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Algebra

Problem Set 5 (due Oct 24, 2006)

5.1. Show that there is (up to isomorphism) only one group of order 255, namely $\mathbb{Z}/255\mathbb{Z}$. *Hint:* Construct suitable subgroups and consider the commutator.

5.2. a) How many (non-isomorphic) abelian groups of order 16200 do exist?

b) Let p_n be the number of abelian groups of order n, and for $\Re s > 1$ let $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ be the Riemann zeta function. Show

$$\sum_{n=1}^{\infty} \frac{p_n}{n^s} = \prod_{j=1}^{\infty} \zeta(js), \quad \Re s > 1. \tag{1}$$

Remark 1: Maybe you want to spend one line on the convergence of the product on the right side of (1). Use the known fact that a product $\prod(1+a_n)$ converges absolutely if and only if $\sum a_n$ converges absolutely (why? take logarithms!).

Remark 2: With standard methods of analytic number theory one can show that (1) implies

$$\sum_{n \leq x} p_n = Cx + O(\sqrt{x}), \quad \text{where } C = \prod_{j=2}^{\infty} \zeta(j) = 2.2948\ldots$$

In other words, "on average" there are about 2.3 abelian groups of given order. One can also show that for any $\varepsilon > 0$, there is a constant c_ε such that $p_n \leq c_\varepsilon n^\varepsilon$.

5.3. Let K be a field and $T \subseteq GL(n, K)$ the subgroup of upper triangular matrices. Show that T is solvable.

5.4 Define the upper central series of a group G by

$$\{1\} \leq Z_1 \leq Z_2 \leq \ldots$$

where $Z_1 := Z(G)$ is the centre, and Z_{j+1} is the unique subgroup of G satisfying $Z_{j+1}/Z_j = Z(G/Z_j)$.

a) Show that this gives a normal series of groups, i.e. Z_j is normal in G for all j.

b) Let G be a nilpotent group of class n, that is, we assume

$$G = G^0 \geq G^1 \geq \ldots \geq G^n = \{1\}$$

where $G^j = [G, G^{j-1}]$. Show by a detailed induction that $Z_n = G$.