

# TOPOLOGICAL ENTROPY OF QUADRATIC POLYNOMIALS AND DIMENSION OF SECTIONS OF THE MANDELBROT SET

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ABSTRACT. A fundamental theme in holomorphic dynamics is that the local geometry of parameter space (e.g. the Mandelbrot set) near a parameter reflects the geometry of the Julia set, hence ultimately the dynamical properties, of the corresponding dynamical system. We establish a new instance of this phenomenon in terms of entropy.

Indeed, we prove an “entropy formula” relating the entropy of a polynomial restricted to its Hubbard tree to the Hausdorff dimension of the set of rays landing on the corresponding vein in the Mandelbrot set. The results contribute to the recent program of W. Thurston of understanding the geometry of the Mandelbrot set via the core entropy.

## 1. INTRODUCTION

Let us consider the family of quadratic polynomials

$$f_c(z) := z^2 + c, \quad \text{with } c \in \mathbb{C}.$$

The *filled Julia set*  $K(f_c)$  of a quadratic polynomial  $f_c$  is the set of points which do not escape to infinity under iteration, and the *Julia set*  $J(f_c)$  is the boundary of  $K(f_c)$ . The *Mandelbrot set*  $\mathcal{M}$  is the connectedness locus of the quadratic family, i.e.

$$\mathcal{M} := \{c \in \mathbb{C} : \text{the Julia set of } f_c \text{ is connected}\}.$$

A fundamental theme in the study of parameter spaces in holomorphic dynamics is that the local geometry of the Mandelbrot set near a parameter  $c$  reflects the geometry of the Julia set  $J(f_c)$ , hence it is related to dynamical properties of  $f_c$ . In this paper we will establish a new instance of this principle, by comparing the topological entropy of quadratic polynomials to the Hausdorff dimension of certain sets of external rays.

The map  $f_c(z) = z^2 + c$  is a degree-two ramified cover of the Riemann sphere  $\hat{\mathbb{C}}$ , hence a generic point has exactly 2 preimages, and the topological entropy of  $f_c$  always equals  $\log 2$ , independently of the parameter [Ly]. If  $c$  is real, however, then  $f_c$  can also be seen as a real interval map, and its restriction to the real line also has a well-defined topological entropy, which we will denote by  $h_{top}(f_c, \mathbb{R})$ . The dependence of  $h_{top}(f_c, \mathbb{R})$  on  $c$  is much more interesting: indeed, it is a continuous, decreasing function of  $c$  [MT], and it is constant on baby Mandelbrot sets not containing the Feigenbaum parameter [Do3] (see Figure 1).

The Riemann map  $\Phi_M : \hat{\mathbb{C}} \setminus \bar{\mathbb{D}} \rightarrow \hat{\mathbb{C}} \setminus \mathcal{M}$  uniformizes the exterior of the Mandelbrot set, and images of radial arcs are called *external rays*. Each angle  $\theta \in \mathbb{R}/\mathbb{Z}$  determines the external ray  $R_M(\theta) := \Phi_M(\rho e^{2\pi i \theta})_{\{\rho > 1\}}$ , which is said to *land* if the limit as  $\rho \rightarrow 1^+$  exists.

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*Key words and phrases.* Mandelbrot set; entropy; Hausdorff dimension; Hubbard trees; complex dynamics.

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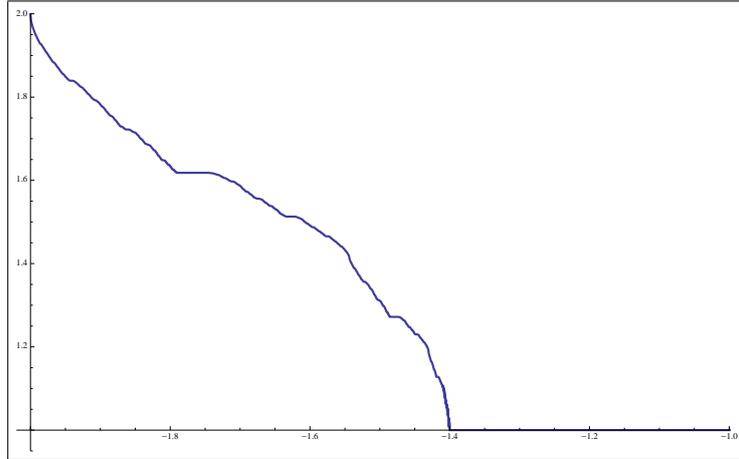


FIGURE 1. Topological entropy of the real quadratic family  $f_c(z) := z^2 + c$ , as a function of  $c$ . For each value of  $c \in [-2, -1]$ , we plot the growth number  $e^{h_{top}(f_c, \mathbb{R})}$ .

Given a subset  $A$  of  $\partial\mathcal{M}$ , one can define the *harmonic measure*  $\nu_M$  as the probability that a random ray from infinity lands on  $A$ :

$$\nu_M(A) := \text{Leb}(\{\theta \in S^1 : R_M(\theta) \text{ lands on } A\}).$$

If one takes  $A := \partial\mathcal{M} \cap \mathbb{R}$  to be the real slice of the boundary of  $\mathcal{M}$ , then the harmonic measure of  $A$  is zero. However, the set of rays which land on the real axis has full Hausdorff dimension [Za2]. (By comparison, the set of rays which land on the main cardioid has zero Hausdorff dimension.) As a consequence, it is more useful to look at Hausdorff dimension than harmonic measure; for each  $c$ , let us consider the section

$$M_c := \{\theta \in S^1 : R_M(\theta) \text{ lands on } \partial\mathcal{M} \cap [c, 1/4]\}$$

of all parameter rays which land on the real axis, to the right of  $c$ . The function

$$c \mapsto \text{H.dim } M_c$$

increases from 0 to 1 as  $c$  moves towards the tip of  $\mathcal{M}$ , as hyperbolic components become more “spread apart” near the tip, thus it is easier for rays to land on the real slice. In the dynamical plane, one can consider the set of rays which land on the real slice of  $J(f_c)$ , and let  $S_c$  be the set of external angles of rays landing on  $J(f_c) \cap \mathbb{R}$ . This way, we construct the function  $c \mapsto \text{H.dim } S_c$ , which we want to compare to the Hausdorff dimension of  $M_c$ .

The first main result is the following identity:

**Theorem 1.1.** *Let  $c \in [-2, 1/4]$ . Then we have*

$$\frac{h_{top}(f_c, \mathbb{R})}{\log 2} = \text{H.dim } S_c = \text{H.dim } M_c.$$

The first equality establishes a relation between entropy, Hausdorff dimension and the Lyapunov exponent of the doubling map (in the spirit of the “entropy formulas” [Ma], [Yo], [LeYo]), while the second equality can be seen as an instance of Douady’s principle relating the local geometry of the Mandelbrot set to the geometry of the corresponding Julia set. Indeed, we can replace  $M_c$  with the set of angles of rays landing on  $[c, c + \epsilon]$  in parameter space, as long as  $[c, c + \epsilon]$  does not lie in a tuned copy of the Mandelbrot set. Note that by Beurling’s theorem [Be], the set of rays which possibly do not land has zero logarithmic capacity, hence

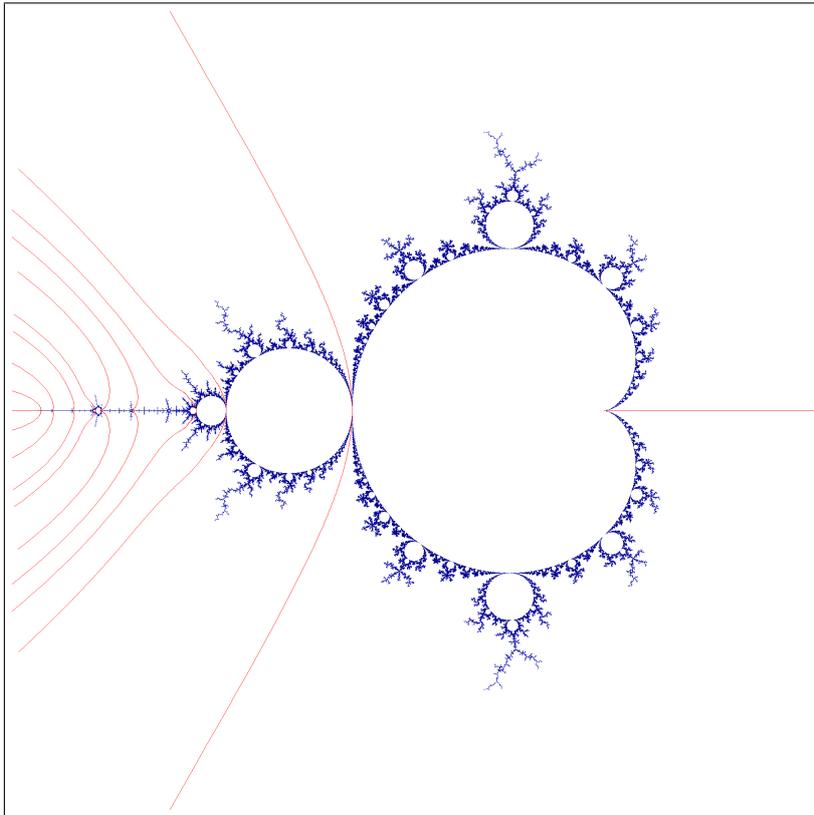


FIGURE 2. A few rays which land on the real slice of the Mandelbrot set.

zero Hausdorff dimension (see e.g. [Po], Theorems 9.19 and 10.3), thus the result is independent of the MLC conjecture.

A first study of the dimension of the set of angles of rays landing on the real axis has been done in [Za2], where it is proven that the set of angles of parameter rays landing on the real slice of  $\mathcal{M}$  has dimension 1. Zakeri also provides estimates on the dimension along the real axis, and specifically asks for dimension bounds for parameters near the Feigenbaum point ( $-1.75 \leq c \leq c_{Feig}$ , see [Za2], Remark 6.9). Our result gives an identity rather than an estimate, and the dimension of  $S_c$  can be exactly computed in the case  $c$  is postcritically finite (see following example).

Recall the dimension of  $S_c$  also equals the dimension of the set  $B_c$  of angles landing at biaccessible points (Proposition 6.1). Smirnov [Sm] first showed that such set has positive Hausdorff dimension for Collet-Eckmann maps. More recent work on biaccessible points is due, among others, to Zakeri [Za3] and Zdunik [Zd]. The first equality in Theorem 1.1 has also been established independently by Bruin-Schleicher [BS].

### Example

The *airplane map* has a superattracting cycle of period 3, and its characteristic angle is  $\theta_c = \frac{3}{7}$ . The topological dynamics of this real map is encoded by the right-hand side diagram: the interval  $A$  is mapped onto  $A \cup B$ , and  $B$  is mapped onto  $A$ . Then the number of laps of  $f_c^n$  is given by the Fibonacci numbers, hence the topological entropy is  $\log_2 \frac{\sqrt{5}+1}{2}$ .

Also note that the set of angles whose rays land on the Hubbard tree (see section 4 for a definition) for  $f_c$  is the set of numbers whose binary expansion does not contain any sequence of three consecutive equal symbols. It is a Cantor set which can be generated by the automaton in Figure 3, and its Hausdorff dimension also equals the logarithm of the golden mean.

It is harder to characterize explicitly the set of parameter rays which land on the boundary of  $\mathcal{M}$  to the right of the characteristic ray: however, as a consequence of Theorem 1.1, the dimension of such set is also  $\log_2 \frac{\sqrt{5}+1}{2}$ .

