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Spectral Methods of Automorphic Forms

Problem Set 5 (due Dec 7)

5.1. a) Let $N(x)$ be the number of integer solutions to $ad - bc = 1$ inside the ball $a^2 + b^2 + c^2 + d^2 \leq x$. Show $N(x) = 6x + O(x^{2/3+\varepsilon})$.

b) Let $r(n)$ be the number of representations of n as a sum of two squares. Show

$$\sum_{n \leq x} r(n)r(n+1) = 8x + O(x^{2/3+\varepsilon})$$

c) (*voluntarily*) Do you have an idea how you to get an asymptotic formula for $\sum_{n \leq x} r(n)r(n+h)$, h odd?

Hints: a) Note that $4u\left(i, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) i\right) = a^2 + b^2 + c^2 + d^2$.

b) Let $\Gamma := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid a \equiv d\,(2), b \equiv c\,(2)\right\} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right) \Gamma_0(2) \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. Change variables $a + d = 2k, a - d = 2l, b + c = 2m, b - c = 2n$, and note that the sum in question counts $\#\{(k, l, m, n) : k^2 - l^2 - m^2 + n^2 = 1, k^2 + l^2 + m^2 + n^2 \leq 2x + 1\}$.

c) Use Hecke operators.

5.2. Let $Q(x, y) = ax^2 + bxy + cy^2 \in \mathbb{Z}[x]$ be a primitive quadratic form of discriminant $d = b^2 - 4ac > 0$, d not a square. Let $\theta_{1,2} \in \mathbb{R}$ be the two roots of $Q(x, 1) = 0$, and let g be the geodesic in $\mathbb{H}$ joining θ_1 and θ_2. Let $\left(\begin{smallmatrix} (t_0 - bu_0)/2 & -cu_0 \\ au_0 & (t_0 + bu_0)/2 \end{smallmatrix}\right)$ be a generator of the stabilizer of Q under $SL_2(\mathbb{Z})$ (cf. problem 3.4), and let $\epsilon_d := (t_0 + u_0\sqrt{d})/2$. Let $h(d)$ be the number of equivalence classes of forms of discriminant d. (Clearly $h(d) = 0$ unless $d \equiv 0, 1 \pmod 4$, and we also define $h(d) = 0$ if d is a square.)

Show that the map $Q \mapsto g$ is a bijection between classes of quadratic forms and primitive conjugacy classes of hyperbolic motions (determined by the geodesic g). Conclude that there are $h(d)$ closed geodesics on $SL_2(\mathbb{Z})\backslash\mathbb{H}$ of length $\log \epsilon_d^2$, and all closed geodesics are accounted for in this way. Conclude

$$\sum_{\epsilon_d \leq \sqrt{x}} 2h(d) \log \epsilon_d = x + O(x^{3/4+\varepsilon})$$

or (by partial summation)

$$\sum_{\epsilon_d \leq x} h(d) = \int_2^{x^2} \frac{dt}{\log t} + O(x^{3/2+\varepsilon}).$$

Of course, ordering class numbers by the size of the regulator is not the most natural choice. It would be more natural to sum over all discriminants up to x, but this sum has restisted any evaluation (or even non-trivial upper bounds) so far...

5.3. Let u be a Maaß cusp form of weight 0 for $\Gamma = SL_2(\mathbb{Z})$ that is an eigenform of all Hecke operators with eigenvalues $\lambda(n)$.

a) Using the multiplicativity of the Hecke operators, show that

$$\sum_n \frac{\lambda(n)^2}{n^s} = \zeta(s) \sum_n \frac{\lambda(n^2)}{n^s}$$

for $\Re s$ sufficiently large.

b) Let

$$\theta(z) := y^{1/4} \sum_{n \in \mathbb{Z}} e(n^2 z)$$

For odd d let $\epsilon_d = 1$ if $d \equiv 1 \pmod 4$ and $\epsilon_d = i$ if $d \equiv 3 \pmod 4$. For odd d let $\left(\frac{c}{d}\right)$ be the quadratic residue symbol, i.e. the standard Jacobi symbol if $(c, d) = 1$, $d > 0$, and $\left(\frac{c}{d}\right) = 0$ if $(c, d) > 1$, and $\left(\frac{c}{d}\right) = \text{sgn}(c) \left(\frac{c}{|d|}\right)$ for $d < 0$ and finally $\left(\frac{0}{\pm 1}\right) = 1$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ let

$$j(\gamma, z) := \epsilon_d^{-1} \left(\frac{c}{d}\right) \left(\frac{|cz + d|}{cz + d}\right)^{-1/2}$$

with the standard branch of the square root. Show that

$$\theta(\gamma z) = j(\gamma, z) f(z).$$

Hint: You can assume that $j(\gamma, z)$ is a decent multiplier, i.e. $j(\alpha\beta, z) = j(\alpha, \beta z) j(\beta, z)$ for all $\alpha, \beta \in \Gamma_0(4)$. Then you only need to show the transformation formula for generators of $\Gamma_0(4)$, e.g. $-I$, T and $ST^{-4}S$ (why do they generate $\Gamma_0(4)$?).

c) Let

$$E(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} j(\gamma, z)^{-1} (\Im \gamma z)^s$$

(this is called a metaplectic Eisenstein series and is the standard Eisenstein series of weight 1/2). Find an integral representation of $\sum \lambda(n^2) n^{-s}$ using u, θ and $E(., s)$.