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Algebra

Problem Set 16 (due Mar 6, 2007)

16.1. Let p be an odd prime number, $a \in \mathbb{Q}$, and assume that $a^{1/p} \notin \mathbb{Q}$. Show $[\mathbb{Q}(a^{1/p}) : \mathbb{Q}] = p$. Show by induction $[\mathbb{Q}(a^{1/p^n}) : \mathbb{Q}] = p^n$.

16.2. a) Let L/K be an algebraic extension, and let $a, b \in L$. Show (one line) that $\deg_K(a + b) \leq \deg_K(a)\deg_K(b)$. More generally, show that $\deg_K(p(a, b)) \leq \deg_K(a)\deg_K(b)$ for any polynomial $p \in K[x, y]$.

b) What is the splitting field of $x^6 + x^3 + 1 \in \mathbb{Q}[x]$?

c) Let L/K be a field extension. Show that L/K is algebraic if and only if every ring M with $K \subseteq M \subseteq L$ is a field.

Remark: The quick proof of part (a) gives an elegant algebraic argument that there is a polynomial over $\mathbb{Q}$ of degree at most 105 having the root $(2^{1/7} + 1)(e^{2\pi i/5} + 3^{1/3})$. However, it will take you at least half an hour to find this polynomial explicitly.

16.3. Let M_1/K, M_2/K be two finite extensions, and let $L := M_1M_2 := M_1(M_2) = M_2(M_1)$ be the composite field. Show that $[L : K] \leq [M_1 : K][M_2 : K]$. Assuming that $[M_1 : K]$ and $[M_2 : K]$ are coprime, show that $M_1 \cap M_2 = K$ and $[L : K] = [M_1 : K][M_2 : K]$.

16.4. Let L/K be a field extension, and $\phi : L \to L$ a K-homomorphism, i.e. $\phi|_K = id$.

a) Show that ϕ is bijective if L/K is algebraic.

b) Show that in general ϕ is not bijective.

c) Give an example of an algebraic extension L/K and a field homomorphism $\phi : L \to L$ that is not bijective (so $\phi|_K \neq id$).

16.5. a) Write down an addition and multiplication table of $\mathbb{F}_8$.

b) Show (two lines) that in a finite field $\mathbb{F}_q$ every element $a \in \mathbb{F}_q$ is a sum of two squares. *Hint:* What is the cardinality of $\#\{x^2 \mid x \in \mathbb{F}_q\}$ and $\#\{a - y^2 \mid y \in \mathbb{F}_q\}$?