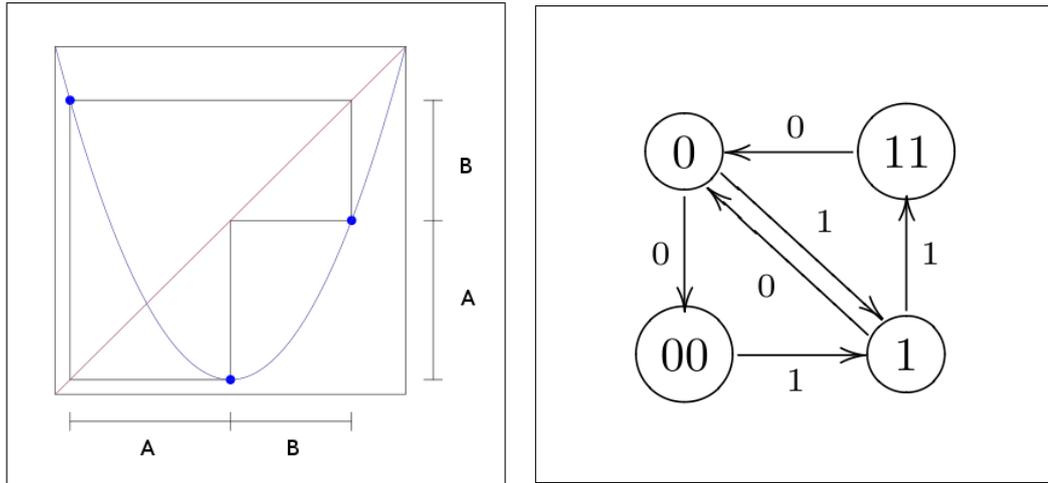


FIGURE 3. To the left: the combinatorics of the airplane map of period 3. To the right: the automaton which produces all symbolic codings of angles landing on its Hubbard tree.

A more complicated example is the *Feigenbaum parameter*  $c_{Feig}$ , the accumulation point of the period doubling cascades. Using estimates on logarithmic capacity, Manning [Ma2] proved that a large set of infinitely renormalizable rays lands in parameter space. As a corollary of Theorem 1.1, we are able to answer a question of Zakeri ([Za2], Remark 6.9):

**Corollary 1.2.** *For the Feigenbaum parameter  $c_{Feig}$  we have*

$$\text{H.dim } S_{c_{Feig}} = 0.$$

*As a corollary, the set of biaccessible angles for the Feigenbaum parameter also has Hausdorff dimension zero.*

**1.1. The complex case.** The result of Theorem 1.1 lends itself to a natural generalization for complex quadratic polynomials, which we will now describe.

In the real case, we related the entropy of the restriction of  $f_c$  on an invariant interval to the Hausdorff dimension of a certain set of angles of external rays landing on the real slice of the Mandelbrot set.

In the case of complex quadratic polynomials, the real axis is no longer invariant, but we can replace it with the Hubbard tree  $T_c$  (see section 4). In particular, we define the polynomial  $f_c$  to be *topologically finite* if the Julia set is connected and locally connected and the Hubbard tree is homeomorphic to a finite tree (see Figure 4, left). We thus define the entropy  $h_{top}(f_c|_{T_c})$  of the restriction of  $f_c$  to the Hubbard tree, and we want to compare it to the Hausdorff dimension of some subset of parameter space. Let  $H_c$  be the set of external rays which land on  $T_c$ .

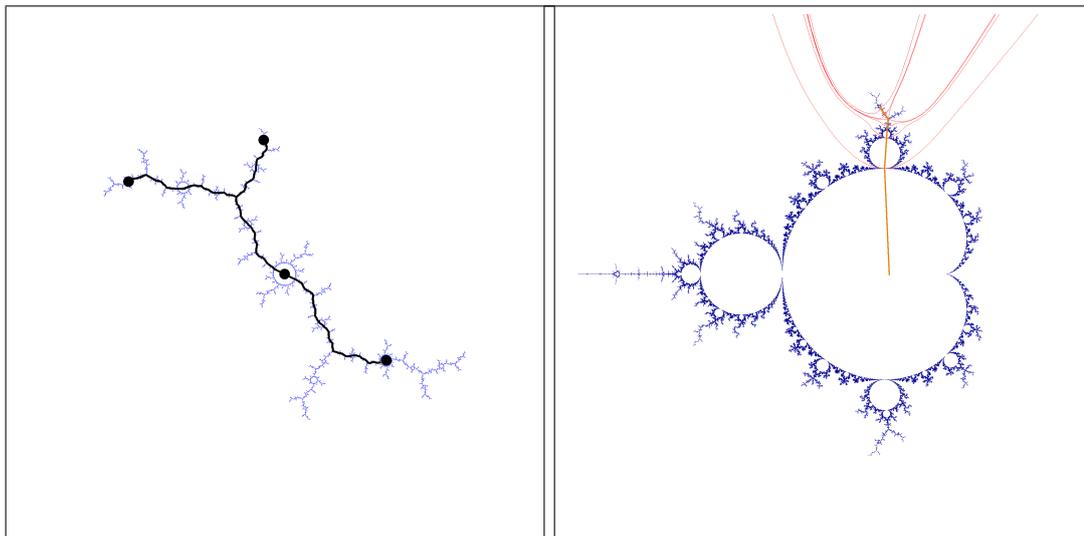


FIGURE 4. To the left: the Hubbard tree of the complex polynomial of period 4 and characteristic angles  $\theta = 3/15, 4/15$ . To the right: the vein joining the center of the main cardioid with the main antenna in the  $1/3$ -limb ( $\theta = 1/4$ ), and external rays landing on it.

In parameter space, a generalization of the real slice is a *vein*: a vein  $v$  is an embedded arc in  $\mathcal{M}$ , joining a parameter  $c \in \partial\mathcal{M}$  with the center of the main cardioid. Given a vein  $v$  and a parameter  $c$  on  $v$ , we can define the set  $M_c$  as the set of external angles of rays which land on  $v$  closer than  $c$  to the main cardioid:

$$M_c := \{\theta \in S^1 : R_M(\theta) \text{ lands on } v \cap [0, c]\}$$

where  $[0, c]$  means the segment of vein joining  $c$  to the center of the main cardioid (see Figure 4, right).

Note that the set of topologically finite parameters contain the postcritically finite ones but it is much larger: indeed, every parameter  $c \in \partial\mathcal{M}$  which is biaccessible (i.e. it belongs to some vein) is topologically finite (see section 4).

In the  $\frac{p}{q}$ -limb, there is a unique parameter  $c_{p/q}$  such that the critical point lands on the  $\beta$  fixed point after  $q$  iterates (i.e.  $f^q(0) = \beta$ ). The vein  $v_{p/q}$  joining  $c_{p/q}$  to  $c = 0$  will be called the *principal vein* of angle  $p/q$ . Note that  $v_{1/2}$  is the real axis, while  $v_{1/3}$  is the vein constructed by Branner and Douady [BD]. We can now extend the result of Theorem 1.1 to principal veins:

**Theorem 1.3.** *Let  $v = v_{p/q}$  be a principal vein in the Mandelbrot set, and  $c \in v \cap \partial\mathcal{M}$  a parameter along the vein. Then we have the equality*

$$\frac{h_{\text{top}}(f_c|_{T_c})}{\log 2} = \text{H.dim } H_c = \text{H.dim } M_c.$$

We conjecture that the previous equality holds along any vein  $v$ . Note that the statement can be given in more symmetric terms in the following way. If one defines for each  $A \subseteq \mathcal{M}$ ,

$$\Theta_M(A) := \{\theta \in S^1 : R_M(\theta) \text{ lands on } A\}$$

and similarly, for each  $A \subseteq J(f_c)$ , the set

$$\Theta_c(A) := \{\theta \in S^1 : R_c(\theta) \text{ lands on } A\}$$

where  $R_c(\theta)$  is the external ray at angle  $\theta$  in the dynamical plane for  $f_c$ , then Theorem 1.3 is equivalent to the statement

$$\text{H.dim } \Theta_c([0, c]) = \text{H.dim } \Theta_M([0, c]).$$

**1.2. Pseudocenters and real hyperbolic windows.** The techniques we use in the proof rely on the combinatorial analysis of the symbolic dynamics, and many ideas come from a connection with the dynamics of continued fractions. Indeed, on a combinatorial level the structure of the real slice of the Mandelbrot set is isomorphic to the structure of the bifurcation set  $\mathcal{E}$  for continued fractions [BCIT], so we can use the combinatorial tools we developed in that case ([CT], [CT2]) to analyze the quadratic family.

For instance, in [CT], a key concept is the *pseudocenter* of an interval, namely the (unique!) rational number with the smallest denominator. When translated to the world of binary expansions, used to describe the parameter space of quadratic polynomials, the definition becomes

**Definition 1.4.** *The pseudocenter of a real interval  $[a, b]$  with  $|a - b| < 1$  is the unique dyadic rational number  $x = \frac{p}{2^q}$  within  $[a, b]$  with shortest binary expansion (i.e. such that  $q$  is minimal).*

It is not hard to check from the definition that the binary expansion of the pseudocenter is obtained by comparing the binary expansions of  $a$  and  $b$  and stopping at the first non-matching digit. E.g., the pseudocenter of the interval  $[\frac{13}{15}, \frac{14}{15}]$  is  $\frac{7}{8} = 0.111$ , since  $\frac{13}{15} = 0.\overline{1101}$  and  $\frac{14}{15} = 0.\overline{1110}$ .

Recall that a *hyperbolic component*  $W \subseteq \mathcal{M}$  is a connected, open subset of parameters  $c$  for which the critical point of  $f_c$  is attracted to a periodic cycle. If  $W$  intersects the real axis, we define the *hyperbolic window* associated to  $W$  to be the interval  $(\theta_2, \theta_1) \subseteq [0, 1/2]$ , where the rays  $R_M(\theta_1)$  and  $R_M(\theta_2)$  land on  $\partial W \cap \mathbb{R}$ .

By translating the bisection algorithm of ([CT], section 2.4) in terms of kneading sequences, we get the following algorithm to generate all real hyperbolic windows (see section 8.2).

**Theorem 1.5.** *The set of all real hyperbolic windows in the Mandelbrot set can be generated as follows. Let  $c_1 < c_2$  be two real parameters on the boundary of  $\mathcal{M}$ , with external angles  $0 \leq \theta_2 < \theta_1 \leq \frac{1}{2}$ . Let  $\theta^*$  be the dyadic pseudocenter of the interval  $(\theta_2, \theta_1)$ , and let*

$$\theta^* = 0.s_1 s_2 \dots s_{n-1} s_n$$

*be its binary expansion, with  $s_n = 1$ . Then the hyperbolic window of smallest period in the interval  $(\theta_2, \theta_1)$  is the interval of external angles  $(\alpha_2, \alpha_1)$  with*

$$\begin{aligned} \alpha_2 &:= \frac{0.\overline{s_1 s_2 \dots s_{n-1}}}{0.s_1 s_2 \dots s_{n-1} \check{s}_1 \check{s}_2 \dots \check{s}_{n-1}} \\ \alpha_1 &:= \end{aligned}$$

*where  $\check{s}_i := 1 - s_i$ . All real hyperbolic windows are obtained by iteration of this algorithm, starting with  $\theta_2 = 0$ ,  $\theta_1 = 1/2$ .*

A similar algorithm is given by Schleicher in the context of the theory of *internal addresses* [Sch].

