Prof. Valentin Blomer

University of Toronto
Department of Mathematics

Algebra

Problem Set 18 (due Mar 20, 2007)

18.1. Let L/K be a finite Galois extension, let M_1, M_2 be intermediate fields, and $H_j := \mathrm{Gal}(L/M_j)$ the corresponding subgroups of $\mathrm{Gal}(L/K)$.

a) Show that M_1, M_2 are K-isomorphic if and only if H_1 and H_2 are conjugated.

b) Show that the composite field M_1M_2 corresponds to $H_1 \cap H_2$, and the intersection $M_1 \cap M_2$ corresponds to $\langle H_1, H_2 \rangle$.

c) Now assume that M_1 and M_2 are Galois over K. Show that M_1M_2 is Galois over K, and $\phi : \mathrm{Gal}(M_1M_2/M_1) \to \mathrm{Gal}(M_2/(M_1 \cap M_2))$, given by $\sigma \mapsto \sigma|_{M_2}$, is bijective.

d) Still assuming that M_1, M_2 are Galois over K, show that $\psi : \mathrm{Gal}(M_1M_2/K) \to \mathrm{Gal}(M_1/K) \times \mathrm{Gal}(M_2/K)$, given by $\sigma \mapsto (\sigma|_{M_1}, \sigma|_{M_2})$, is injective. If in addition $M_1 \cap M_2 = K$, ψ is also bijective.

18.2. Let $p_1, \dots, p_n$ be distinct prime numbers. Show that $L := \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n})$ is Galois over $\mathbb{Q}$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^n$.

18.3. Let a, b be rational numbers with $b > a^2$. Consider $p = (x^2 - a)^2 - b \in \mathbb{Q}[x]$ with zeros $\pm\sqrt{a \pm \sqrt{b}}$, and assume that p is irreducible. Let L be the splitting field of p over $\mathbb{Q}$. What is the Galois group of L/K? Use $\sigma : \sqrt{a + \sqrt{b}} \mapsto \sqrt{a - \sqrt{b}}$ and $\tau : \sqrt{a + \sqrt{b}} \mapsto -\sqrt{a + \sqrt{b}}$ as generators. Write down a field diagram with all subfields of L and the corresponding Galois groups.