Irregular behavior for semigroup actions

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Dynamical Systems Seminar - University of Toronto

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Plan of the talk

1. Ergodic theorems for (semi)group actions

   ('quenched and annealed averaging')

2. Irregular points
3. Main results and application to linear cocycles
4. Some ideas in the proofs

Ergodic theorem ($\mathbb{N}$ and $\mathbb{Z}$ actions)

- Birkhoff (1931)

If $f\colon (X,\mu) \to (X,\mu)$ is a measure preserving map and $\varphi \in L^1(\mu)$ then

$$\tilde{\varphi}(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

exists for μ-a.e. x, and

$$\int \tilde{\varphi}\, d\mu = \int \varphi\, d\mu.$$

If, in addition, μ is ergodic then the time averages converge a.e. to $\int \varphi\, d\mu$.

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- Assume f is continuous and X is a compact metric space. The basin of attraction of $\mu \in \mathcal{M}_{erg}(f)$

$$B(\mu) := \left\{ x \in X\colon \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \to^{w^*} \mu \right\}$$

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- Rmk: The ergodic theorem holds non-stationary identically distributed dynamical systems:

• if $(f_t)_{t\in T}$ preserve (X,μ) and $\varphi \in L^1(\mu)$ then

$$\varphi_n^{\underline{t}} := \varphi(f_{t_n} \circ \cdots \circ f_{t_2} \circ f_{t_1})$$

are identically distributed r.v. (depending on $\underline{t} = (t_1, t_2, \dots)$)

• if ν is a probability measure on T then, for $\nu^{\mathbb{N}}$ a.e. $\underline{t} = (t_1, t_2, \dots)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_n^{\underline{t}}(x) \text{ exists } \mu\text{-a.e.}$$

Ergodic theorems

Rmk:

• The skew-product

$$\begin{array}{rccc} F: & T^{\mathbb{N}} \times X & \to & T^{\mathbb{N}} \times X \\ & ((t_1, t_2, \dots), x) & \mapsto & (\sigma(t_1, t_2, \dots), f_{t_1}(x)) \end{array}$$

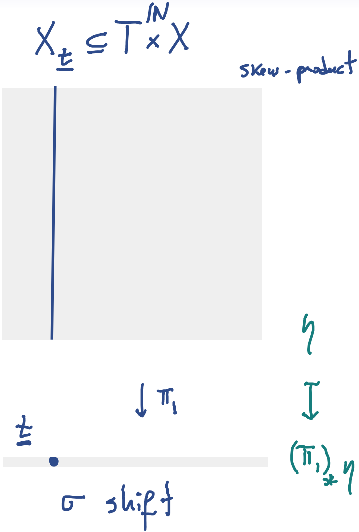
preserves $\nu^{\mathbb{N}} \times \mu$

• Take $\hat{\varphi}(\underline{t}, x) = \varphi(x)$. By the ergodic and Fubini theorems, for $\nu^{\mathbb{N}}$-a.e. $\underline{t}$ there exists $X_{\underline{t}} \subset X$ of full μ-measure so that

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi_n^{\underline{t}}(x) = \frac{1}{n} \sum_{j=0}^{n-1} \hat{\varphi}(F^j(\underline{t}, x))$$

exists for every $x \in X_{\underline{t}}$.

Ergodic theorems



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RMK:

- If ζ is F-invariant and $\hat{\varphi} \in L^1(\zeta)$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\varphi}(F^j(\underline{t}, x)) \text{ exists}$$

for $(\pi_1)_*\zeta$-a.e. $\underline{t}$ and for every $x \in X_{\underline{t}}$, where $X_{\underline{t}} \subset X$ is a full $\mu_{\underline{t}}$-measure set

Ergodic theorems (more general group actions)

G finitely generated (semi)group

$G_1 = \{g_1, g_2, \ldots, g_\kappa\}$ generating set (or $G_1 = \{g_1^{\pm 1}, g_2^{\pm 1}, \ldots, g_\kappa^{\pm 1}\}$)

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Assume $S : G \times X \to X$ is a continuous group action:

(i) for every $g \in G$, the map $S_g := S(g, \cdot) : X \to X$ is continuous,

(ii) $S_{hg} = S_h \circ S_g$ for every $g, h \in G$.

