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Algebra

Problem Set 15 (due Feb 27, 2007)

15.1. Let $q > 2$ be an integer.

a) Write down explicitly the q characters $G \to \mathbb{C}^*$ of the additive group $G = \mathbb{Z}/q\mathbb{Z}$ in terms of simple functions, and write down the orthogonality relations in this case.

b) Let now G be the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$, and let $(a, q) = 1$. Since G is abelian, every function $f : G \to \mathbb{C}$ is a class function, in particular $f(n) := 1$ if $n \equiv a \pmod q$, and $f(n) := 0$ otherwise. The function f is a linear combination $\sum_\chi a(\chi)\chi$ of characters $\chi : G \to \mathbb{C}$. Evaluate the coefficients $a(\chi)$.

15.2. a) Show that every continuous group homomorphism $\psi : \mathbb{R} \to \mathbb{R}$ is of the form $\psi(x) = cx$ for some $c \in \mathbb{R}$.

b) Conclude that every continuous group homomorphism $\phi : \mathbb{R} \to U_1 = S^1$ is of the form $\phi(x) = e^{icx}$ for some $x \in \mathbb{R}$.

c) Show that all continuous one-dimensional representations of $\rho : U_1 = SO_2(\mathbb{R}) = S^1 \to \mathbb{C}^*$ are unitary (i.e. their image is in U_1) and of the form $x \to x^n$ for $n \in \mathbb{Z}$. *Remark:* One can show that all irreducible continuous representations of U_1 are one-dimensional.

15.3. a) Let $\mathfrak{A}$ be a finite-dimensional algebra over a field K, and let $\mathfrak{gl}(A)$ be the vector space $End_K(A)$ of K-vector space homomorphisms $A \to A$ together with the Lie bracket $[X, Y] := X \circ Y - Y \circ X$ for $X, Y \in End_K(A)$. Let $\mathfrak{D}(A) := \{f : A \to A \text{ linear } \mid f(xy) = xf(y) + f(x)y\}$ the set of all "derivatives" of A. Show that $\mathfrak{D}(A)$ is a sub-Lie-algebra of $\mathfrak{gl}(A)$.

b) Show that for $A \in R^{n\times n}$ the matrix $e^{tA} \in SL_n(\mathbb{R})$ for all $t \in \mathbb{R}$ if and only if $trA = 0$, so $\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \text{ tr}A = 0\}$.

Alternatively, show that the matrix $e^{tA} \in O_n(\mathbb{R})$ for all $t \in \mathbb{R}$ if and only if A is skew symmetric, so $\mathfrak{o}_n(\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid A \text{ skew symmetric}\}$.

15.4. a) Let L/K be a field extension of finite degree. Show that $[L(x) : K(x)] = [L : K]$.

b) Show that $\mathbb{R}$ is transcendental over $\mathbb{Q}$. *Hint:* You can use that $\mathbb{R}$ is uncountable.

c) What is the degree of $2i + \sqrt{11} \in \mathbb{C}$ over $\mathbb{Q}$ and over $\mathbb{Q}(i)$?

d) Find the minimal polynomial of $2^{1/3} + 3^{1/2}$ over $\mathbb{Q}$.