**1.3. Thurston's point of view.** The results of this paper relate to recent work of W. Thurston, who looked at the entropy of Hubbard trees (which he calls *core entropy*) as a function of the external angle. Indeed, every external angle  $\theta$  of the Mandelbrot set combinatorially determines a lamination (see section 3) and the lamination determines an abstract Hubbard tree, of which we can compute the entropy  $h(\theta)$ .

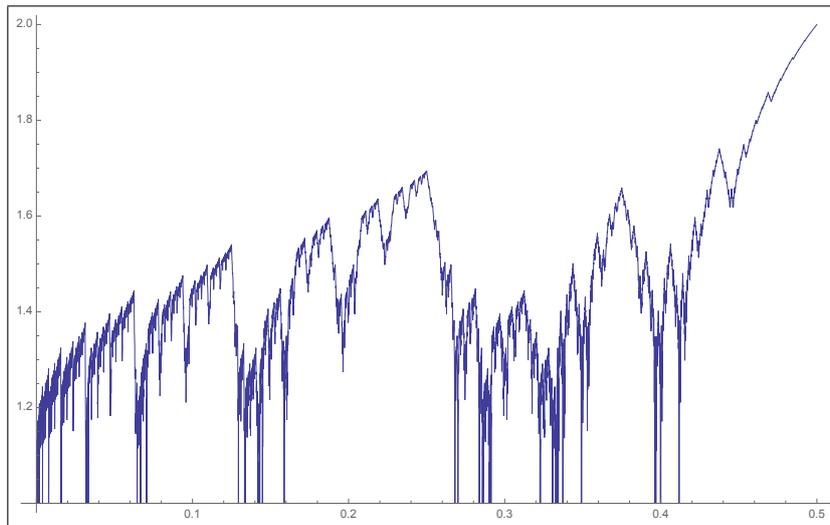


FIGURE 5. Entropy of Hubbard trees as a function of the external angle (by W. Thurston).

Thurston produced very interesting pictures (Figure 5), suggesting that the complexity of the Mandelbrot set is encoded in the combinatorics of the Hubbard tree, and the variation in entropy reflects the geometry of  $\mathcal{M}$ .

In this sense, Theorems 1.1 and 1.3 contribute to this program: in fact, the entropy grows as one goes further from the center of  $\mathcal{M}$  (see also [Li]), and our results make precise the relationship between entropy and landing of rays near the tips.

Note that Thurston’s approach is in some sense dual to ours, since we look at the variation of entropy along the veins, i.e. from “inside” the Mandelbrot set as opposed to from “outside” as a function of the external angle.

We point out that the idea of the pseudocenter described in the introduction seems also to be fruitful to study the entropy of the Hubbard tree as a function of the external angle: indeed, we conjecture that the maximum of the entropy on any wake is achieved precisely at its pseudocenter. Let us denote by  $h(\theta)$  the entropy of the Hubbard tree corresponding to the parameter of external angle  $\theta$ .

**Conjecture 1.6.** *Let  $\theta_1 < \theta_2$  be two external angles whose rays  $R_M(\theta_1)$ ,  $R_M(\theta_2)$  land on the same parameter in the boundary of the Mandelbrot set. Then the maximum of entropy on the interval  $[\theta_1, \theta_2]$  is attained at its pseudocenter:*

$$\max_{\theta \in [\theta_1, \theta_2]} h(\theta) = h(\theta^*)$$

where  $\theta^*$  is the pseudocenter of the interval  $[\theta_1, \theta_2]$ .

**1.4. Sketch of the argument.** The proof of Theorem 1.1 is carried out in two steps. Let  $f_c$  be a topologically finite polynomial, and denote  $T_c$  its Hubbard tree (for a precise definition, see Section 4); moreover, let  $H_c$  be the set of angles of external rays landing on the Hubbard tree:

$$H_c := \{\theta \in S^1 : R_c(\theta) \text{ lands on } T_c\}.$$

We first prove (Theorem 7.1 in section 7) the relationship between topological entropy  $h_{top}(f_c|_{T_c})$  of the map restricted to the Hubbard tree and the Hausdorff dimension of the set  $H_c$ , for all topologically finite polynomials  $f_c$ .

In order to deal with rays landing in parameter space, we shall denote  $P_c$  the set of external angles of rays whose impression intersects the segment  $[c, 0]$ ; this set can be easily described combinatorially (see Section 8), and conjecturally equals  $M_c$ . However, it can be proven to have the same dimension as  $M_c$  (see Proposition 13.12). The bulk of the argument is thus proving the identity of Hausdorff dimensions between the real Julia set and the combinatorial slices of  $\mathcal{M}$ :

**Theorem 1.7.** *For any  $c \in [-2, \frac{1}{4}]$ , we have the equality*

$$\text{H.dim } S_c = \text{H.dim } P_c.$$

It is not hard to show that  $P_c \subseteq H_c \subseteq S_c$  for any real parameter  $c$  (Corollary 8.6); it is much harder to give a lower bound for the dimension of  $P_c$  in terms of the dimension of  $H_c$ ; indeed, it seems impossible to include a copy of  $H_c$  in  $P_c$  when  $c$  belongs to some *tuning window*, i.e. to some baby Mandelbrot set. However, for non-renormalizable parameters we can prove the following:

**Proposition 1.8.** *Given a non-renormalizable, real parameter  $c \in \mathcal{M}$  and another real parameter  $c' > c$ , there exists a piecewise linear map  $F : S^1 \rightarrow S^1$  such that*

$$F(H_{c'}) \subseteq P_c.$$

The proposition implies equality of dimension for all non-renormalizable parameters. By applying tuning operators, we then get equality for all finitely-renormalizable parameters, which are dense hence the result follows from continuity.

Proposition 1.8 will be proved in section 10. Its proof relies on the definition of a class of parameters, which we call *dominant*, which are a subset of the set of periodic, non-renormalizable parameters. We will show that for these parameters (which can be defined purely combinatorially) it is easier to construct an inclusion of the Hubbard tree into parameter space; finally, the most technical part will be proving that such parameters are dense in the set of non-renormalizable angles.

In order to establish the result for complex veins, we first prove continuity of entropy along veins by a version of kneading theory for Hubbard trees (section 12). Finally, we transfer the inclusion of Proposition 1.8 from the real vein to the other principal vein via a combinatorial version of the Branner-Douady surgery (section 13).

**1.5. Remarks and acknowledgements.** The history of this paper is quite interesting. After the discovery of the connection between continued fractions and the real slice of  $\mathcal{M}$  [BCIT], the statement for the real case (Theorem 1.1) came out of discussions with Carlo Carminati in spring 2011, as an application of our combinatorial techniques (indeed, many proofs follow the arguments developed in [CT2] for continued fractions). At about the same time, I have been informed of the recent work of W. Thurston on the entropy of Hubbard trees, which sparked new interest and inspired the generalization to complex veins.

I especially wish to thank Tan Lei and C.T. McMullen for several useful conversations, as well as A.M. Benini, W. Jung and S. Koch; moreover I wish to thank D. Schleicher for pointing out reference [Ri]. Some of the pictures have been created with the software `mandel` of W. Jung.

## 2. EXTERNAL RAYS

Let  $f(z) = z^2 + c$ . Recall that the *filled Julia set*  $K(f)$  is the set of points which do not escape to infinity under iteration:

$$K(f) := \{z \in \mathbb{C} : f^n(z) \text{ does not tend to } \infty \text{ as } n \rightarrow \infty\}.$$

The *Julia set*  $J(f)$  is the boundary of  $K(f)$ . If  $K(f)$  is connected, then the complement of  $K(f)$  in the Riemann sphere is simply connected, so it can be uniformized by the Riemann mapping  $\Phi : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \hat{\mathbb{C}} \setminus K(f)$  which maps the exterior of the closed unit disk  $\mathbb{D}$  to the exterior of  $K(f)$ . The Riemann mapping is unique once we impose  $\Phi(\infty) = \infty$  and  $\Phi'(\infty) = 1$ . With this choice,  $\Phi$  conjugates the action of  $f$  on the exterior of the filled Julia set to the map  $z \mapsto z^2$ , i.e.

$$(1) \quad f(\Phi(z)) = \Phi(z^2).$$

By Carathéodory's theorem (see e.g. [Po]), the Riemann mapping extends to a continuous map  $\overline{\Phi}$  on the boundary  $\overline{\Phi} : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \hat{\mathbb{C}} \setminus \text{int } K(f)$  if and only if the Julia set is locally connected. If this is the case, the restriction of  $\overline{\Phi}$  to the boundary is sometimes called the *Carathéodory loop* and it will be denoted as

$$\gamma : S^1 \rightarrow J(f).$$

As a consequence of the eq. (1), the action of  $f$  on the set of angles is semiconjugate to multiplication by 2 (mod 1):

$$(2) \quad \gamma(2 \cdot \theta) = f(\gamma(\theta)) \quad \text{for each } \theta \in \mathbb{R}/\mathbb{Z}.$$

In the following, we will denote as

$$D(\theta) := 2 \cdot \theta \pmod{1}$$

the doubling map of the circle. Moreover, we will add the subscript  $c$  when we need to make the dependence on the polynomial  $f_c$  more explicit. Given  $\theta \in \mathbb{R}/\mathbb{Z}$ , the *external ray*  $R_c(\theta)$  is the image of the radial arc at angle  $2\pi\theta$  via the Riemann mapping  $\Phi_c : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \hat{\mathbb{C}} \setminus K(f_c)$ :

$$R_c(\theta) := \{\Phi_c(\rho e^{2\pi i\theta})\}_{\rho > 1}.$$

The ray  $R_c(\theta)$  is said to *land* at  $x$  if

$$\lim_{\rho \rightarrow 1^+} \Phi_c(\rho e^{2\pi i\theta}) = x.$$

If the Julia set is locally connected, then all rays land; in general, by Fatou's theorem, the set of angles for which  $R_c(\theta)$  does not land has zero Lebesgue measure, and indeed by Beurling's theorem [Be] it also has zero logarithmic capacity and hence zero Hausdorff dimension (see e.g. [Po], Theorems 9.19 and 10.3). It is however known that there exist non-locally connected Julia sets for polynomials [Mi2]. The ray  $R_c(0)$  always lands on a fixed point of  $f_c$  which is traditionally called the  $\beta$  fixed point and denoted as  $\beta$ . The other fixed point of  $f_c$  is called the  $\alpha$  fixed point. Note that in the case  $c = \frac{1}{4}$  one has  $\alpha = \beta$ . Finally, the critical point of  $f_c$  will be denoted by 0, and the critical value by  $c$ .

Analogously to the Julia sets, the exterior of the Mandelbrot set can be uniformized by the Riemann mapping

$$\Phi_M : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \hat{\mathbb{C}} \setminus \mathcal{M}$$

with  $\Phi_M(\infty) = \infty$ , and  $\Phi_M'(\infty) = 1$ , and images of radial arcs are called *external rays*. Every angle  $\theta \in \mathbb{R}/\mathbb{Z}$  determines an external ray

$$R_M(\theta) := \Phi_M(\{\rho e^{2\pi i\theta} : \rho > 1\})$$

which is said to *land* at  $x$  if the limit  $\lim_{\rho \rightarrow 1^+} \Phi_M(\rho e^{2\pi i\theta})$  exists. According to the *MLC conjecture* [DH], the Mandelbrot set is locally connected, and therefore all rays land on some point of the boundary of  $\mathcal{M}$ .

