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Spectral Methods of Automorphic Forms

Problem Set 4 (due Nov 30)

4.1. Let $\Gamma = \Gamma_0(N)$, $k \geq 4$ an (even) integer, m a positive integer, and define

$$P_m(z) := \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma} (cz + d)^{-k} e(m\gamma z) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} j_\gamma(z)^{-k} e(m\gamma z), \text{ say.}$$

a) Show the Fourier expansion

$$P_m(z) = \sum_{n=1}^{\infty} \left(\frac{n}{m}\right)^{(k-1)/2} \left(\delta_{nm} + 2\pi i^{-k} \sum_{N|c} \frac{S(n, m, c)}{c} J_{k-1} \left(\frac{4\pi\sqrt{nm}}{c} \right) \right) e(nz).$$

Hint: At some point you might encounter the integral

$$\int_{\Im v = y} (cv)^{-k} e\left(-\frac{m}{c^2 v} - nv\right) dv$$

for $n \in \mathbb{Z}$. Show that this integral vanishes if $n \leq 0$ by moving the line of integration upwards. If $n > 0$, use that this is $2\pi i^{-k} c^{-1} (n/m)^{(k-1)/2} J_{k-1}(4\pi\sqrt{nm}/c)$.

b) Let $F(z) = \sum_{n=1}^{\infty} \hat{F}(n) e(nz) \in S_k(N)$ be a holomorphic cusp form (that is, $F(\gamma z) = j_\gamma(z)^k F(z)$ for $\gamma \in \Gamma$). Show that

$$\langle F, P_m \rangle = \int_{\Gamma \setminus \mathbb{H}} F(z) \overline{P_m(z)} y^k d\mu(z) = \frac{\Gamma(k-1)!}{(4\pi m)^{k-1}} \hat{F}(m)$$

4.2. Let $\lambda_j = s_j(1-s_j) = 1/4 + t_j^2$ denote the finite (probably empty) sequence of exceptional eigenvalues of Δ (of weight 0) on $\Gamma_0(N) \setminus \mathbb{H}$.

Let $u_j(z) = \sum_{n \neq 0} \rho_j(n) W_{0, it_j}(4\pi|n|y) e(nx)$ denote the corresponding eigenfunction.

a) Let $X \geq 1$ be a (large) parameter. Show

$$\sum_j |\rho_j(1)|^2 X^{2s_j-1} \ll 1 + \frac{X^{1/2}}{N} (NX)^\varepsilon$$

and conclude $\lambda_j \geq 3/16$. *Hint:* Take $h(t) = (X^{it} + X^{-it})^2 / (t^2 + 1)^2$ in the Kuznetsov formula.

b) Choose a basis of the space of exceptional eigencuspforms as follows (you don't need to prove the details): If the subspace corresponding to λ_j is one-dimensional we can assume that u_j is a Hecke eigenform, in particular $\rho_j(1)$ does not vanish and is in fact $\gg N^{-1/2-\varepsilon}$ (since $\text{vol}(\Gamma_0(N)\backslash\mathbb{H}) \ll N^{1+\varepsilon}$). If the subspace corresponding to λ_j has more than one dimension, we might get oldforms, but not more than $\tau(N) \ll N^\varepsilon$ for each newform. We can therefore assume

$$\sum_j N^{-1-\varepsilon} X^{2s_j-1} \ll \sum_j |\rho_j(1)|^2 X^{2s_j-1}.$$

Use this to show the density estimate

$$\#\{j : s_j > \alpha\} \ll N^{3-4\alpha+\varepsilon}.$$

For $\alpha > 1/2$ this is small compared to the covolume of $\Gamma_0(N)$, but probably only a weak bound for the empty set...

7.3. Show the following “large-sieve”-type inequality for Fourier coefficients of holomorphic cusp forms: Let $k \geq 4$ be even, $\Gamma = \Gamma_0(N)$, and let $\{F_j = \sum_n \hat{F}_j(n)e(nz)\}$ be an orthonormal basis of $S_k(N)$ (it has size about $kN/12$). Show

$$\sum_j \frac{N\Gamma(k-1)}{(4\pi)^{k-1}} \left| \sum_{n \leq x} a_n n^{-(k-1)/2} \hat{F}_j(n) \right|^2 \ll (N+x) \sum_{n \leq x} |a_n|^2$$

for *any* sequence a_n of complex numbers.

This estimate displays clearly the “quasi-orthogonality” of cusp form coefficients if the level is large: Cauchy-Schwarz would give

$$\sum_{n \leq x} |a_n|^2 N \sum_j \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{m \leq x} m^{-(k-1)} |\hat{F}_j(m)|^2 \ll Nx \sum_{n \leq x} |a_n|^2.$$

Since obviously Cauchy-Schwarz loses a lot, the above estimate shows that one can't make a_n close to proportional to all $\hat{F}_j(n)$; even stronger, if the a_n conspire with $\hat{F}_j(n)$ for one j they behave randomly with respect to most others.

Hint: Call the left hand side $L(N)$. Open the square and apply Petersson. If $x \leq N/(2\pi)$ use the power series expansion for J_{k-1} and estimate trivially. Otherwise embed $S_k(N) \subseteq S_k(Np)$ for a suitable prime p of size about $4\pi x/N$, say. Since $[\Gamma_0(Np) : \Gamma_0(N)] \leq p+1$, conclude $L(N) \leq (1+1/p)L(Np)$.