MATH 1045HF TOPICS IN ERGODIC THEORY: INTRODUCTION TO RANDOM WALKS ON GROUPS

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This class will focus on properties of group actions from a probabilistic point of view, investigating the relations between the dynamics, measure theory and geometry of groups.

Overview. We will start with a brieft introduction to ergodic theory, discussing measurable transformations and the basic ergodic theorems.

Then we will approach random walks on matrix groups and lattices in Lie groups, following the work of Furstenberg. Topics of discussion will be: positivity of drift and Lyapunov exponents. Stationary measures. Geodesic tracking. Entropy of random walks. The Poisson-Furstenberg boundary. Applications to rigidity.

We will then turn to a similar study of group actions which do not arise from homogeneous spaces, but which display some features of negatively curved spaces: for instance, hyperbolic groups (in the sense of Gromov) and groups acting on hyperbolic spaces. This will lead us to applications to geometric topology: in particular, to the study of mapping class groups and $Out(F_N)$.

Timetable and zoom link. Lecture schedule:

(1) Monday, 12-1
(2) Friday, 12-2

At least for the first two weeks, the course will run online on Zoom. After that, we will decide depending on the needs of the audience.
Zoom link: `https://utoronto.zoom.us/j/84960814260`
Passcode: `051379`

Prerequisites. An introduction to measure theory and/or probability, basic topology and basic group theory. No previous knowledge of geometric group theory or Teichmüller theory is needed.

Lecture plan.

(1) Introduction to the class
(2) The ergodic theorems: Birkhoff, Kingman, the martingale convergence theorem
(3) Random walks on abelian groups
Boundary theory
(4) Stationary measures
(5) Convergence to the boundary: Furstenberg's theorem
(6) Definitions of the Poisson boundary
(7) Entropy of random walks
(8) Triviality of Poisson boundary: the entropy criterion
(9) Identification of the Poisson boundary
(10) Ray approximation and strip approximation

(11) Furstenberg's rigidity for lattices in $SL_n(\mathbb{R})$
(12) The Martin boundary
Sublinear tracking
(13) The multiplicative ergodic theorem
(14) Sublinear tracking in CAT(0) spaces: Karlsson-Margulis
(15) Sublinear tracking in Teichmüller space
(16) Positive drift for random walks
Applications to topology
(17) Hyperbolic geometry
(18) Introduction to Gromov-hyperbolic spaces
(19) Applications: free groups
(20) The mapping class group
(21) The curve complex and its boundary
(22) The group $Out(F_N)$ and the complexes on which it acts
(23) Horofunction compactification
(24) Random walks on weakly hyperbolic groups

Bibliography.

At least for the first part of the class, the main reference will be

- A. Furman, *Random walks on groups and random transformations*, in *Handbook of dynamical systems, Vol. I.* They are also available online at `http://homepages.math.uic.edu/~furman/preprints/hb.pdf`.

For the boundary theory, we will mainly rely on the survey

- V. Kaimanovich, *Boundaries of invariant Markov operators: the identification problem*, in *Ergodic theory of $\mathbb{Z}^d$ actions*, eds. M. Pollicott, K. Schmidt. Also available online.

The later part of applications to geometric group theory and topology is somewhat new, so we will rely more heavily on research papers.

(1) General texts on random walks on groups
  (a) Y. Benoist, J.-F. Quint, *Random walks on reductive groups*
  (b) P. Bougerol, J. Lacroix, *Products of random matrices with applications to Schrödinger operators*
  (c) W. Woess, *Random walks on infinite graphs and groups*
(2) Boundary theory
  (a) H. Furstenberg, *Non-commuting random products*
  (b) H. Furstenberg, *Poisson boundaries and envelopes of discrete groups*
  (c) V. Kaimanovich, A. Vershik, *Random walks on discrete groups: boundary and entropy*
  (d) V. Kaimanovich, *The Poisson formula for groups with hyperbolic properties*
(3) Hyperbolic geometry
  (a) M. Bridson, A. Haefliger, *Metric spaces of nonpositive curvature*
  (b) B. Farb, D. Margalit, *A primer on the mapping class group*
(4) Applications
  (a) A. Karlsson, G. Margulis, *A Multiplicative Ergodic Theorem and Nonpositively Curved Spaces*
  (b) V. Kaimanovich, H. Masur, *The Poisson boundary of the mapping class group*

(c) J. Maher, G. Tiozzo, *Random walks on weakly hyperbolic groups*