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- Templeman (1967) Lindenstrauss (2001)
If G is an amenable group acting by measure preserving maps, $\varphi \in L^1(\mu)$ and $(F_n)_{n \geqslant 1}$ is a tempered Følner sequence then

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g \in F_n} \varphi(g(x))$$

exists for μ-a.e. x

A Følner sequence is tempered if $\exists C > 0$ s.t.

$$\Big| \bigcup_{1 \leqslant k < n} F_k^{-1} F_n \Big| \leqslant C |F_n| \qquad \forall n \in \mathbb{N}.$$

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- Guivarch (1969), Nevo & Stein (1994), Bufetov (2002)

If the free group $G = \mathbb{F}_\kappa$ acts by measure preserving maps and $\varphi \in L^p(\mu)$ ($p > 1$) then

$$\lim_{n\to\infty} \frac{1}{2\kappa(2\kappa-1)^n} \sum_{|g|=n} \varphi(g(x))$$

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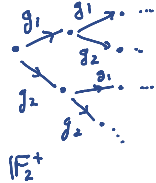
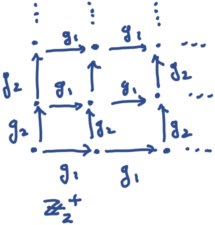
- Pathwise ergodic theorem

Cayley graph $\mathcal{G}$





Coding of infinite paths:

$\mathbb{F}_\kappa \to \mathcal{G}$ (or $\{1, 2, \ldots, \kappa\}^{\mathbb{Z}} \to \mathcal{G}$)

$\mathbb{P}$ random walk on $\mathbb{F}_\kappa$

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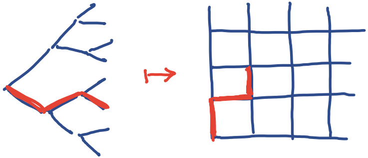

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If G acts by measure preserving maps and $\varphi \in L^1(\mu)$ then 'almost all' infinite paths in the Cayley graph $\mathcal{G}$ are so that

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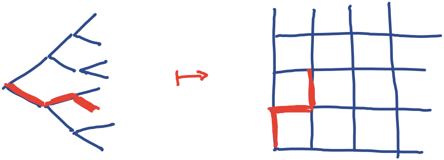

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- Rmk: Ghys (2001) proved that a Baire generic pair $(f, g) \in \mathrm{Homeo}(M)$ generates a free group

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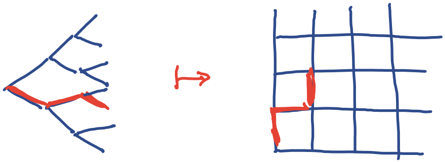

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Irregular behavior (a.k.a. non-typical or historical behavior)

- $x \in X$ is *φ-irregular* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \text{ does not exist}$$

- $I_\varphi(f)$ is the set of φ-irregular points
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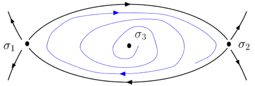

Figure: Irregular behavior on Bowen's eye

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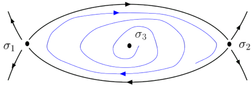

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- A dichotomy (Takens 94', 08', Barreira, Schmeling 00', Chen, Küpper, Shu 05', Li, Wu 13',...)

If $f : \mathbb{S}^1 \to \mathbb{S}^1$ is $C^{1+\alpha}$-expanding map and $\varphi : \mathbb{S}^1 \to \mathbb{R}$ is Hölder then

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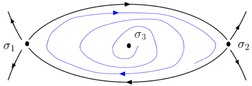

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Questions:

1. Are there simple criteria to detect when $I_\varphi(f)$ is Baire generic?
2. Can one expect such dichotomies in the context of group actions?
3. Can one describe the irregular sets of typical group actions (Birkhoff and group averaging)?

Main Results ($\mathbb{N}$ and $\mathbb{R}_+$ continuous actions)

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Let f be a continuous map on a compact metric space X. Given $\varphi \in C(X)$, consider the first integral

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Assume there exist $\alpha, \beta \in \mathbb{R}$ and dense sets $X_\alpha, X_\beta \subset X$ so that $L_\varphi(x) = \alpha < \beta = L_\varphi(y)$ for every $x \in X_\alpha$ and $y \in X_\beta$. Then $I_\varphi(f)$ is a Baire generic subset of X.

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Rmk: The assumptions are verified whenever there exist two distinct ergodic measures whose basins are dense in X (even if these are not fully supported)

Examples: Hyperbolic sets, continuous maps with specification, homoclinic classes, minimal non-uniquely ergodic maps, Lorenz attractors, singular hyperbolic flows, ...