**2.1. Biaccessibility and regulated arcs.** A point  $z \in J(f_c)$  is called *accessible* if it is the landing point of at least one external ray. It is called *biaccessible* if it is the landing point of at least two rays, i.e. there exist  $\theta_1, \theta_2$  two distinct angles such that  $R_c(\theta_1)$  and  $R_c(\theta_2)$  both land at  $z$ . This is equivalent to say that  $J(f_c) \setminus \{z\}$  is disconnected.

Let  $K = K(f_c)$  be the filled Julia set of  $f_c$ . Assume  $K$  is connected and locally connected. Then it is also path-connected (see e.g. [Wi], Chapter 8), so given any two points  $x, y$  in  $K$ , there exists an arc in  $K$  with endpoints  $x, y$ .

If  $K$  has no interior, then the arc is uniquely determined by its endpoints  $x, y$ . Let us now describe how to choose a canonical representative inside the Fatou components in the case  $K$  has interior. In this case, each bounded Fatou component eventually maps to a periodic Fatou component, which either contains an attracting cycle, or it contains a parabolic cycle on its boundary, or it is a periodic Siegel disk.

Since we will not deal with the Siegel disk case in the rest of the paper, let us assume we are in one of the first two cases. Then there exists a Fatou component  $U_0$  which contains the critical point, and a biholomorphism  $\phi_0 : U_0 \rightarrow \mathbb{D}$  to the unit disk mapping the critical point to 0. The preimages  $\phi_0^{-1}(\{\rho e^{2\pi i \theta} : 0 \leq \rho < 1\})$  of radial arcs in the unit disk are called *radial arcs* in  $U_0$ . Any other bounded Fatou component  $U$  is eventually mapped to  $U_0$ ; let  $k \geq 0$  be the smallest integer such that  $f_c^k(U) = U_0$ . Then the map  $\phi := \phi_0 \circ f_c^k$  is a biholomorphism of  $U$  onto the unit disk, and we define radial arcs to be preimages under  $\phi$  of radial arcs in the unit disk.

An embedded arc  $I$  in  $K$  is called *regulated* (or *legal* in Douady's terminology [Do2]) if the intersection between  $I$  and the closure of any bounded Fatou component is contained in the union of at most two radial arcs. With this choice, given any two points  $x, y$  in  $K$ , there exists a unique regulated arc in  $K$  with endpoints  $x, y$  ([Za], Lemma 1). Such an arc will be denoted by  $[x, y]$ , and the corresponding open arc by  $(x, y) := [x, y] \setminus \{x, y\}$ . A *regulated tree* inside  $K$  is a finite tree whose edges are regulated arcs. Note that, in the case  $K$  has non-empty interior, regulated trees as defined need not be invariant for the dynamics, because  $f_c$  need not map radial arcs to radial arcs. However, by construction, radial arcs in any bounded Fatou component  $U$  different from  $U_0$  map to radial arcs in  $f_c(U)$ . In order to deal with  $U_0$ , we need one further hypothesis. Namely, we will assume that  $f_c$  has an attracting or parabolic cycle of period  $p$  with real multiplier. Then we can find a parametrization  $\phi_0 : U_0 \rightarrow \mathbb{D}$  such that the interval  $I := \phi_0^{-1}((-1, 1))$  is preserved by the  $p$ -th iterate of  $f_c$ , i.e.  $f_c^p(I) \subseteq I$ . The interval  $I$  will be called the *bisector* of  $U_0$ . Now note that, if the regulated arc  $[x, y]$  does not contain 0 in its interior and it only intersects the critical Fatou component  $U_0$  in its bisector, then we have

$$f_c([x, y]) = [f_c(x), f_c(y)].$$

The *spine* of  $f_c$  is the regulated arc  $[-\beta, \beta]$  joining the  $\beta$  fixed point to its preimage  $-\beta$ . The biaccessible points are related to the points which lie on the spine by the following lemma.

**Lemma 2.1.** *Let  $f_c(z) = z^2 + c$  be a quadratic polynomial whose Julia set is connected and locally connected. Then the set of biaccessible points is*

$$\mathcal{B} = J(f_c) \cap \bigcup_{n \geq 0} f_c^{-n}((-\beta, \beta)).$$

*Proof.* Let  $f = f_c$ , and  $x \in J(f) \cap (-\beta, \beta)$ . The set  $V := R_c(0) \cup [-\beta, \beta] \cup R_c(1/2)$  disconnects the plane in two parts,  $\mathbb{C} \setminus V = A_1 \cup A_2$ . We claim that  $x$  is the limit of points in the basin of infinity  $U_\infty$  on both sides of  $V$ , i.e. for each  $i = 1, 2$  there exists a sequence  $\{x_n\}_{n \in \mathbb{N}} \subseteq A_i \cap U_\infty$  with  $x_n \rightarrow x$ ; since the Riemann mapping  $\Phi$

extends continuously to the boundary, this is enough to prove that there exist two external angles  $\theta_1 \in (0, 1/2)$  and  $\theta_2 \in (1/2, 1)$  such that  $R_c(\theta_1)$  and  $R_c(\theta_2)$  both land on  $x$ . Let us now prove the claim; if it is not true, then there exists an open neighborhood  $\Omega$  of  $x$  and an index  $i \in \{1, 2\}$  such that  $\Omega \cap A_i$  is connected and contained in the interior of the filled Julia set  $K(f)$ , hence  $\Omega \cap A_i$  is contained in some bounded Fatou component. This implies that  $\Omega \cap V$  lies in the closure of a bounded Fatou component, and  $x$  on its boundary. However, this contradicts the definition of regulated arc, because if  $U$  is a bounded Fatou component intersecting a regulated arc  $I$ , then  $\partial U \cap I$  does not disconnect  $\bar{U} \cap I$ . Suppose now that  $x \in J(f)$  is such that  $f^n(x)$  belongs to  $(-\beta, \beta)$  for some  $n$ . Then by the previous argument  $f^n(x)$  is biaccessible, and since  $f$  is a local homeomorphism outside the spine,  $x$  is also biaccessible.

Conversely, suppose  $x$  is biaccessible, and the two rays at angles  $\theta_1$  and  $\theta_2$  land on  $x$ , with  $0 < \theta_1 < \theta_2 < 1$ . Then there exists some  $n$  for which  $1/2 \leq D^n(\theta_2) - D^n(\theta_1) < 1$ , hence  $R_c(D^n(\theta_1))$  and  $R_c(D^n(\theta_2))$  must lie on opposite sides with respect to the spine, and since they both land on  $f^n(x)$ , then  $f^n(x)$  belongs to the spine. Since the point  $\beta$  is not biaccessible ([Mc], Theorem 6.10),  $f^n(x)$  must belong to  $(-\beta, \beta)$ .  $\square$

**Lemma 2.2.** *We have that  $\alpha \in [0, c]$ .*

*Proof.* Indeed, since  $\alpha \in (-\beta, 0)$  ([Za], Lemma 5), we have  $-\alpha \in (\beta, 0)$  and  $\alpha = f(-\alpha) \in (\beta, c)$ . Thus, since  $0 \in (\alpha, \beta)$  we have  $\alpha \in (0, c)$ .  $\square$

**Lemma 2.3.** *For  $x \in [0, \beta)$ , we have  $x \in (f(x), \beta)$ .*

*Proof.* Note that  $f$  maps the set  $[0, \beta]$  bijectively onto  $[c, \beta]$ , which is homeomorphic to an interval. Moreover, since  $0$  belongs to  $[c, \beta]$ , we have  $[c, \beta] \supseteq [0, \beta]$ . Thus,  $f$  is a homeomorphism between two intervals  $I = [0, \beta]$  and  $J = [c, \beta]$ , with  $J \supseteq I$ , and which fixes one endpoint of  $I$ . Moreover,  $f$  does not fix any other point of  $I$ , since  $\alpha$  does not belong to  $I$  as seen from the previous lemma. Thus, either

$$f(x) \in (\beta, x) \quad \text{for each } x \in I \setminus \{\beta\}$$

or

$$x \in (\beta, f(x)) \quad \text{for each } x \in I \setminus \{\beta\}.$$

In the first case, one has  $\lim_{n \rightarrow \infty} f^n(x) = \beta$  for each  $x \in I$ , but this cannot happen since the  $\beta$  fixed point is repelling. Hence we must be in the second case, proving the claim.  $\square$

For more general properties of biaccessibility we refer to [Za].

### 3. LAMINATIONS

A powerful tool to construct topological models of Julia sets and the Mandelbrot set is given by laminations, following Thurston's approach. As we will see, laminations represent equivalence relations on the boundary of the disk arising from external rays which land on the same point. We now give the basic definitions, and refer to [Th] for further details.

A *geodesic lamination*  $\lambda$  is a set of hyperbolic geodesics in the closed unit disk  $\bar{\mathbb{D}}$ , called the *leaves* of  $\lambda$ , such that no two leaves intersect in  $\mathbb{D}$ , and the union of all leaves is closed.

A *gap* of a lamination  $\lambda$  is the closure of a component of the complement of the union of all leaves. In order to represent Julia sets of quadratic polynomials, we need to restrict ourselves to *invariant laminations*.

Let  $d \geq 2$ . The map  $g(z) := z^d$  acts on the boundary of the unit disk, hence it induces a dynamics on the set of leaves. Namely, the image of a leaf  $\overline{pq}$  is defined

as the leaf joining the images of the endpoint:  $g(\overline{pq}) = \overline{g(p)g(q)}$ . A lamination  $\lambda$  is *forward invariant* if the image of any leaf  $L$  of  $\lambda$  still belongs to  $\lambda$ . Note that the image leaf may be *degenerate*, i.e. consist of a single point on the boundary of the disk.

A lamination is *invariant* if in addition to being forward invariant it satisfies the additional conditions:

- *Backward invariance*: if  $\overline{pq}$  is in  $\lambda$ , then there exists a collection of  $d$  disjoint leaves in  $\lambda$ , each joining a preimage of  $p$  to a preimage of  $q$ .
- *Gap invariance*: for any gap  $G$ , the hyperbolic convex hull of the image of  $G_0 = \overline{G} \cap S^1$  is either a gap, a leaf, or a single point.

In this paper we will only deal with quadratic polynomials, so  $d = 2$  and the invariant laminations for the map  $g(z) = z^2$  will be called *invariant quadratic laminations*. A leaf of maximal length in a lamination is called a *major leaf*, and its image a *minor leaf*. Typically, a quadratic invariant lamination has 2 major leaves, but the minor leaf is always unique.

If  $J(f_c)$  is a Julia set of a quadratic polynomial, one can define the equivalence relation  $\sim_c$  on the unit circle  $\partial\mathbb{D}$  by saying that  $\theta_1 \sim_c \theta_2$  if the rays  $R_c(\theta_1)$  and  $R_c(\theta_2)$  land on the same point.

