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Theory of Character Sums

Problem Set 5 (due Apr 7, 2009)

5.1. (30%) Let χ be a primitive character mod q. Show that

$$\max_{M,N} \left| \sum_{n=M+1}^{M+N} \chi(n) \right| \geq \frac{\sqrt{q}}{\pi}.$$

Hint: Start with the inequalities

$$\max_{M} \left| \sum_{n=M+1}^{M+N} \chi(n) \right| \geq \frac{1}{q} \sum_{M=1}^{q} \left| \sum_{n=M+1}^{M+N} \chi(n) \right| \geq \frac{1}{q} \left| \sum_{M=1}^{q} e\left(\frac{M}{q}\right) \sum_{n=M+1}^{M+N} \chi(n) \right|,$$

compute the double sum, and choose N suitably.

5.2. (10% - one line) Let χ be a nontrivial character mod q, let $b \in \mathbb{Z}$, and let a be coprime to q. Show

$$\sum_{n=M+1}^{M+N} \chi(an+b) \ll \sqrt{q} \log q$$

with an absolute implied constant (independent of a and b).

5.3. (30%) "In every interval of length slightly larger than $p^{1/4}$, quadratic residues and non-residues mod p are equally distributed." Make this statement precise (including all ε's etc.) and prove it. *Hint:* There is a simple expression for the characteristic function on quadratic residues and non-residues.

5.4. (35%) a) Let c_n, $1 \leq n \leq q$, be a sequence of complex numbers, and let

$$\widehat{c}_k := \sum_{n=1}^{q} c_n e(kn/q)$$

be its Fourier transform. Show that

$$\sum_{M=1}^{q} \left| \sum_{n=M+1}^{M+N} c_n - \frac{N}{q} \sum_{n=1}^{q} c_n \right|^2 = \frac{1}{q} \sum_{k=1}^{q-1} |\widehat{c}_k|^2 \frac{\sin^2(\pi N k/q)}{\sin^2(\pi k/q)}$$

Hint: Use Parseval's formula $\sum_{M=1}^{q} |a_M|^2 = \frac{1}{q} \sum_{k=1}^{q} |\widehat{a}_k|^2$.

b) Apply the preceding formula for $c_n = \chi(n)$ where χ is a nontrivial character mod q to deduce that

$$\sum_{M=1}^{q} \left| \sum_{n=M+1}^{M+N} \chi(n) \right|^2 \leq Nq.$$

Explain in what sense this gives square-root cancellation for arbitrarily short character sums.

c) Apply the formula for $c_n = 1$, $1 \leq n \leq q-1$, $c_q = 0$ to deduce the cute formula

$$\sum_{k=1}^{q-1} \frac{\sin^2(\pi Nk/q)}{\sin^2(\pi k/q)} = (q-N)N.$$

Specialize $q = 2N$ and write $k = 2n-1$ to show that

$$\sum_{n=1}^{N} \left(N \sin\left(\pi \frac{2n-1}{2N} \right) \right)^{-2} = 1.$$

d) Deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \pi^2/8$, and conclude $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$. *Hint:* Let N go to infinity, use a simple approximation for $\sin x$ for small x, and observe that only $n \ll N^{3/4}$, say, contribute to the sum in an essential way.