Main Results (Group averaging)

Theorem 2 (Carvalho, Coelho, Salgado, V., preprint 2021')
Let X be a Baire metric space, $\Phi = (\varphi_n)_{n\geqslant 1} \in C(X)^{\mathbb{N}}$ and

$$\mathcal{W}_\Phi(x) = \mathrm{acc}(\varphi_n(x))_{n\geqslant 1}$$

Assume there exist dense subsets $\mathcal{D}_1, \mathcal{D}_2 \subset X$ and $\varepsilon > 0$ such that

$$\sup_{x\in D_1, y\in D_2} \; \sup_{a\in\mathcal{W}_\Phi(x), b\in\mathcal{W}_\Phi(y)} |a-b| > \varepsilon.$$

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$g_1, g_2 : \mathbb{S}^1 \to \mathbb{S}^1$
$g_1(x) = 2x \,(\mathrm{mod}\ 1)$
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$\mathcal{O}^-(p)$ and $\mathcal{O}^-(q)$ are dense in $\mathbb{S}^1$

Main Results (Group averaging)

Theorem 2 (Carvalho, Coelho, Salgado, V., preprint 2021')
Let X be a Baire metric space, $\Phi = (\varphi_n)_{n\geqslant 1} \in C(X)^{\mathbb{N}}$ and

$$\mathcal{W}_\Phi(x) = \operatorname{acc}(\varphi_n(x))_{n\geqslant 1}$$

Assume there exist dense subsets $\mathcal{D}_1, \mathcal{D}_2 \subset X$ and $\varepsilon > 0$ such that

$$\sup_{x\in D_1, y\in D_2} \sup_{a\in\mathcal{W}_\Phi(x), b\in\mathcal{W}_\Phi(y)} |a-b| > \varepsilon.$$

Then $I_\Phi(f)$ is a Baire generic subset of X.

Example:

$g_1, g_2 : \mathbb{S}^1 \to \mathbb{S}^1$
$g_1(x) = 2x \,(\text{mod } 1)$
$g_2(x) = 3x \,(\text{mod } 1)$
p, q common periodic points
$\varphi \in C(\mathbb{S}^1)$ s.t. $\int \varphi \, d\mu_p \neq \int \varphi \, d\mu_q$
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If $x \in \mathcal{O}^-(p)$ then

$$\frac{1}{n^2} \sum_{i,j=0}^{n} \varphi(g_1^i g_2^j(x))$$

and

$$\frac{1}{2^n} \sum_{|g|=n} \varphi(g(x)) = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \varphi(g_1^i g_2^{n-i}(x))$$

converge to $\int \varphi \, d\mu_p$

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Corollary: The sets

$$\left\{x \in \mathbb{S}^1 \colon \frac{1}{n^2}\sum_{i,j=0}^{n} \varphi(g_1^i g_2^j(x)) \text{ diverges}\right\}$$

$$\left\{x \in \mathbb{S}^1 : \frac{1}{2^n}\sum_{|g|=n} \varphi(g(x)) \text{ diverges}\right\}$$

are Baire residual subsets of $\mathbb{S}^1$.

Main Results (Averaging along paths)

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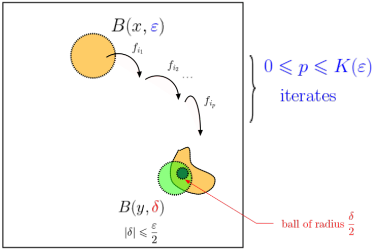
X compact metric space
$G_1 = \{id, f_1, f_2, \ldots, f_\kappa\}$ generators
G (semi)group generated by G_1
$S : G \times X \to X$ continuous semigroup action

S has frequent hitting times if $\forall \varepsilon > 0\ \exists K(\varepsilon) > 0$ so that the following holds:

given $B_1, B_2 \subset X$ balls of radius ε and $0 < \delta \leqslant \frac{\varepsilon}{2}$, respectively, there exists $0 \leqslant p \leqslant K(\varepsilon)$, $\underline{\omega} \in \Sigma_\kappa := \{1, 2, \ldots, \kappa\}^{\mathbb{N}}$ and a ball $B_2' \subset B_2$ of radius $\delta/2$ so that $f_{\underline{\omega}}^p(B_1) \supset B_2'$.