From the equivalence relation  $\sim_c$  one can construct a quadratic invariant lamination in the following way. Let  $E$  be an equivalence class for  $\sim_c$ . If  $E = \{\theta_1, \theta_2\}$  contains two elements, then we define the leaf  $L_E$  as  $L_E := (\theta_1, \theta_2)$ . If  $E = \{\theta\}$  is a singleton, then we define  $L_E$  to be the degenerate leaf  $L_E := \{\theta\}$ . Finally, if  $E = \{\theta_1, \dots, \theta_k\}$  contains more than two elements, with  $0 \leq \theta_1 < \theta_2 < \dots < \theta_k < 1$ , then we define  $L_E$  to be the union of the leaves  $L_E := (\theta_1, \theta_2) \cup (\theta_2, \theta_3) \cup \dots \cup (\theta_k, \theta_1)$ . Finally, we let the associated lamination  $\lambda_c$  be

$$\lambda_c := \bigcup_{E \text{ equiv. class of } \sim_c} L_E.$$

The lamination  $\lambda_c$  is an invariant quadratic lamination. The equivalence relation  $\sim_c$  can be extended to a relation  $\cong_c$  on the closed disk  $\overline{\mathbb{D}}$  by taking convex hulls, and the quotient of the disk by  $\cong_c$  is a model for the Julia set:

**Theorem 3.1** ([Do2]). *If the Julia set  $J(f_c)$  is connected and locally connected, then it is homeomorphic to the quotient of  $\overline{\mathbb{D}}$  by the equivalence relation  $\cong_c$ .*

We define the *characteristic leaf* of a quadratic polynomial  $f_c$  with Julia set connected and locally connected to be the minor leaf of the invariant lamination  $\lambda_c$ . The endpoints of the characteristic leaf are called *characteristic angles*.

**3.1. The abstract Mandelbrot set.** In order to construct a model for the Mandelbrot set, Thurston [Th] defined the *quadratic minor lamination QML* as the union of the minor leaves of all quadratic invariant laminations (see Figure 6).

As in the Julia set case, the lamination determines an equivalence relation  $\cong_M$  on  $\overline{\mathbb{D}}$  by identifying points on the same leaf, and also points in the interior of finite ideal polygons whose sides are leaves. The quotient

$$\mathcal{M}_{abs} := \overline{\mathbb{D}} / \cong_M$$

is called *abstract Mandelbrot set* (for a complete treatment, we refer to [Ke]). It is a compact, connected and locally connected space. Douady [Do2] constructed a continuous surjection

$$\pi_M : \mathcal{M} \rightarrow \mathcal{M}_{abs}$$

which is injective if and only if  $\mathcal{M}$  is locally connected.

The idea behind the construction is that leaves of *QML* connect external angles whose corresponding rays in parameter space land on the same point. However,

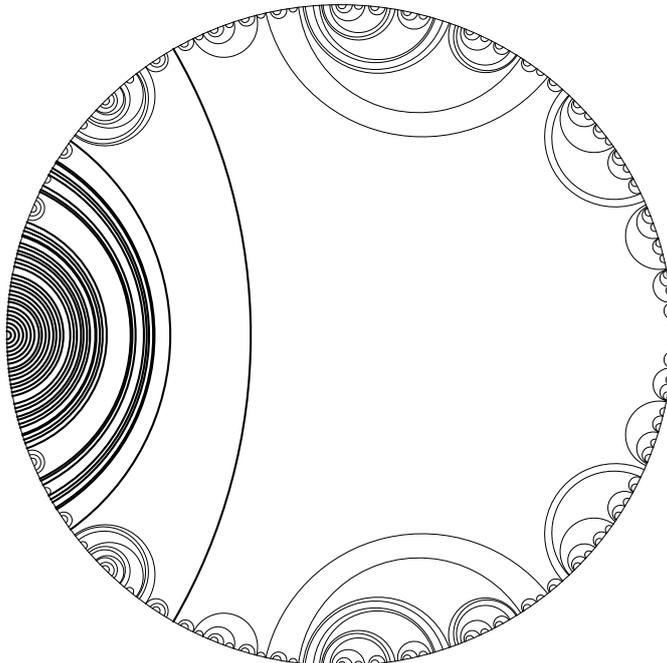


FIGURE 6. Thurston's *quadratic minor lamination*. The quotient of the unit disk by the equivalence relation given by the lamination is a topological model for  $\mathcal{M}$ . Leaves which are symmetric with respect to complex conjugation (displayed thicker) correspond to rays landing on the real axis.

since we do not know whether  $\mathcal{M}$  is locally connected, additional care is required. Indeed, let  $\sim_M$  denote the equivalence relation on  $\partial\mathbb{D}$  induced by the lamination  $QML$ , and  $\theta_1 \asymp_M \theta_2$  denote that the external rays  $R_M(\theta_1)$  and  $R_M(\theta_2)$  land on the same point. The following theorem summarizes a few key results comparing the analytic and combinatorial models of the Mandelbrot set:

**Theorem 3.2.** *Let  $\theta_1, \theta_2 \in S^1$  be two angles. Then the following are true:*

- (1) *if  $\theta_1 \asymp_M \theta_2$ , then  $\theta_1 \sim_M \theta_2$ ;*
- (2) *if  $\theta_1 \sim_M \theta_2$  and  $\theta_1, \theta_2$  are rational, then  $\theta_1 \asymp_M \theta_2$ ;*
- (3) *if  $\theta_1 \sim_M \theta_2$  and  $\theta_1, \theta_2$  are not infinitely renormalizable, then  $\theta_1 \asymp_M \theta_2$ .*

*Proof.* (1) and (2) are contained in ([Th], Theorem A.3). (3) follows from Yoccoz's theorem on landing of rays at finitely renormalizable parameters (see [Hu] for the proof). Indeed, Yoccoz proves that external rays  $R_M(\theta)$  with non-infinitely renormalizable combinatorics land, and moreover that the intersections of nested parapuzzle pieces contain a single point. Along the boundary of each puzzle piece lie pairs of external rays with rational angles (see also [Hu], sections 5 and 12) which land on the same point, and since the intersection of the nested sequence of puzzle pieces is a single point  $c \in \partial\mathcal{M}$ , the rays  $\theta_1$  and  $\theta_2$  land on the same point  $c$ .  $\square$

The following criterion makes it possible to check whether a leaf belongs to the quadratic minor lamination by looking at its dynamics under the doubling map:

**Proposition 3.3** ([Th]). *A leaf  $m$  is the minor leaf of some invariant quadratic lamination (i.e. it belongs to  $QML$ ) if and only if the following three conditions are met:*

- (a) all forward images of  $m$  have disjoint interiors;
- (b) the length of any forward image of  $m$  is never less than the length of  $m$ ;
- (c) if  $m$  is a non-degenerate leaf, then  $m$  and all leaves on the forward orbit of  $m$  are disjoint from the interiors of the two preimage leaves of  $m$  of length at least  $1/3$ .

For the rest of the paper we shall work with the abstract, locally connected model of  $\mathcal{M}$  and study its dimension using combinatorial techniques; only at the very end (Proposition 13.12) we shall compare the analytical and combinatorial models and prove that our results hold for the actual Mandelbrot set even without assuming the MLC conjecture.

#### 4. HUBBARD TREES

Assume now that the polynomial  $f = f_c(z) = z^2 + c$  has Julia set which is connected (i.e.  $c \in \mathcal{M}$ ) and locally-connected, and no attracting fixed point (i.e.  $c$  lies outside the main cardioid). The *critical orbit* of  $f$  is the set  $\text{Crit}(f) := \{f^k(0)\}_{k \geq 0}$ . Let us now give the fundamental

**Definition 4.1.** *The Hubbard tree  $T$  for  $f$  is the smallest regulated tree which contains the critical orbit, i.e.*

$$T := \bigcup_{i,j \geq 0} [f^i(0), f^j(0)].$$

Note that, according to this definition, the set  $T$  need not be closed in general. We shall establish a few fundamental properties of Hubbard trees.

**Lemma 4.2.** *The following properties hold:*

- (1)  $T$  is the smallest forward-invariant set which contains the regulated arc  $[\alpha, 0]$ ;
- (2)  $T = \bigcup_{n \geq 0} [\alpha, f^n(0)]$ .

*Proof.* Let now  $T_1$  be the smallest forward-invariant set which contains the regulated arc  $[\alpha, 0]$ . By definition,  $T$  is forward-invariant and contains  $[\alpha, 0]$  since  $\alpha \in [0, c]$ , so  $T_1 \subseteq T$ . Let now

$$T_2 := \bigcup_{n \geq 0} [\alpha, f^n(0)].$$

Since  $[f^i(0), f^j(0)] \subseteq [\alpha, f^i(0)] \cup [\alpha, f^j(0)]$ , then  $T \subseteq T_2$ . By definition,

$$T_1 = \bigcup_{n \geq 0} f^n([\alpha, 0]).$$

Since  $f^i([\alpha, 0]) \supseteq [\alpha, f^i(0)]$ , then  $T_2 \subseteq T_1$ , hence  $T = T_1 = T_2$ .  $\square$

The tree thus defined need not have finitely many edges. However, in the following we will restrict ourself to the case when  $T$  is a finite tree. Let us introduce the definition:

**Definition 4.3.** *A polynomial  $f$  is topologically finite if the Julia set is locally connected and the Hubbard tree  $T$  is homeomorphic to a tree with finitely many edges.*

Recall that a polynomial is called *postcritically finite* if the critical orbit is finite. Postcritically finite polynomials are also topologically finite, but it turns out that the class of topologically finite polynomials is much bigger and indeed it contains all polynomials along the veins of the Mandelbrot set (see also section 11.1).

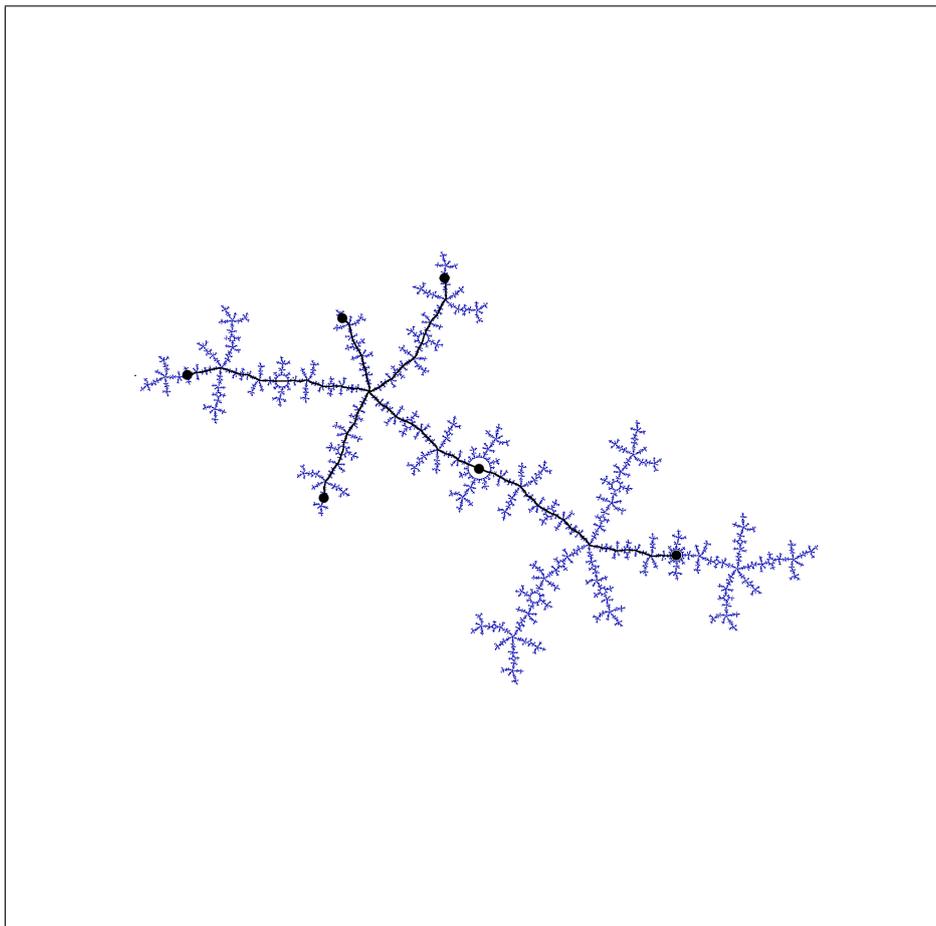


FIGURE 7. The Hubbard tree of the quadratic polynomial with characteristic leaf  $(19/63, 20/63)$ . The map  $f_c$  is postcritically finite, and the critical point belongs to a cycle of period 6. The parameter  $c$  belongs to the principal vein in the  $2/5$ -limb.

