Statistical properties for coarse expanding dynamical systems

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Quasiworld workshop
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Summary

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joint with T. Das, F. Przytycki, M. Urbański, A. Zdunik
Thurston maps
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- $f(z) := \frac{p(z)}{q(z)}$ a postcritically finite rational map.
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Examples

- \( f(z) := \frac{p(z)}{q(z)} \) a postcritically finite rational map.
- A Lattés map \( g(z) = 2z \) as \( g : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z} \) and \( z \sim -z \)

\[
\begin{array}{ccc}
\mathbb{C}/\Lambda & \xrightarrow{g} & \mathbb{C}/\Lambda \\
\downarrow & & \downarrow \\
\hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}}
\end{array}
\]
Lattés maps

\[ g(z) = 4 \frac{z(1 - z^2)}{(1 + z^2)^2} \]
Pillow maps with flaps

Modification of Lattés maps: add a “flap”
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Visual metric on the sphere:
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**Visual metric** on the sphere:

\[
\text{diam(\text{piece of depth } n)} \approx \lambda^{-n}
\]
History

- **Bonk-Meyer**: Expanding Thurston maps.
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- **Z. Li**: ergodic theory of expanding Thurston maps. Existence and uniqueness of equilibrium measures. Note: he works directly on the sphere, estimating the PF operator.
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Finite branched coverings

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where $U$ ranges over all open neighborhoods of $y$. A point $y$ is **critical** if $\deg(f; y) > 1$. 

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We define the **branch set** as $B_f := \{ y \in Y : \deg(f; y) > 1 \}$
The setting

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If $\mathcal{U}$ is a cover of $W_1$, then

$$\mathcal{U}_n := \{\text{connected components of } f^{-n}(\mathcal{U})\}$$
Coarse expanding conformal (cxc) systems

Haïssinsky-Pilgrim axioms
Coarse expanding conformal (cxc) systems

Haïssinsky-Pilgrim axioms

A finite branched cover \( f : \mathcal{W}_1 \to \mathcal{W}_0 \) which satisfies:

- **Expanding**: There exists a cover \( U \) of \( \mathcal{W}_1 \) such that
  \[
  \lim_{n \to \infty} \text{diam}(U^n) = 0
  \]

Formally: For any other cover \( V \) there exists \( N \) such that for any \( n \geq N \), every element of \( U^n \) is contained in some element of \( V \).
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- **[Irreducibility]**: For any \( x \in X \) and any open set \( W \ni x \), there exists \( n \) such that \( f^n(W) \supseteq X \).
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- **[Degree]**: \( \exists C > 0 \) s.t.

  \[
  \deg(f^k : U \to V) \leq C
  \]

  for any \( U \in \mathcal{U}_{n+k}, \ V \in \mathcal{U}_n \).
Weakly coarse expanding (wxc) systems

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Note: If $W_1 \subseteq S_2$, then [Finiteness] is automatic (Whyburn).
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Definition

A metric $\rho$ on $X$ is exponentially contracting if $\exists C, \alpha > 0$ such that

$$\text{diam}_{\rho}(U) \leq Ce^{-\alpha n}$$

for any $U \in \mathcal{U}_n$. 

Lemma

For any weakly coarse expanding system $f: \mathcal{W}_1 \to \mathcal{W}_0$, there exists an exponentially contracting metric $\rho$ on the repellor $X$. 

Definition

A metric $\rho$ on $X$ is a visual metric if $\exists C_1, C_2, \alpha > 0$ such that

$$C_1 e^{-\alpha n} \leq \text{diam}_{\rho}(U) \leq C_2 e^{-\alpha n}$$

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From topological to metric

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Examples of weakly coarse expanding systems

- Expanding Thurston maps

Subhyperbolic rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also cxc.

Collet-Eckmann rational maps:

\[ |(f^n)'(f(c))| \geq C \lambda^n \]
for \( \lambda > 1 \) if \( c \) does not map to another critical point, and there are no parabolic cycles.

Topological Collet-Eckmann rational maps: there exist \( M \geq 0 \), \( P \geq 1 \), \( r > 0 \) such that for every \( x \in X \) there exists a sequence \( n_j \) with \( n_j \leq P \cdot j \) and for each \( j \neq \{0 \leq i < n_j: \text{Comp} f_i(x) f^{-n_j + i} B(f^{n_j}(x), r) \cap \text{Crit} f \neq \emptyset} \leq M \). These are wcx but not cxc.
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  for \( \lambda > 1 \) if \( c \) does not map to another critical point, and there are no parabolic cycles
- **Topological Collet-Eckmann** rational maps:
Examples of weakly coarse expanding systems

- **Expanding Thurston maps**

- **Subhyperbolic** rational maps: the critical points either converge to attracting cycles, or eventually map to a repelling periodic cycle. These are also **cxc**.

