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Algebra

Problem Set 8 (due Nov 21, 2006)

8.1. a) Let R be commutative ring and $\mathfrak{a} \subseteq R$ an ideal. Let $\emptyset \neq H \subseteq R$ be a nonempty subset satisfying $H \cap \mathfrak{a} = \emptyset$ and $h_1 h_2 \in H$ for $h_1, h_2 \in H$. Show that there is a prime ideal $\mathfrak{p} \in R$ such that $\mathfrak{p} \cap H = \emptyset$ and $\mathfrak{a} \subseteq \mathfrak{p}$. *Hint:* Zorn's lemma.

b) Let $\mathcal{N}(R)$ be the nilradical of R (cf. exercise 7.1c) and assume there exists $a \in R \setminus \mathcal{N}(R)$. Show that there is a prime ideal $\mathfrak{p} \in R$ with $a \notin \mathfrak{p}$. *Hint:* Use part a).

c) Show that $\mathcal{N}(R)$ is the intersection of all prime ideals of R (the empty intersection being R itself).

8.2. For a set $I \subseteq \mathbb{R}$ let $\mathcal{R}(I)$ be the set of bounded, continuous functions $f : I \to \mathbb{R}$. For $c \in I$ let $\mathfrak{m}_c$ be the maximal ideal $\{f \in \mathcal{R}(I) \mid f(c) = 0\}$.

a) Let $\mathfrak{m} \subseteq \mathcal{R}([0,1])$ be a maximal ideal. Show that there exists a unique $c \in [0,1]$ such that $\mathfrak{m} = \mathfrak{m}_c$.

b) Let $c \in [0,1]$. Show that the ideal $\mathfrak{m}_c \in \mathcal{R}([0,1])$ is not finitely generated. *Hint:* This requires some ε's and δ's.

c) Let $\mathfrak{a} \subseteq \mathcal{R}(\mathbb{R})$ be the set of all bounded continuous functions having compact support (i.e. $f(x) = 0$ for $|x|$ sufficiently large). Show that $\mathfrak{a}$ is an ideal that is not a prime ideal.

d) Show that $\mathcal{R}(\mathbb{R})$ has a maximal ideal that is not of the form $\mathfrak{m}_c$ for any $c \in \mathbb{R}$.

Remark: The map $c \mapsto \mathfrak{m}_c$ gives an embedding $I \hookrightarrow \mathcal{M}(I) := \{\text{maximal ideals of } \mathcal{R}(I)\}$. The latter set can be equipped with a topology such that it becomes compact, and I with the topology inherited from $\mathbb{R}$ becomes a subset. This is called the Stone-Cech-compactification of I.

8.3 Let R be a Noetherian ring.

a) Let $\mathfrak{a} \subseteq R$ be an ideal. Show that $R/\mathfrak{a}$ is Noetherian.

b) Assume that R has a unity, and let $f : R \to R$ be a ring epimorphism. Show that f is injective. *Hint:* Iterate f.

8.4. Fix a prime p. The group $\mathbb{Z}_p = \{(x_1, x_2 \ldots) \mid x_j \in \mathbb{Z}/p^j\mathbb{Z}, x_{j+1} \equiv x_j \pmod{p^j}\}$ becomes a ring via componentwise multiplication (a subring of the direct product $\prod_{j=1}^{\infty} \mathbb{Z}/p^j\mathbb{Z}$).

a) Show that $\mathbb{Z}_p$ is a domain.

b) Show that $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$.

c) Let $j \in \mathbb{N}$. Show that the map $f_j : \mathbb{Z}_p \to (p^j) = p^j\mathbb{Z}_p$, given by $f_j((x_n)) := (p^j x_n)$, is an isomorphism of additive groups.

d) Show that $\mathbb{Z}_p$ has exactly one maximal ideal, namely (p) (which by the preceding part is as a group isomorphic to $\mathbb{Z}_p$ itself).

e) Let $\phi : \mathbb{Z}_p \setminus \{0\} \to \mathbb{N}$ be given by $\phi((x_n)) = \min\{n \in \mathbb{N} \mid x_n \neq 0\}$. Show that $\mathbb{Z}_p$ is Euclidean with respect to ϕ. What are the prime (=irreducible) elements? *Hint:* Use parts b and c.