**Proposition 4.4.** *Let  $f$  have locally connected Julia set. Suppose there is an integer  $n \geq 1$  such that  $f^n(0)$  lies on the regulated arc  $[\alpha, \beta]$ , and let  $N$  be the smallest such integer. Then  $f$  is topologically finite, and the Hubbard tree  $T$  of  $f$  is given by*

$$T = \bigcup_{i=0}^N [\alpha, f^i(0)].$$

*Proof.* Let  $T_N := \bigcup_{i=0}^N [\alpha, f^i(0)]$ . By Lemma 4.2 (2),  $T_N \subseteq T$ . Note now that for each  $i$  we have

$$f([\alpha, f^i(0)]) \subseteq [\alpha, c] \cup [\alpha, f^{i+1}(0)]$$

thus

$$f(T_N) \subseteq T_N \cup [\alpha, f^{N+1}(0)].$$

Now, either  $f^N(0)$  lies in  $[\alpha, -\alpha]$ , or by Lemma 2.3,  $f^N(0)$  lies between  $\beta$  and  $f^{N+1}(0)$ . In the first case,  $[\alpha, f^{N+1}(0)] \subseteq [\alpha, c]$  and in the second case  $[\alpha, f^{N+1}(0)] \subseteq [\alpha, f^N(0)]$ ; in both cases,  $[\alpha, f^{N+1}(0)] \subseteq T_N$ , so  $T_N$  is forward-invariant and it contains  $[\alpha, 0]$ , so it contains  $T$  by Lemma 4.2 (1).  $\square$

**Proposition 4.5.** *If the Julia set of  $f$  is locally connected and the critical value  $c$  is biaccessible, then  $f$  is topologically finite.*

*Proof.* Since  $c$  is biaccessible, by Lemma 2.1 there exists  $n \geq 0$  such that  $f^n(c)$  belongs to the spine  $[-\beta, \beta]$  of the Julia set. Then either  $f^n(c)$  or  $f^{n+1}(c)$  lie on  $[\alpha, \beta]$ , so  $f$  is topologically finite by Proposition 4.4.  $\square$

Let us define the *extended Hubbard tree*  $\tilde{T}$  to be the union of the Hubbard tree and the spine:

$$\tilde{T} := T \cup [-\beta, \beta].$$

Note the extended tree is also forward invariant, i.e.  $f(\tilde{T}) \subseteq \tilde{T}$ . Moreover, it is related to the usual Hubbard tree in the following way:

**Lemma 4.6.** *The extended Hubbard tree eventually maps to the Hubbard tree:*

$$\tilde{T} \setminus \{\beta, -\beta\} \subseteq \bigcup_{n \geq 0} f^{-n}(T).$$

*Proof.* Since  $f([\alpha, -\beta]) = [\alpha, \beta]$ , we just need to check that every element  $z \in [\alpha, \beta]$  eventually maps to the Hubbard tree. Indeed, either there exists  $n \geq 0$  such that  $f^n(z) \in [\alpha, c] \subseteq T$ , or, by Lemma 2.3, the sequence  $\{f^n(z)\}_{n \geq 0}$  all lies on  $[0, \beta]$  and it is ordered along the segment, i.e. for each  $n$ ,  $f^{n+1}(z)$  lies in between 0 and  $f^n(z)$ . Then the sequence must have a limit point, and such limit point would be a fixed point of  $f$ . However,  $f$  has no fixed points on  $[0, \beta]$ , contradiction.  $\square$

**4.1. Valence.** If  $T$  is a finite tree, then the *degree* of a point  $x \in T$  is the number of connected components of  $T \setminus \{x\}$ , and is denoted by  $\deg(x)$ . Moreover, let us denote by  $\deg(T)$  denote the largest degree of a point on the tree:

$$\deg(T) := \max\{\deg(x) : x \in T\}.$$

On the other hand, for each  $z \in J(f)$ , we call *valence* of  $z$  the number of external rays which land on  $z$  and denote it as

$$\text{val}(z) := \#\{\theta \in S^1 : R_c(\theta) \text{ lands on } z\}.$$

The valence of  $z$  also equals the number of connected components of  $J(f) \setminus \{z\}$  ([Mc], Theorem 6.6), also known as the *Urysohn-Menger index* of  $J(f)$  at  $z$ .

**Proposition 4.7.** *Let  $\tilde{T}$  be the extended Hubbard tree for a topologically finite quadratic polynomial  $f$ . Then the number of rays  $M$  landing on  $x \in \tilde{T}$  is bounded above by*

$$M \leq 2 \cdot \deg(\tilde{T}).$$

The proposition follows easily from the

**Lemma 4.8.** *Let  $\tilde{T}$  be the extended Hubbard tree for  $f$ , and  $x \in \tilde{T}$  a point on the tree which never maps to the critical point. Then the number of rays  $M$  landing on  $x$  is bounded above by*

$$M \leq \max\{\deg(f^n(x)) : n \geq 0\}.$$

*Proof.* Note that, since the forward orbit of  $x$  does not contain the critical point,  $f^n$  is a local homeomorphism in a neighborhood of  $x$ ; thus, for each  $n \geq 0$ ,  $\text{val}(f^n(x)) = \text{val}(x)$  and  $\deg(f^n(x)) \geq \deg(x)$ . Suppose now the claim is false: let  $N$  be such that  $\deg(f^N(x)) = \max\{\deg(f^n(x)) : n \geq 0\} < \text{val}(x)$ , and denote  $y = f^N(x)$ . Then there are two angles  $\theta_1, \theta_2$  such that the rays  $R_c(\theta_1)$  and  $R_c(\theta_2)$  both land at  $y$ , and the sector between  $R_c(\theta_1)$  and  $R_c(\theta_2)$  does not intersect the tree. Then, there exists  $M \geq 0$  such that the rays  $R_c(D^M(\theta_1))$  and  $R_c(D^M(\theta_2))$  lie on opposite sides

of the spine, thus their common landing point  $z := f^M(y)$  must lie on the spine. Moreover, since  $val(z) = val(x) \geq 2$  while only one ray lands on the  $\beta$  fixed point,  $z$  must lie in the interior of the spine. This means that the sector between the rays  $R_c(D^M(\theta_1))$  and  $R_c(D^M(\theta_2))$  intersects the spine, so  $deg(f^M(y)) > deg(y)$ , contradicting the maximality of  $N$ .  $\square$

*Proof of Proposition 4.7.* If  $val(x) > 0$ , then  $x$  lies in the Julia set  $J(f)$ . Now, if the forward orbit of  $x$  does not contain the critical point, the claim follows immediately from the Lemma. Otherwise, let  $n \geq 0$  be such that  $f^n(x) = 0$  is the critical point. Note that this  $n$  is unique, because otherwise the critical point would be periodic, so it would not lie in the Julia set. Hence, by applying the Lemma to the critical value  $f^{n+1}(x)$ , we have

$$val(f^{n+1}(x)) \leq deg(\tilde{T}).$$

Finally, since the map  $f$  is locally a double cover at the critical point,

$$val(x) = val(f^n(x)) = 2 \cdot val(f^{n+1}(x)) \leq 2 \cdot deg(\tilde{T}).$$

$\square$

## 5. TOPOLOGICAL ENTROPY

Let  $f : X \rightarrow X$  be a continuous map of a compact metric space  $(X, d)$ . A measure of the complexity of the orbits of the map is given by its *topological entropy*. Let us now recall its definition. Useful references are [dMvS] and [CFS].

Given  $x \in X$ ,  $\epsilon > 0$  and  $n$  an integer, we define the ball  $B_f(x, \epsilon, n)$  as the set of points whose orbit remains close to the orbit of  $x$  for the first  $n$  iterates:

$$B_f(x, \epsilon, n) := \{y \in X : d(f^i(x), f^i(y)) < \epsilon \forall 0 \leq i \leq n\}.$$

A set  $E \subseteq X$  is called  $(n, \epsilon)$ -*spanning* if every point of  $X$  remains close to some point of  $E$  for the first  $n$  iterates, i.e. if  $X = \bigcup_{x \in E} B_f(x, \epsilon, n)$ . Let  $N(n, \epsilon)$  be the minimal cardinality of a  $(n, \epsilon)$ -spanning set. The topological entropy is the growth rate of  $N(n, \epsilon)$  as a function of  $n$ :

**Definition 5.1.** *The topological entropy of the map  $f : X \rightarrow X$  is defined as*

$$h_{top}(f) := \lim_{\epsilon \rightarrow 0^+} \lim_{n \rightarrow \infty} \frac{1}{n} \log N(n, \epsilon).$$

When  $f$  is a piecewise monotone map of a real interval, it is easier to compute the entropy by looking at the number of laps. Recall the *lap number*  $L(g)$  of a piecewise monotone interval map  $g : I \rightarrow I$  is the smallest cardinality of a partition of  $I$  in intervals such that the restriction of  $g$  to any such interval is monotone. The following result of Misiurewicz and Szlenk relates the topological entropy to the growth rate of the lap number of the iterates of  $f$ :

**Theorem 5.2** ([MS]). *Let  $f : I \rightarrow I$  be a piecewise monotone map of a close bounded interval  $I$ , and let  $L(f^n)$  be the lap number of the iterate  $f^n$ . Then the following equality holds:*

$$h(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log L(f^n).$$

Another useful property of topological entropy is that it is invariant under dynamical extensions of bounded degree:

**Proposition 5.3** ([Bo2]). *Let  $f : X \rightarrow X$  and  $g : Y \rightarrow Y$  be two continuous maps of compact metric spaces, and let  $\pi : X \rightarrow Y$  a continuous, surjective map such that  $g \circ \pi = \pi \circ f$ . Then*

$$h_{top}(g) \leq h_{top}(f).$$

Moreover, if there exists a finite number  $d$  such that for each  $y \in Y$  the fiber  $\pi^{-1}(y)$  has cardinality always smaller than  $d$ , then

$$h_{top}(g) = h_{top}(f).$$

In order to resolve the ambiguities arising from considering different restrictions of the same map, if  $K$  is an  $f$ -invariant set we shall use the notation  $h_{top}(f, K)$  to denote the topological entropy of the restriction of  $f$  to  $K$ . Recall that the *non-wandering set* for a map  $f$  of a compact metric space  $X$  is the set of points  $x$  in  $X$  such that for each neighborhood  $U$  of  $x$  there exists  $n \geq 1$  with  $f^n(U) \cap U \neq \emptyset$ . The topological entropy of a map equals the topological entropy of its restriction to the non-wandering set:

**Proposition 5.4** ([Bo]). *Let  $f : X \rightarrow X$  a continuous map of a compact metric space, and let  $Y \subseteq X$  its non-wandering set. Then  $h_{top}(f, Y) = h_{top}(f, X)$ .*

The following proposition is the fundamental step to relate entropy and Hausdorff dimension of invariant subsets of the circle ([Fu], Proposition III.1; see also [Bi]):

**Proposition 5.5.** *Let  $d \geq 1$ , and  $\Omega \subset \mathbb{R}/\mathbb{Z}$  be a closed, invariant set for the map  $Q(x) := dx \pmod{1}$ . Then the topological entropy of the restriction of  $Q$  to  $\Omega$  is related to the Hausdorff dimension of  $\Omega$  in the following way:*

$$\text{H.dim } \Omega = \frac{h_{top}(Q, \Omega)}{\log d}.$$

## 6. INVARIANT SETS OF EXTERNAL ANGLES

Let  $f_c$  be a topologically finite quadratic polynomial, and  $T_c$  its Hubbard tree. One of the main players in the rest of the paper is the set  $H_c$  of angles of external rays landing on the Hubbard tree:

$$H_c := \{\theta \in S^1 : R_c(\theta) \text{ lands on } T_c\}.$$

Note that, since  $T_c$  is compact and the Carathéodory loop is continuous by local connectivity,  $H_c$  is a closed subset of the circle. Moreover, since  $T_c \cap J(f_c)$  is  $f_c$ -invariant, then  $H_c$  is (forward-)invariant for the doubling map, i.e.  $D(H_c) \subseteq H_c$ .