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\[
\#\{0 \leq i < n_j : C_{\text{comp}} f_{i(x)} f^{-(n_j-i)} B(f^{n_j}(x), r) \cap \text{Crit } f \neq \emptyset\} \leq M
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Examples of weakly coarse expanding systems - II

- **Polymodials** (Blokh-Cleveland-Misiurewicz)

\[ f(z) = \lambda \frac{z^2}{|z|} + 1 \]
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Finiteness holds under certain conditions: e.g. if

\[ f_1(z) = z^2 + 2, \quad f_2(z) = z^2 - 2 \]
Thermodynamic formalism

Let $f : X \to X$ the dynamics, and $\varphi : (X, \rho) \to \mathbb{R}$ be Hölder function.
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If $\mu$ is an $f$-invariant measure on $X$, 

$P_\mu(\varphi) := h_\mu(f) + \int \varphi \, d\mu$

and the topological pressure is

$P_{\text{top}}(\varphi) = \sup_{\mu \in \mathcal{M}(f)} P_\mu(\varphi)$

Definition

A measure $\mu$ which achieves the sup is called an equilibrium state.

$\varphi = 0$: $P_{\text{top}}(\varphi)$ is top. entropy, $\mu$ is measure of maximal entropy

$\varphi = -s \log |f'|$: $\mu$ is a conformal measure of dimension $s$ ⇒ compute Hausdorff dimension of $X$.
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Statement of results

Theorem

Let $f : W_1 \to W_0$ be a wcx system with $W_1 \subseteq S^2$, 

\[
\text{there exists a unique equilibrium state } \mu_{\psi} \text{ for } \psi \text{ on } X.
\]

Let $\psi : (X, \rho) \to \mathbb{R}$ be H"older continuous observable, and denote $S^n \psi(x) := \sum_{k=0}^{n-1} \psi(f^k(x))$. Then there exists the finite limit

\[
\sigma_2 := \lim_{n \to \infty} \frac{1}{n} \int_X (S_n \psi(x) - n \int \psi \, d\mu_{\psi})^2 \, d\mu_{\psi} \geq 0
\]

such that the following statistical laws hold:
Statement of results

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Let $f : W_1 \to W_0$ be a wcx system with $W_1 \subseteq S^2$, let $X$ be its repellor, $\rho$ an exp. contr. metric on $X$, $\psi : (X, \rho) \to \mathbb{R}$ be H"older, then:

1. there exists a unique equilibrium state $\mu_{\psi}$ for $\psi$ on $X$.

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such that the following statistical laws hold:
Statement of results - II

Theorem

1. (Central Limit Theorem, CLT) For any $a < b$,

$$
\mu_\varphi \left( \left\{ x \in X : \frac{S_n\psi(x) - n \int_X \psi \ d\mu_\varphi}{\sqrt{n}} \in [a, b] \right\} \right) \rightarrow \int_a^b \frac{e^{-t^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \ dt
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If \( \sigma = 0 \), it converges to a \( \delta \)-mass at 0.
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*If \( \sigma = 0 \), it converges to a \( \delta \)-mass at 0.*

2. *(Law of Iterated Logarithm, LIL)* For \( \mu_\varphi \)-a.e. \( x \in X \),

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\limsup_{n \to \infty} \frac{S_n\psi(x) - n \int_X \psi \, d\mu_\varphi}{\sqrt{n \log \log n}} = \sqrt{2\sigma^2}.
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$$

3. (Exponential Decay of Correlations, EDC) For any $\chi \in L^1(X, \mu_\varphi)$ there exist $\alpha > 0$, $C \geq 0$ such that

$$
\left| \int_X \psi \cdot (\chi \circ f^n) \ d\mu_\varphi - \int_X \psi \ d\mu_\varphi \cdot \int_X \chi \ d\mu_\varphi \right| \leq Ce^{-n\alpha}.
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\]
Statement of results - III

Theorem

1. *(Large Deviations, LD)*

2. \( \sigma = 0 \) if and only if there exists a continuous \( u : X \to \mathbb{R} \) such that \( \psi - \int_X \psi \, d\mu = u \circ f - u \).

3. \( \mu_1 = \mu_2 \) if and only if there exists \( K \in \mathbb{R} \) and a continuous \( u : X \to \mathbb{R} \) such that \( \phi_1 - \phi_2 = u \circ f - u + K \).

