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Algebra

Problem Set 17 (due Mar 13, 2007)

17.1. Prove the following Theorem of Wedderburn (1905): Every finite skew field (= divison ring) K is commutative.

Preparation: Let $Z := \{x \in K \mid xy = yx \text{ for all } y \in K\}$, $q := \#Z$, and for $a \in K$ let $N(a) := \{x \in K \mid xa = ax\}$. Then K and all $N(a)$ are a vector spaces over the (commutative) field Z, hence $\#K = q^n, \#N(a) = q^{n(a)}$ for some $n, n(a) \in \mathbb{N}$. Let $\Phi_n \in \mathbb{Z}[x]$ be the n-th cyclotomic polynomial. The group K^* acts on itself by conjugation.

Write down the class equation for K^* and show that $\Phi_n(q) \mid q-1$. Conclude $n = 1$.

17.2. a) Prove the following Theorem of Liouville (1844): Let $a \in \mathbb{R}$ be algebraic over $\mathbb{Q}$ with degree $m \geq 2$. Show that there is a constant $c = c(a) > 0$ such that $|a - p/q| \geq cq^{-m}$ for all $p, q \in \mathbb{Z}$, $q > 0$. *Remark:* In other words, algebraic numbers cannot be approximated arbitrarily well by rational numbers.

b) Let $a \in \mathbb{R}$. Assume that for every $n \in \mathbb{N}$ there is $y = y(n) = p/q \in \mathbb{Q}$ (with $q \geq 2$) such that $0 < |a - y| < q^{-n}$ then a is transcendental over $\mathbb{Q}$.

c) Let $b \geq 2$ be an integer. Show that $a := \sum_{j=1}^{\infty} b^{-j!}$ is transcendental over $\mathbb{Q}$.

Hints: a) Let $f \in \mathbb{Q}[x]$ be the minimal polynomial of a. Start with $q^{-m} \leq |f(p/q)| \leq c'|a - p/q|$ for some constant $c' > 0$ and p/q close to a.

b) You need to show that the assumption implies that a is irrational.

17.3. a) Let L/K be an algebraic extension of fields.

a) Show that there is a smallest field N containing L such that N/K is normal. Show that N is unique up to isomorphism, that $[N : K] < \infty$ if $[L : K] < \infty$, and that $[N : K]$ is separable if $[L : K]$ is separable.

b) Let $L_s := \{x \in L \mid x \text{ separable over } K\}$ be the separable closure of K in L. Show that L_s is a subfield of L, and L/L_s is *purely inseparable*, i.e. for all $x \in L$ there there is $r \in \mathbb{N}_0$ such that $x^{p^r} \in L_s$ (where $p = \text{char}K$).

17.4. Let L/K be a field extension. Consider the following two statements:
A: "L perfect implies K perfect." B: "K perfect implies L perfect."
Prove or disprove A and B assuming
a) $L = K(x_1, \ldots, x_n)$ for some $n \in \mathbb{N}$.
b) L/K is algebraic;
c) L/K is separable;
d) L/K is purely inseparable (see problem 17.3b);
e) L/K is finite;
f) L/K is arbitrary.