $2^k(2^k - 1)$ for a Mersenne prime $p = 2^k - 1$. It not known if odd perfect numbers exist. If so, they must be astronomically large. It is not known if there are finitely or infinitely many Mersenne primes. Currently somewhat more than 40 Mersenne primes are known.

1.3. a) Calculate $\texttt{gcd}(12345, 67890)$.

b) Calculate $d = \texttt{gcd}(1422, 2946)$ and write d as linear combination of 1422 and 2946.

c) Define recursively the sequence (f_n) by $f_1 = 1, f_2 = 1, f_{n+1} = f_n + f_{n-1}$. These are the Fibonacci numbers. Write down the first 10 Fibonacci numbers, and prove by induction for all $n \geq 2$ that f_{n+1} is coprime to both f_n and f_{n-1}.