In (2) and (3), \( u \) is Hölder continuous w.r.t. the visual metric.
Statement of results - III

Theorem

1. (Large Deviations, LD) For every $t \in \mathbb{R}$,

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\lim_{n \to \infty} \frac{1}{n} \log \mu_\varphi \left\{ x \in X : \text{sgn}(t) S_n \psi(x) \geq \text{sgn}(t) n \int_X \psi \, d\mu_{\varphi+t\psi} \right\} =
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2. $\sigma = 0$ if and only if there exists a continuous $u : X \to \mathbb{R}$ such that

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$$= -t \int_X \psi \, d\mu_{\varphi + t\psi} + P_{\text{top}}(\varphi + t \psi) - P_{\text{top}}(\varphi).$$
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3. \( \mu_{\varphi_1} = \mu_{\varphi_2} \) if and only if there exists \( K \in \mathbb{R} \) and a continuous \( u : X \to \mathbb{R} \) such that

\[
\varphi_1 - \varphi_2 = u \circ f - u + K.
\]

In (2) and (3), \( u \) is Hölder continuous w.r.t. the visual metric.
Theorem

Let $f : W_1 \to W_0$ be a wcx system with no periodic critical points,
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Let $f : W_1 \rightarrow W_0$ be a wcx system with no periodic critical points, let $X$ be its repellor, $\rho$ an exp. contr. metric on $X$, 

1. there exists a unique equilibrium state $\mu_{\varphi}$ for $\varphi$ on $X$.
2. Central limit theorem
3. Law of Iterated Logarithm
4. Exponential Decay of Correlations
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Proof: the geometric coding

Let $f : W_1 \rightarrow W_0$ be wcx of degree $d$. 
Proof: the geometric coding

Let \( f : \mathcal{W}_1 \to \mathcal{W}_0 \) be wcx of degree \( d \). Let \( \Sigma = \{1, \ldots, d\}^\mathbb{N} \) and \( \sigma : \Sigma \to \Sigma \) the shift.
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Proof: no periodic critical points $\rightarrow$ no entropy drop

Lemma (No entropy drop)

For any invariant measure $\mu$, we have

$$h_{\mu}(\sigma) = h_{\sigma_*\mu}(f)$$
Proof: no periodic critical points $\rightarrow$ no entropy drop

Lemma (No entropy drop)

For any invariant measure $\mu$, we have

$$h_\mu(\sigma) = h_{\sigma^*\mu}(f)$$

For a finite orbit $(x_i)$, we call an $\epsilon$-singular time an index $i$ s.t. exists $y$ with $f(x_i) = f(y)$ and $d(x_i, y) < \epsilon$. 
Proof: no periodic critical points \(\rightarrow\) no entropy drop

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Lemma

For any \(0 < \zeta < 1\) there exists \(\epsilon\) s.t.

\[
\#\{i \leq n : x_i \text{ is } \epsilon - \text{singular} \} \leq \zeta n
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Lemma $\Rightarrow S_{n,x} := \#\{n\text{-cylinders intersecting } \pi^{-1}(x)\}$
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“fiber is subexponentially small” $\rightarrow$ no entropy drop
Periodic critical points

**Warning**: for periodic critical points entropy may drop for some measure $\mu$. 
The blowup procedure

Suppose $f(p) = p$ is a critical point.
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Construct \( \pi : Y \to X \)
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Suppose $f(p) = p$ is a critical point. Construct $\pi : Y \to X$ by blowing up critical point to a circle.
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Define $g : Y \rightarrow Y$
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Suppose $f(p) = p$ is a critical point.
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Define $g : Y \to Y$ as $g(\theta) := d\theta$ on $\pi^{-1}(p)$. 
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$Y \cong$ Siérpinski carpet $\to X \cong$ sphere
The blowup procedure

Suppose \( f(p) = p \) is a critical point.
Construct \( \pi : Y \rightarrow X \) by blowing up critical point to a circle
Define \( g : Y \rightarrow Y \) as \( g(\theta) := d\theta \) on \( \pi^{-1}(p) \).
Extend the dynamics on the preimage of the critical orbit.

\[ Y \cong \text{Siépinski carpet} \quad \rightarrow \quad X \cong \text{sphere} \]
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Hard part: define a metric $\tilde{\rho}$ on $Y$ which is exponentially contracting for $g$. 
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- Since \((Y, \bar{\rho})\) is weakly coarse expanding, one constructs semiconjugacies:

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- Then \(g : Y \to Y\) is wcx without periodic critical points
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- The general theorem follows.
Further directions

- Do you have \textit{equidistribution} of preimages?
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\[ \frac{1}{\# f^{-n}(x)} \sum_{y \in f^{-n}(x)} \delta_y \to \mu \]
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- (D. Meyer) Can you identify the Lebesgue measure from the thermodynamics?
The end

Thank you!