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Algebra

Problem Set 4 (due Oct 17, 2006)

4.1. Let $\mathbb{H} := \{z \in \mathbb{C} \mid \Im z > 0\}$. Show that $SL(2,\mathbb{R})$ acts transitively on $\mathbb{H}$ by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}.$$

What quotient of $SL(2,\mathbb{Z})$ acts faithfully on $\mathbb{H}$?

4.2. List all finite groups having exactly three conjugacy classes.

4.3. a) Let N be a normal subgroup of a finite group G, and let $\mathcal{C}$ be a conjugacy class of G that is contained in N. Let $x \in \mathcal{C}$, and let Z_x be its centralizer. Let $k := [G : NZ_x]$. Show that $\mathcal{C}$ is the disjoint union of k conjugacy classes of equal size *in* N.

b) Apply part a) to the special case $G = S_5$, $N = A_5$. How many conjugacy classes are there in A_5? What are their sizes? Write down the class equation.

4.4. For a subgroup U of G define the normalizer $N(U) := \{g \in G \mid gU = Ug\}$.

a) Show that $N(U)$ is the largest subgroup in which U is normal.

b) Let U be a p-subgroup of a finite group G. Show that $[N(U) : U] \equiv [G : U] \pmod{p}$. *Hint:* Consider a suitable group action.