Similarly, we will denote by  $S_c$  the set of angles of rays landing on the spine  $[-\beta, \beta]$ , and  $B_c$  the set of angles of rays landing on the set of biaccessible points.

**Proposition 6.1.** *Let  $f_c$  be a topologically finite quadratic polynomial. Then*

$$\text{H.dim } H_c = \text{H.dim } S_c = \text{H.dim } B_c.$$

*Proof.* Lemma 2.1 implies the inclusion

$$S_c \setminus \{0, 1/2\} \subseteq B_c \subseteq \bigcup_{n \geq 0} D^{-n}(S_c)$$

hence

$$\text{H.dim } S_c \leq \text{H.dim } B_c \leq \sup_{n \geq 0} \text{H.dim } D^{-n}(S_c) = \text{H.dim } S_c.$$

Moreover, it is clear that  $H_c \subseteq B_c$ , and by Lemma 4.6 one also has

$$S_c \setminus \{0, 1/2\} \subseteq \bigcup_{n \geq 0} D^{-n}(H_c)$$

hence  $\text{H.dim } S_c \leq \text{H.dim } H_c \leq \text{H.dim } B_c$ . □

We will now characterize the set  $H_c$  and other similar sets of angles purely in terms of the dynamics of the doubling map on the circle, as the set of points whose orbit never hits certain open intervals.

In order to do so, we will make use of the following lemma:

**Lemma 6.2.** *Let  $X \subseteq S^1$  be a closed, forward invariant set for the doubling map  $D$ , so that  $D(X) \subseteq X$ , and let  $U \subseteq S^1$  be an open set, disjoint from  $X$ . Suppose moreover that*

- (1)  $D^{-1}(X) \setminus X \subseteq U$ ;
- (2)  $\partial U \subseteq X$ .

*Then  $X$  equals the set of points whose orbit never hits  $U$ :*

$$X = \{\theta \in S^1 : D^n(\theta) \notin U \quad \forall n \geq 0\}.$$

*Proof.* Let  $\theta$  belong to  $X$ . By forward invariance,  $D^n(\theta) \in X$  for each  $n \geq 0$ , and since  $X$  and  $U$  are disjoint, then  $D^n(\theta) \notin U$  for all  $n$ . Conversely, let us suppose that  $\theta$  does not belong to  $X$ , and let  $V$  be the connected component of the complement of  $X$  containing  $\theta$ ; since the doubling map is uniformly expanding, there exists some  $n$  such that  $f^n(V)$  is the whole circle, hence there exists an integer  $k \geq 1$  such that  $D^k(V) \cap X \neq \emptyset$ , but  $D^{k-1}(V) \cap X = \emptyset$ ; then,  $D^{k-1}(V)$  intersects  $D^{-1}(X) \setminus X$ , so by (1) it intersects  $U$ . Moreover, since  $\partial U \subseteq X$  we have  $D^{k-1}(V) \cap \partial U = \emptyset$ , so  $D^{k-1}(V)$  is an open set which intersects  $U$  but does not intersect its boundary, hence  $D^{k-1}(V) \subseteq U$  and, since  $\theta \in V$ , we have  $D^{k-1}(\theta) \in U$ .  $\square$

Let us now describe combinatorially the set of angles of rays landing on the Hubbard tree. Let  $T_c$  be the Hubbard tree of  $f_c$ ; since  $T_c$  is a compact set, then  $H_c = \gamma^{-1}(T_c)$  is a closed subset of the circle. Among all connected components of the complement of  $H_c$ , there are finitely many  $U_1, U_2, \dots, U_r$  which contain rays which land on the preimage  $f_c^{-1}(T_c)$ . The angles of rays landing on the Hubbard tree are precisely the angles whose future trajectory for the doubling map never hits the  $U_i$ :

**Proposition 6.3** ([Li]). *Let  $T_c$  be the Hubbard tree of  $f_c$ , and  $U_1, U_2, \dots, U_r$  be the connected components of the complement of  $H_c$  which contain rays landing on  $f_c^{-1}(T_c)$ . Then the set  $H_c$  of angles of rays landing on  $T_c$  equals*

$$H_c = \{\theta \in S^1 : D^n(\theta) \notin U_i \quad \forall n \geq 0 \quad \forall i = 1, \dots, r\}.$$

*Proof.* It follows from Lemma 6.2 applied to  $X = H_c$  and  $U = U_1 \cup \dots \cup U_r$ . Indeed,  $D(H_c) \subseteq H_c$  since  $T_c \cap J(f_c)$  is forward-invariant under  $f_c$ . The set  $U$  is disjoint from  $H_c$  by definition of the  $U_i$ . Moreover, if  $\theta$  belongs to  $D^{-1}(H_c) \setminus H_c$ , then  $R_c(\theta)$  lands on  $f_c^{-1}(T_c)$ , so  $\theta$  belongs to some  $U_i$ . Finally, let us check that for each  $i$  we have the inclusion  $\partial U_i \subseteq H_c$ . Indeed, if  $U$  is non-empty then  $H_c$  has no interior (since it is invariant for the doubling map and does not coincide with the whole circle), so angles on the boundary of  $U_i$  are limits of angles in  $H_c$ , so their corresponding rays land on the Hubbard tree by continuity of the Riemann mapping on the boundary.  $\square$

## 7. ENTROPY OF HUBBARD TREES

We are now ready to prove the relationship between the topological entropy of a topologically finite quadratic polynomial  $f_c$  and the Hausdorff dimension of the set of rays which land on the Hubbard tree  $T_c$ :

**Theorem 7.1.** *Let  $f_c(z) = z^2 + c$  be a topologically finite quadratic polynomial, let  $T_c$  be its Hubbard tree and  $H_c$  the set of external angles of rays which land on the Hubbard tree. Then we have the identity*

$$\frac{h_{top}(f_c |_{T_c})}{\log 2} = \text{H.dim } H_c.$$

*Proof.* Let  $\gamma : S^1 \rightarrow J(f_c)$  the Carathéodory loop. We know that

$$\gamma(D(\theta)) = f_c(\gamma(\theta)).$$

By Proposition 4.7, the cardinality of the preimage of any point is bounded; hence, by Theorem 5.3, we have

$$h_{top}(f_c, J(f_c) \cap T_c) = h_{top}(D, \gamma^{-1}(J(f_c) \cap T_c)) = h_{top}(D, H_c).$$

Moreover, Proposition 5.4 implies

$$h_{top}(f_c, J(f_c) \cap T_c) = h_{top}(f_c, T_c).$$

Then we conclude, by the dimension formula of Proposition 5.5, that

$$\text{H.dim } H_c = \frac{h_{top}(D, H_c)}{\log 2}.$$

□

The exact same argument applies to any compact, forward invariant set  $X$  in the Julia set:

**Theorem 7.2.** *Let  $f_c$  be a topologically finite quadratic polynomial, and  $X \subseteq J(f_c)$  compact and invariant (i.e.  $f_c(X) \subseteq X$ ). Let define the set*

$$\Theta_c(X) := \{\theta \in S^1 : R_c(\theta) \text{ lands on } X\};$$

*then we have the equality*

$$\frac{h_{top}(f_c | X)}{\log 2} = \text{H.dim } \Theta_c(X).$$

## 8. COMBINATORIAL DESCRIPTION: THE REAL CASE

Suppose  $c \in \partial\mathcal{M} \cap \mathbb{R}$ . By definition, the *dynamic root*  $r_c$  of  $f_c$  is the critical value  $c$  if  $c$  belongs to the Julia set, otherwise it is the smallest value of  $J(f_c) \cap \mathbb{R}$  larger than  $c$ . This means that  $r_c$  lies on the boundary of the bounded Fatou component containing  $c$ .

Recall that the *impression* of a parameter ray  $R_M(\theta)$  is the set of all  $c \in \partial\mathcal{M}$  for which there is a sequence  $\{w_n\}$  such that  $|w_n| > 1$ ,  $w_n \rightarrow e^{2\pi i\theta}$ , and  $\Phi_M^{-1}(w_n) \rightarrow c$ . We denote the impression of  $R_M(\theta)$  by  $\hat{R}_M(\theta)$ . It is a non-empty, compact, connected subset of  $\partial\mathcal{M}$ . Every point of  $\partial\mathcal{M}$  belongs to the impression of at least one parameter ray. Conjecturally, every parameter ray  $R_M(\theta)$  lands at a well-defined point  $c(\theta) \in \partial\mathcal{M}$  and  $\hat{R}_M(\theta) = c(\theta)$ .

In the real case, much more is known to be true. First of all, every real Julia set is locally connected [LvS]. The following result summarizes the situation for real maps.

**Theorem 8.1** ([Za2], Theorem 3.3). *Let  $c \in \partial\mathcal{M} \cap \mathbb{R}$ . Then there exists a unique angle  $\theta_c \in [0, 1/2]$  such that the rays  $R_c(\pm\theta_c)$  land at the dynamic root  $r_c$  of  $f_c$ . In the parameter plane, the two rays  $R_M(\pm\theta_c)$ , and only these rays, contain  $c$  in their impression.*

The theorem builds on the previous results of Douady-Hubbard [DH] and Tan Lei [TanL] for the case of periodic and preperiodic critical points and uses density of hyperbolicity in the real quadratic family to get the claim for all real maps.