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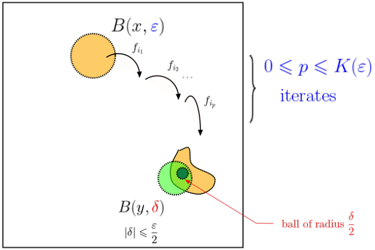

Rmks:

- Every minimal action by isometries has frequent hitting times
- The frequent hitting times condition implies that the sequence of return times to balls of radius ε are syndetic (with uniform constant)

Main Results (Averaging along paths)

Theorem 3 (Ferreira, V., 2021')
Let X be a compact metric space and $S : G \times X \to X$ be a semigroup action generated by bi-Lipschitz homeomorphisms $G_1 = \{f_1, f_2, \dots, f_\kappa\}$. If S has frequent hitting times and $\varphi \in C(X)$ is not a coboundary for some f_i then the set

$$I_\varphi(\mathbb{S}) := \Big\{x \in X : \frac{1}{n}\sum_{j=0}^{n-1} \varphi(g_\omega^j(x)) \text{ diverges} \\ \text{along some infinite path in G}\Big\}$$

is Baire generic in X. Moreover:

(i) $h^{GLW}(\mathbb{S}, I_{\mathbb{S}}(\varphi)) \geqslant H^{\text{Pinsker}}(\varphi)$

(ii) $h^B(\mathbb{S}, I_{\mathbb{S}}(\varphi)) \geqslant h_*(\varphi) - \log\kappa$

$h^{GLW}(\mathbb{S}, \cdot)$ = Ghys-Langevin-Walczak's entropy (1988)
$h^B(\mathbb{S}, \cdot)$ = Bufetov's entropy (1999)
$H^{\text{Pinsker}}(\varphi) = c$ if $\forall \varepsilon > 0\, \exists \mu_1, \mu_2 \in \mathcal{M}_{erg}(F)$ that distinguish φ and $h_{\mu_i}(F \mid \sigma) > c - \varepsilon$

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Some Notions of Entropy:

X compact metric space

$G_1 = \{id, g_1, g_2, \dots, g_\kappa\}$ continuous, $G = \bigcup_{n\geqslant 1} G_n$ semigroup

- $x, y \in X$ are (n, ε)-*separated* along the path $g_{\omega_n} \circ \cdots \circ g_{\omega_2} \circ g_{\omega_1}$ if there exists $1 \leqslant j \leqslant n$ s.t. $d(g^j_\omega(x), g^j_\omega(y)) > \varepsilon$
- Entropy of infinite path $\mathcal{F}_\omega = (g^j_\omega)_j$ in G (Kolyada-Snoha 96'):
$$h(\mathcal{F}_\omega) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s(\omega, n, \varepsilon)$$
where $s(\omega, n, \varepsilon) =$ max. card. of (n, ε)-separated points along path
- GLW-entropy of semigroup action (Ghys-Langevin-Walczak 88'):
$$h^{GLW}(\mathbb{S}) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s(G, n, \varepsilon)$$
where $s(G, n, \varepsilon) =$ max. card. of points separated by G_n elements
- B-entropy of free semigroup action (Bufetov 99'):
$$h^{B}(\mathbb{S}) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \left(\frac{1}{\kappa^n} \sum_{g\in G_n} s(\omega, n, \varepsilon) \right)$$

Main Results (Averaging along paths)

Theorem 4 (Ferreira, V., 2021')
Let X be a compact metric space and $S : G \times X \to X$ be a semigroup action generated by bi-Lipschitz homeomorphisms $G_1 = \{f_1, f_2, \ldots, f_\kappa\}$. If S has frequent hitting times, some f_i is minimal and $\varphi \in C(X)$ is not a coboundary for some f_i then

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satisfies:

(i) $\{\omega \in \Sigma_\kappa : I_\omega(\varphi) \text{ Baire generic in X}\}$ is Baire generic in Σ_κ

(ii) $\sup_{\omega \in \Sigma_\kappa} \; h_{I_\omega(\psi)}(\mathcal{F}_\omega) \geqslant H^{\text{Pinsker}}(\psi)$

(iii) $\{\omega \in \Sigma_\kappa : h_{I_\omega(\psi)}(\mathcal{F}_\omega) \geqslant H^{\text{Pinsker}}(\psi)\}$ has entropy $\geqslant H_\sigma^{\text{Pinsker}}(\psi)$

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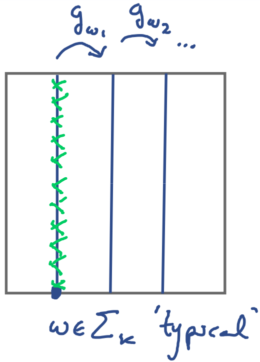

Rmk: Item (i) still holds without the minimality assumption. Previous results by Nakano (2017) on random circle expanding maps.

