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Algebra

Problem Set 10 (due Jan 9, 2007)

10.1. a) Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots x_n]$ be an ideal, and let $\mathfrak{m}$ be a maximal ideal of $R := \mathbb{C}[x_1, \ldots, x_n]/\mathfrak{a}$. Show that $R/\mathfrak{m} \cong \mathbb{C}$.

b) Let R be a commutative Noetherian ring with unity. Show that $R[[x]]$ is Noetherian. *Hint:* Proceed similarly as in the proof of Hilbert's Basis Theorem. The role of the leading coefficients is now played by the trailing coefficients.

10.2 Which of the following six ideals $(x_1, x_2), (x_1, x_2, 2), (x_1+x_2), (x_1x_2), (x_1), (x_1+x_2^2, x_2 + x_1^2)$ are principal ideal, maximal ideals, prime ideals in (a) $\mathbb{Z}[x_1, x_2]$, (b) $\mathbb{Q}[x_1, x_2]$.

10.3. a) Let $n \in \mathbb{N}$. Show that $-1 + \prod_{j=1}^{n}(x - j) \in \mathbb{Z}[x]$ is irreducible.

b) Show that $x^3 + 12x^2 + 18x + 6 \in (\mathbb{Z}[i])[x]$ is irreducible. *Hint:* Eisenstein

10.4. Let $n \in \mathbb{N}$, and let K be a field. For $\mathfrak{a} \subseteq K[x_1, \ldots, x_n]$ an ideal, $V \subseteq K^n$ let $\mathcal{Z}(\mathfrak{a}) \subseteq K^n$ be the zero set and $\mathcal{I}(V)$ the vanishing ideal.

a) Show that $V \subseteq \mathcal{Z}(\mathcal{I}(V))$ and $\mathfrak{a} \subseteq \mathcal{I}(Z(\mathfrak{a}))$.

b) Show that in the first part of part a) equality holds if V is an algebraic set, and that in the second part equality holds if $\mathfrak{a}$ is the vanishing ideal of some set $V \in K^n$.

c) Let $n = 1$, and $f \in K[x]$ be a nonconstant polynomial. Describe the nilradical of $K[x]/(f)$ and the zero set $\mathcal{Z}(f)$ in terms of the prime factorization of f. When is $(f) = \mathcal{I}(\mathcal{Z}(f))$?

d) Let $f, g \in K[x_1, x_2]$, and assume that they do not have a nonconstant common divisor. Show that $\mathcal{Z}((f, g)) \subseteq K^2$ is finite.

Hint: Show that (f, g) contains a nonzero polynomial $h_1 \in K[x]$ by applying Gauß' lemma to show that f and g are relatively prime in $K(x)[y]$. Similarly show that (f, g) contains a nonzero polynomial $h_2 \in K[y]$.

Remark: One can show that $\#\mathcal{Z}((f, g)) \leq \deg f \cdot \deg g$.