Prof. Valentin Blomer

University of Toronto
Department of Mathematics

Algebra

Problem Set 6 (due Oct 31, 2006)

6.1. Show that $O(n)$ is a Lie group. Conclude (one line) that $SO(n)$ is a Lie group. What is its dimension? *Hint:* Identify $O(n)$ as a preimage of a suitable map and consider the differential.

6.2. a) Let K be any field. Show that $SL_2(K)$ is generated by the set of matrices of the form $\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix}\right)$ with $b, c \in K$.

b) Now let K be a field with at least 5 elements and $\text{char} K \neq 2$. Show that the centre of $SL_2(K)$ is $\{\pm E\}$ and that $PSL_2(K) = SL_2(K)/\{\pm E\}$ is simple. As a suggestion, you can proceed as follows:

Let N be a normal subgroup of $SL_2(K)$ different from $\{\pm E\}$.

- Show that N contains a matrix of the form $\left(\begin{smallmatrix} & b \\ c & d \end{smallmatrix}\right)$.
- Show that N contains an upper triangular matrix. Probably you have to use that K has at least 5 elements, so there is $k \in K$ with $k^4 \neq 1$.
- Show that N contains a matrix of the form $\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)$.
- Show that N contains all matrices of the form $\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)$. Use that $\text{char} K \neq 2$.
- Conclude the claim.

6.3. Let Γ be any group. Show that there is a group G such that Γ is normal in G, and any automorphism $\sigma \in \text{Aut}(\Gamma)$ is given by $\sigma(x) = gxg^{-1}$ for some suitable $g \in G$ (that is, in some sense, $\text{Aut}(\Gamma) \subseteq \text{Inn}(G)$).

6.4. Fix a prime p. Show that the p-adic integers contain not only $\mathbb{Z}$, but also large parts of $\mathbb{Q}$. More precisely, let $U := \{\frac{a}{b} \mid a, b \in \mathbb{Z}, p \nmid b\} \subseteq \mathbb{Q}$. It is easy to see that U is a subgroup of the additive group of $\mathbb{Q}$. Show that there is an embedding $U \hookrightarrow \mathbb{Z}_p$. Why can the rest of $\mathbb{Q}$ not be embedded into $\mathbb{Z}_p$?

6.5. It is easy to see that in any abelian group the elements of finite order form a subgroup. This is not true in general. To find a counterexample,

a) look at $SL(2, \mathbb{R})$;

b) construct a suitable quotient of a free group, e.g. generated by two elements.

Which way do you prefer?