Application: Lyapunov irregular behavior

Given $A_1, A_2, \ldots, A_\kappa \in SL(d, \mathbb{R})$ and $\omega \in \Sigma_\kappa$ set $A^{(n)}(\omega) := A_{\omega_n} \ldots A_{\omega_2} A_{\omega_1}$

- Furstenberg-Kesten (1960) if $\mu = \nu^{\mathbb{Z}}$ the top Lyapunov exponent is (μ-a.e.)

$$\lambda_+(A, \nu) = \lim_{n \to \infty} \frac{1}{n} \log \|A^{(n)}(\omega)\|$$

- Furstenberg (1963) if $\mu = \nu^{\mathbb{Z}}$, the semigroup generated by matrices is *non-compact* and *strongly irreducible* on $\operatorname{supp} \nu$ then $\lambda_+(A, \nu) > 0$.

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- Skew-product

$$\begin{array}{rcccl} F_A\colon & \Sigma_\kappa \times \mathbb{R}^d & \longrightarrow & \Sigma_\kappa \times \mathbb{R}^d \\ & (x, v) & \mapsto & (f(x), A(x) \cdot v) \end{array}$$

- Projective cocycle

$$\begin{array}{rcccl} P_A\colon & \Sigma_\kappa \times \mathbf{P}\mathbb{R}^d & \longrightarrow & \Sigma_\kappa \times \mathbf{P}\mathbb{R}^d \\ & (\omega, v) & \mapsto & \left(\sigma(\omega), \frac{A(\omega) \cdot v}{\|A(\omega) \cdot v\|}\right) \end{array}$$

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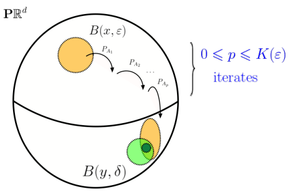
- Sumi- V.-Yamamoto (2016) these skew-products do not satisfy the specification property
- In low dimension the linear cocycle is 'often' strongly projectively accessible

(i.e. the projective semigroup action on $X = \mathbf{P}\mathbb{R}^d$ has frequent hitting times)



Example: $SO(3,\mathbb{R})$ matrices ($\alpha,\beta \notin \mathbb{Q}$)

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}$$

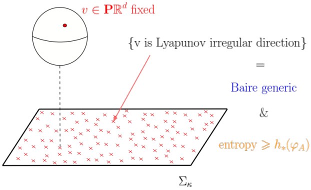
Application (Lyapunov irregular behavior)

Theorem 5 (Ferreira, V., 2021')
If $A_1, A_2, \dots, A_\kappa \in SL(d, \mathbb{R})$ generate a non-compact and strongly projectively accessible semigroup then:

- for each $v \in \mathbf{P}\mathbb{R}^d$ there exists $\mathcal{R}_v \subset \Sigma_\kappa$ Baire generic, with entropy at least $h_*(\varphi_A)$ s.t. for every $\omega \in \mathcal{R}_v$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\| < \limsup_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\| \quad (\star)$$

- there exists a Baire residual subset $\mathcal{R} \subset \Sigma_\kappa$ and a dense subset $\mathcal{D} \subset \mathbf{P}\mathbb{R}^d$ so that $(\star)$ holds for every $\omega \in \mathcal{R}$ and every $v \in \mathcal{D}$.



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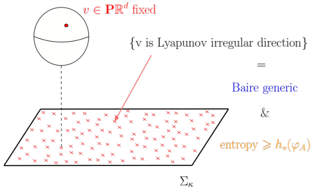

Rmk: Previous results on irregular behavior for the *top Lyapunov exponent* of Hölder continuous cocycles: Herman (1981), Furman (1997), Tian (2015, 2017) These rely on very different techniques: (i) u.s.c. of $\mu \mapsto \lambda_+(A, f, \mu)$, (ii) bounded distortion for linear cocycles by Kalinin (2011)

Application (Lyapunov irregular behavior)

