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Algebra

Problem Set 1 (due Sept 19, 2006)

1.1. a) We define on $\mathbb{Q} \setminus \{1\}$ a binary operation by

$$x \odot y := x + y - xy.$$

Show that $(\mathbb{Q} \setminus \{1\}, \odot)$ is a group.

b) Let c be the speed of light. According to Einstein, two velocities v_1, v_2 pointing in the same directions are added by the rule

$$v_1 \oplus v_2 := \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}.$$

(Imagine a person walking inside a moving train from the back to the front, then his own speed and the speed of the train are added as above.) Show that this defines a group law on the interval $I = (-c, c)$. In other words, show that $\oplus$ is a binary operation on I which satisfies the group axioms. You can take the associative law for granted and don't need to prove it.

Remark: The interesting fact about this is that (a) this seemingly complicated law is governed by a group structure and (b) this implies that the addition of two speeds can never exceed $-c$ or c.

1.2 Let G be a group, and $\emptyset \neq U \subseteq G$ a subset.
a) Show that U is a subgroup if and only if $uv^{-1} \in U$ for any $u, v \in U$.
b) Assume that G is finite. Show that U is a subgroup if and only if $uv \in U$ for all $u, v \in U$.

1.3 Show that all subgroups of $\mathbb{Z}$ are of the form $m\mathbb{Z}$ for some $m \in \mathbb{Z}$. (*Hint:* Given a subgroup U of $\mathbb{Z}$, consider the smallest positive element in U.)

1.4 Let G a group, $a, b \in G$. Show that the order of ab is the order of ba (as an element in $\mathbb{N} \cup \{\infty\}$).