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Algebra

Problem Set 11 (due Jan 23, 2007)

11.1. Let R, S be commutative rings with unity and $\phi : R \rightarrow S$ a homomorphism. Show that ϕ induces continuous map $\phi^* : \text{spec}S \rightarrow \text{spec}R$, given by $\phi^*(\mathfrak{p}) = \phi^{-1}(\mathfrak{p}) \subseteq R$. Here $\text{spec}R$ and $\text{spec}S$ are equipped with the Zariski topology.

11.2. Let R be a commutative ring with unity.

a) Show that the following are equivalent:

(1) R is a local ring with maximal ideal \mathfrak{m} .

(2) $\mathfrak{m} := R \setminus R^*$ is an ideal.

(3) There is a maximal ideal $\mathfrak{m} \subseteq R$ such that $1 + \mathfrak{m} := \{1 + m \mid m \in \mathfrak{m}\} \subseteq R^*$.

b) Let $S \subseteq R$ be a multiplicative subset containing 1, but not 0. Consider the map $\phi : \mathfrak{a} \mapsto S^{-1}\mathfrak{a}$ for ideals $\mathfrak{a} \subseteq R$. Show that ϕ is well behaved: it maps ideals to ideals, it commutes with taking sums or products of ideals and it preserves inclusions and intersections.

11.3. Let K be a field, V a finite-dimensional vector space over K , $A \in \text{Hom}_K(V)$ be a K -linear map.

a) Show that V becomes a unitary $K[x]$ -module via $(\sum a_j x^j, v) \mapsto \sum a_j A^j(v)$.

b) Show that the $K[x]$ -submodules of V are exactly the A -invariant subspaces of V .

c) Show that the $K[x]$ -module homomorphisms $B : V \rightarrow V$ are exactly the K -linear maps that commute with A .

d) Show that every element of the $K[x]$ -module V is a torsion element.

11.4. Here is an example that bases of free modules over non-commutative rings need not have the same cardinality. Let K be a field, and V a countably infinite dimensional vector space of K with basis $\{x_1, x_2, \dots\}$. Let $W := V \oplus V$ be another vector space with a countably infinite dimensional basis. Let $f = (f_1, f_2) : V \rightarrow W = V \oplus V$ be an isomorphism, and let $R := \text{End}_K(V)$. Define $g : R \oplus R \rightarrow R$ by

$$g(\phi, \psi)(v) := \phi(f_1(v)) + \psi(f_2(v))$$

for $\phi, \psi \in R, v \in V$. Show that g is an R -module-isomorphism, and so more generally $R^n \cong R^m$ for any $n, m \in \mathbb{N}$.