Corollary (Ferreira, V., 2021')
Let $\mathcal{H} \subset C^0_{loc}(\Sigma_\kappa, SL(3,\mathbb{R}))$ be the set of hyperbolic cocycles. There exist C^0-open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ so that $\mathcal{U}_1 \cup \mathcal{U}_2$ is dense in $\mathcal{H}$ and:

1. if $B \in \mathcal{U}_1$ then the set of Lyapunov irregular points in Σ_κ is Baire generic and has full entropy
2. there exists $\mathcal{R} \subset \mathcal{U}_2$ C^0-*Baire residual and full Haar measure* s.t. if $B \in \mathcal{R}$ then

$$\liminf_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)v\| < \limsup_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)v\|.$$

for generic $\omega \in \Sigma_\kappa$ and a dense set of vectors $\mathcal{D}_\omega \subset E^s_\omega$.

A basic strategy (= quantitative control on recurrence)

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- take μ_1, μ_2 ergodic so that $\int \psi \, d\mu_1 \neq \int \psi \, d\mu_2$
- pick x_1, x_2 so that $\frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(x_i)) \to \int \psi \, d\mu_i$ $(i = 1, 2)$
- $n_1 \ll n_2 \ll n_3 \ll n_4 \ll \dots$ (arbitrary choice)
- *uniform continuity* + *specification* $\Rightarrow$ there exists z_k which approximates well the finite orbits of x_1 and x_2 alternatively

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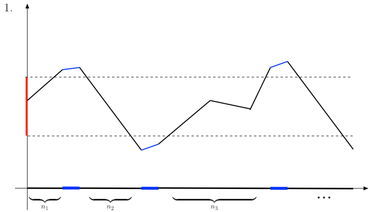
$$\underbrace{00\cdots0}_{n_1}\underbrace{*\cdots*}_{\leqslant p}\underbrace{111\cdots1}_{n_2}\underbrace{*\cdots*}_{\leqslant p}\underbrace{0000\cdots0}_{n_3}\underbrace{*\cdots*}_{\leqslant p}\dots\underbrace{11111\cdots1}_{n_k}$$

z_{k+1} is a point that $1/2^k$-shadows z_k and the next finite piece of orbit.

- $z = \lim_{k\to\infty} z_k$ is φ-irregular



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- if μ_1, μ_2 ergodic large entropy s.t. $\int \psi \, d\mu_1 \neq \int \psi \, d\mu_2$
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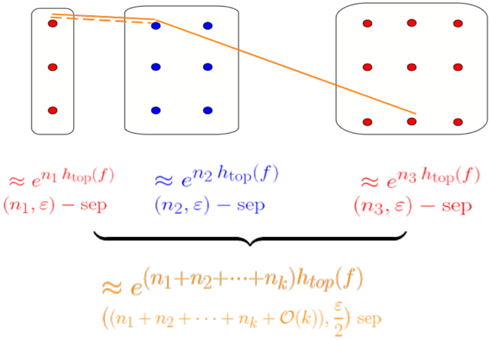

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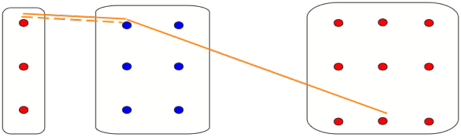


$\approx e^{n_1 h_{\text{top}}(f)}$ $\quad$ $\approx e^{n_2 h_{\text{top}}(f)}$ $\quad$ $\approx e^{n_3 h_{\text{top}}(f)}$

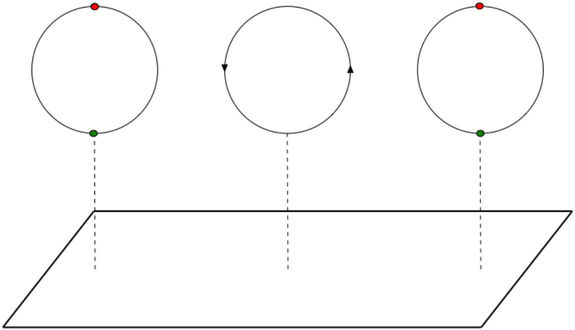
$(n_1, \varepsilon) - \text{sep}$ $\quad$ $(n_2, \varepsilon) - \text{sep}$ $\quad$ $(n_3, \varepsilon) - \text{sep}$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\substack{\approx e^{(n_1+n_2+\cdots+n_k)h_{top}(f)} \\ ((n_1+n_2+\cdots+n_k+\mathcal{O}(k)), \frac{\varepsilon}{2}) \text{ sep}}}$$

- Similar reasoning yields a Baire generic set

However: *This argument requires bounded transitions!*

Toy model



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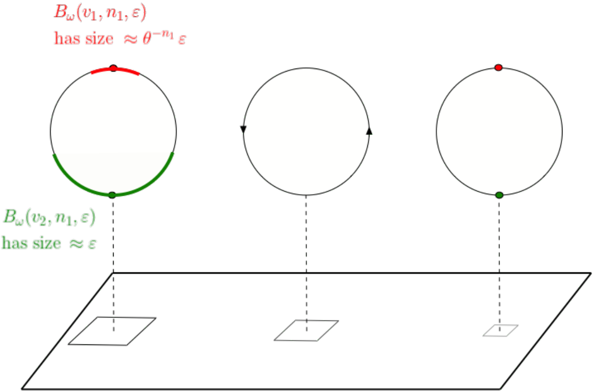
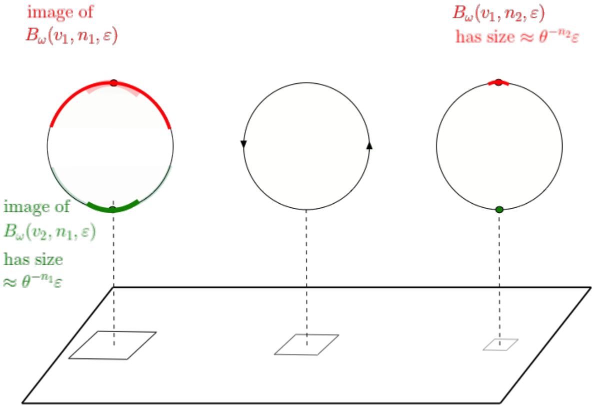

$B_\omega(v_1, n_1, \varepsilon)$
has size $\approx \theta^{-n_1}\,\varepsilon$
$B_\omega(v_2, n_1, \varepsilon)$
has size $\approx \varepsilon$

Toy model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Toy model

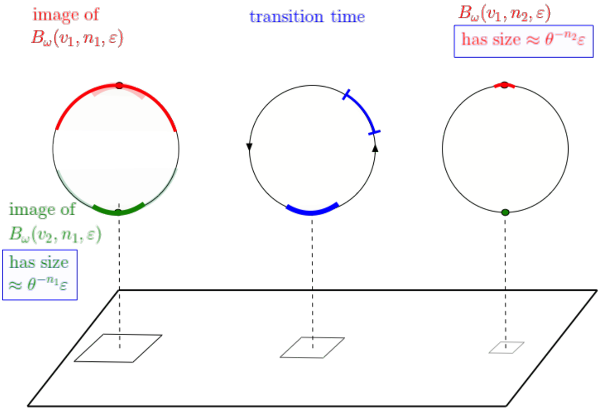
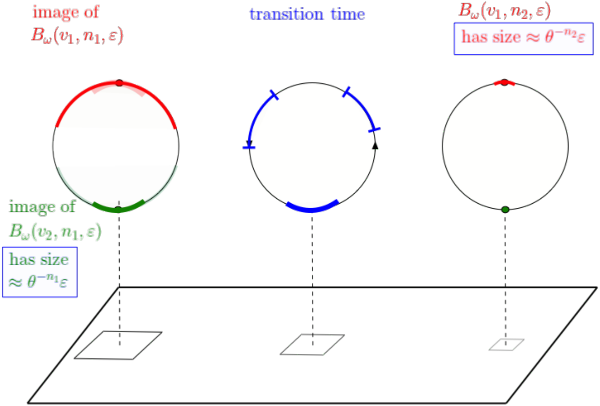


image of
$B_\omega(v_1, n_1, \varepsilon)$
transition time
$B_\omega(v_1, n_2, \varepsilon)$
has size $\approx \theta^{-n_2}\varepsilon$
image of
$B_\omega(v_2, n_1, \varepsilon)$
has size
$\approx \theta^{-n_1}\varepsilon$

Toy model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$B_\omega(v_1, n_1, \varepsilon)$
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$B_\omega(v_2, n_1, \varepsilon)$
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Toy model

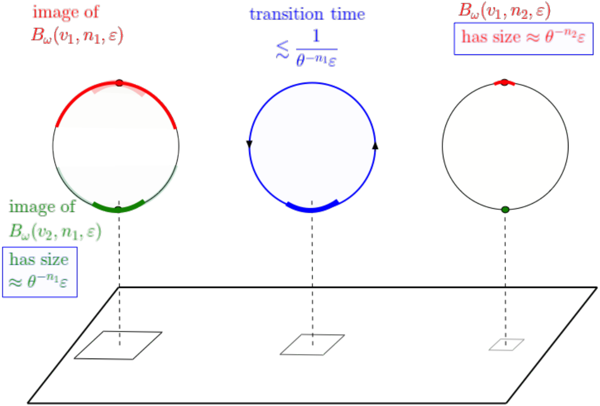


image of
$B_\omega(v_1, n_1, \varepsilon)$
transition time
$\lesssim \frac{1}{\theta^{-n_1}\varepsilon}$
$B_\omega(v_1, n_2, \varepsilon)$
has size $\approx \theta^{-n_2}\varepsilon$
image of
$B_\omega(v_2, n_1, \varepsilon)$
has size
$\approx \theta^{-n_1}\varepsilon$

Toy model

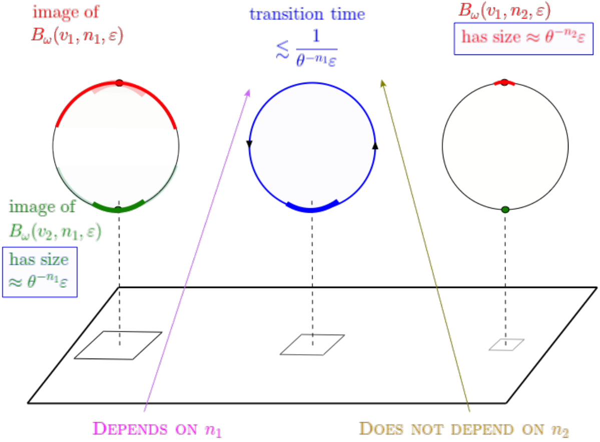
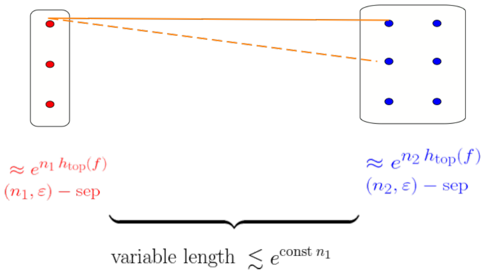


image of
$B_\omega(v_1, n_1, \varepsilon)$
transition time
$\lesssim \frac{1}{\theta^{-n_1}\varepsilon}$
$B_\omega(v_1, n_2, \varepsilon)$
has size $\approx \theta^{-n_2}\varepsilon$
image of
$B_\omega(v_2, n_1, \varepsilon)$
has size
$\approx \theta^{-n_1}\varepsilon$
Depends on n_1
Does not depend on n_2

A Different Approach

- if μ_1, μ_2 ergodic large entropy s.t. $\int \psi \, d\mu_1 \neq \int \psi \, d\mu_2$
- find many points x_i so that $\frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x_i)) \to \int \psi \, d\mu_i$



A Different Approach

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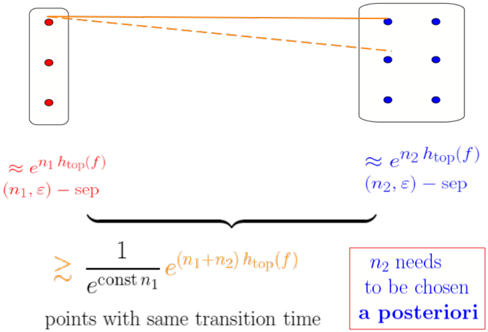

A Different Approach

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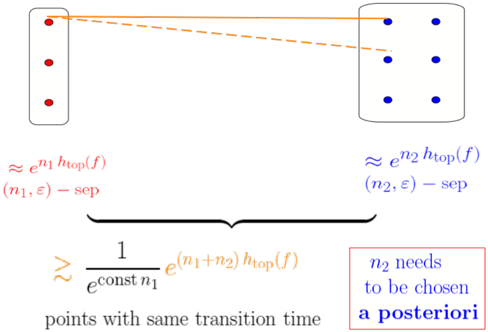

- $n_1 \ll n_2 \ll n_3 \ll n_4 \ll \ldots$ **(properly chosen)**
- bridge between linear cocycles and projective dynamics, build Moran sets with large entropy, ...

References

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Thank you