To each angle  $\theta \in S^1$  we can associate a *length*  $\ell(\theta)$  as the length (along the circle) of the chord delimited by the leaf joining  $\theta$  to  $1 - \theta$  and containing the angle  $\theta = 0$ . In formulas, it is easy to check that

$$\ell(\theta) := \begin{cases} 2\theta & \text{if } 0 \leq \theta < \frac{1}{2} \\ 2 - 2\theta & \text{if } \frac{1}{2} \leq \theta < 1. \end{cases}$$

In order to make notation lighter, we shall sometimes write  $\ell_\theta$  instead of  $\ell(\theta)$ . For a real parameter  $c$ , we will denote as  $\ell_c$  the length of the characteristic leaf

$$\ell_c := \ell(\theta_c).$$

The key to analyzing the symbolic dynamics of  $f_c$  is the following interpretation in terms of the dynamics of the tent map. Since all real Julia sets are locally connected, for  $c$  real all dynamical rays  $R_c(\theta)$  have a well-defined limit  $\gamma_c(\theta)$ , which belongs to  $J(f_c)$ . Let us moreover denote by  $T$  the full tent map on the interval  $[0, 1]$ , defined as  $T(x) := \min\{2x, 2 - 2x\}$ . The following diagram is commutative:

$$\begin{array}{ccc} \begin{array}{c} \curvearrowright \\ D \\ \downarrow \\ S^1 \end{array} & \xrightarrow{\gamma_c} & \begin{array}{c} \curvearrowright \\ f_c \\ \downarrow \\ J(f_c) \end{array} \\ \downarrow \ell & & \downarrow T \\ [0, 1] & & \end{array}$$

This means that we can understand the dynamics of  $f_c$  on the Julia set in terms of the dynamics of the tent map on the space of lengths. First of all, the set of external angles corresponding to rays which land on the real slice of the Julia set can be given the following characterization:

**Proposition 8.2.** *Let  $c \in [-2, \frac{1}{4}]$ . Then the set  $S_c$  of external angles of rays which land on the real slice  $J(f_c) \cap \mathbb{R}$  of the Julia set is*

$$S_c = \{\theta \in S^1 : T^n(\ell_\theta) \leq \ell_c \quad \forall n \geq 1\}.$$

*Proof.* Let  $X$  be the set of angles of rays landing on the segment  $[c, \beta]$ . Since  $f_c^{-1}([c, \beta]) = [-\beta, \beta]$ , then  $D^{-1}(X)$  is the set of angles landing on the spine. Thus, if we set  $U := (\theta_c, 1 - \theta_c)$  then the hypotheses of Lemma 6.2 hold, hence we get the following description:

$$S_c = \{\theta \in S^1 : D^n(\theta) \notin (\theta_c, 1 - \theta_c) \quad \forall n \geq 1\}$$

hence by taking the length on both sides

$$\theta \in S_c \iff \ell(D^n(\theta)) \leq \ell(\theta_c) \quad \forall n \geq 1$$

and by the commutative diagram we have  $\ell(D^n(\theta)) = T^n(\ell(\theta))$ , which, when substituted into the previous equation, yields the claim.  $\square$

Recall that for a real polynomial  $f_c$  the Hubbard tree is the segment  $[c, f_c(c)]$ . Let us denote as  $L_c := \ell(D(\theta_c))$  the length of the leaf which corresponds to  $f_c(c) = c^2 + c$ . The set of angles which land on the Hubbard tree can be characterized as:

**Proposition 8.3.** *The set  $H_c$  of angles of external rays which land on the Hubbard tree for  $f_c$  is:*

$$H_c := \{\theta \in S^1 : T^n(\ell_\theta) \geq L_c \quad \forall n \geq 0\}.$$

*Proof.* Since the Hubbard tree is  $[c, f_c(c)]$  and its preimage is  $[0, c]$ , one can take  $U = (D(\theta_c), 1 - D(\theta_c))$  (where we mean the interval containing zero) and  $X = H_c$ , and we get by Lemma 6.2

$$H_c = \{\theta \in S^1 : D^n(\theta) \notin U \quad \forall n \geq 0\}$$

hence in terms of length

$$H_c = \{\theta \in S^1 : \ell(D^n(\theta)) \geq \ell(D(\theta_c)) \quad \forall n \geq 0\}$$

which yields the result when you substitute  $\ell(D^n(\theta)) = T^n(\ell(\theta))$  and  $L_c = \ell(D(\theta_c))$ .  $\square$

**8.1. The real slice of the Mandelbrot set.** Let us now turn to parameter space. We are looking for a combinatorial description of the set of rays which land on the real axis. However, in order to account for the fact that some rays might not land, let us define the set  $\mathcal{R}$  of *real parameter angles* as the set of angles of rays whose prime-end impression intersects the real axis:

$$\mathcal{R} := \{\theta \in S^1 : \hat{R}_M(\theta) \cap \mathbb{R} \neq \emptyset\}.$$

The set  $\mathcal{R}$  is also the closure (in  $S^1$ ) of the union of the angles of rays landing on the boundaries of all real hyperbolic components. Combinatorially, elements of  $\mathcal{R}$  correspond to leaves which are maximal in their orbit under the dynamics of the tent map:

**Proposition 8.4.** *The set  $\mathcal{R}$  of real parameter angles can be characterized as*

$$\mathcal{R} = \{\theta \in S^1 : T^n(\ell_\theta) \leq \ell_\theta \quad \forall n \geq 0\}.$$

*Proof.* Let  $\theta_c$  be the characteristic angle of a real quadratic polynomial. Since the corresponding dynamical ray  $R_c(\theta)$  lands on the spine, by Proposition 8.2 applied to  $\ell(\theta_c) = \ell_c$  we have for each  $n \geq 0$

$$T^n(\ell(\theta_c)) \leq \ell(\theta_c).$$

Conversely, if  $\theta$  does not belong to  $\mathcal{R}$  then it belongs to the opening of some real hyperbolic component  $W$ . By symmetry, we can assume  $\theta$  belongs to  $[0, 1/2]$ : then  $\theta$  must belong to the interval  $(\alpha, \omega)$ , whose endpoints have binary expansion

$$\begin{aligned} \alpha &= 0.\overline{s_1 \dots s_n} \\ \omega &= 0.s_1 \dots s_n \overline{\check{s}_1 \dots \check{s}_n} \end{aligned}$$

where  $n$  is the period of  $W$ , and  $s_1 = 0$  (recall the notation  $\check{s}_i := 1 - s_i$ ); in this case it is easy to check that both  $\ell(\alpha) = 2\alpha$  and  $\ell(\omega) = 2\omega$  are fixed points of  $T^n$ , and  $T^n(x) > x$  if  $x \in (2\alpha, 2\omega)$ . The description is equivalent to the one given in ([Za2], Theorem 3.7).  $\square$

In the following it will be useful to introduce the following slice of  $\mathcal{R}$ , by taking for each  $c \in [-2, 1/4]$  the set of angles of rays whose impression intersects the real axis to the right of  $c$ .

**Definition 8.5.** *Let  $c \in [-2, 1/4]$ . Then we define the set*

$$P_c := \mathcal{R} \cap [1 - \theta_c, \theta_c]$$

where  $\theta_c \in [0, 1/2]$  is the characteristic ray of  $f_c$ , and  $[1 - \theta_c, \theta_c]$  is the interval containing 0.

Let us note that by Proposition 8.4 we get the following characterization of  $P_c$ :

$$P_c := \{\theta \in S^1 : \ell_\theta \leq \ell_c \text{ and } T^n(\ell_\theta) \leq \ell_\theta \quad \forall n \geq 0\}.$$

A corollary of the previous description is that real parameter rays to the right of  $c$  also land on the Hubbard tree of  $f_c$ :

**Corollary 8.6.** *Let  $c \in [-2, 1/4]$ . Then the inclusion*

$$P_c \setminus \{0\} \subseteq H_c$$

*holds.*

*Proof.* By the above characterization, if  $\theta$  belongs to  $P_c$  we have

$$T^n(\ell_\theta) \leq \ell_\theta \leq \ell_c$$

hence  $\theta$  belongs to  $S_c$  by Proposition 8.2. Moreover, if  $\theta \neq 0$ , then  $\ell_\theta \geq 1/2$ , hence  $\theta$  belongs to  $S_c \cap [1/4, 3/4] = H_c \cap [1/4, 3/4]$  as required.  $\square$

**8.2. The bisection algorithm.** Let us now describe an algorithm to generate all real hyperbolic windows (see Figure 8).

**Theorem 8.7.** *The set of all real hyperbolic windows in the Mandelbrot set can be generated as follows. Let  $c_1 < c_2$  be two real parameters on the boundary of  $\mathcal{M}$ , with external angles  $0 \leq \theta_2 < \theta_1 \leq \frac{1}{2}$ . Let  $\theta^*$  be the dyadic pseudocenter of the interval  $(\theta_2, \theta_1)$ , and let*

$$\theta^* = 0.s_1s_2\dots s_{n-1}s_n$$

*be its binary expansion, with  $s_n = 1$ . Then the hyperbolic window of smallest period in the interval  $(\theta_2, \theta_1)$  is the interval of external angles  $(\alpha_2, \alpha_1)$  with*

$$\begin{aligned} \alpha_2 &:= \frac{0.\bar{s}_1\bar{s}_2\dots\bar{s}_{n-1}}{0.s_1s_2\dots s_{n-1}\bar{s}_1\bar{s}_2\dots\bar{s}_{n-1}} \\ \alpha_1 &:= \frac{0.\bar{s}_1\bar{s}_2\dots\bar{s}_{n-1}}{0.s_1s_2\dots s_{n-1}\bar{s}_1\bar{s}_2\dots\bar{s}_{n-1}} \end{aligned}$$

*(where  $\bar{s}_i := 1 - s_i$ ). All hyperbolic windows are obtained by iteration of this algorithm, starting with  $\theta_2 = 0$ ,  $\theta_1 = 1/2$ .*

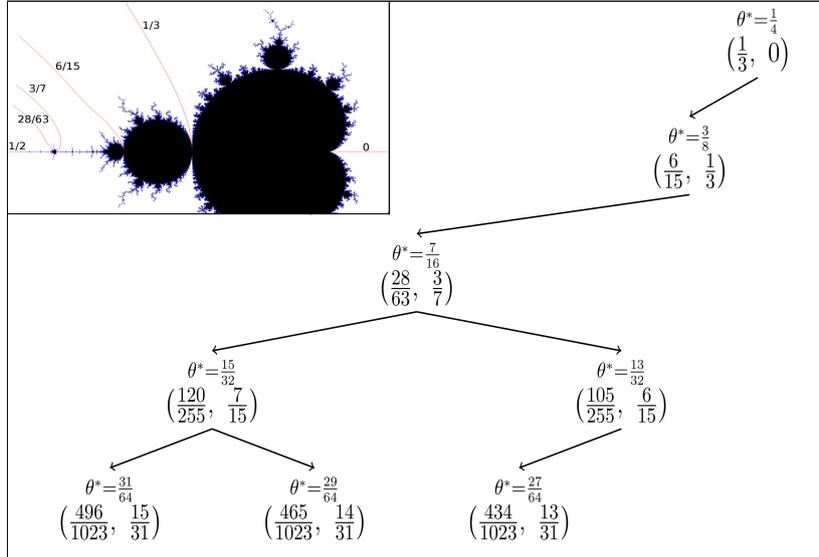


FIGURE 8. The first few generations of the bisection algorithm which produces all real hyperbolic windows between external angles 0 and  $\frac{1}{2}$ . Every interval represents a hyperbolic component, and we display the angles of rays landing at the endpoints as well as the pseudocenter  $\theta^*$ . The root of the tree ( $\theta^* = \frac{1}{4}$ ) corresponds to the real slice of the main cardioid, its child is the “basilica” component of period 2 ( $\theta^* = \frac{3}{8}$ ), then  $\theta^* = \frac{7}{16}$  corresponds to the “airplane” component of period 3 etc. Some branches of the tree do not appear because some pairs of components have an endpoint in common (due to period doubling).

*Proof of Theorem 8.7.* The theorem is a rephrasing, in the language of complex dynamics, of ([BCIT], Proposition 3). Indeed, in [BCIT] it is analyzed the set

$$\Lambda := \{x \in [0, 1] : T^n(x) \leq x \quad \forall n \geq 0\}$$

which by Proposition 8.4 is closely related to the set  $\mathcal{R}$  of real parameter angles, namely we have the equality  $\mathcal{R} \cap [0, 1/2] = \frac{1}{2}\Lambda$  (see also [BCIT], Proposition 7). Moreover, the intervals  $J_d = (r^-, r^+)$  of ([BCIT], Section 4.1) determine exactly

the hyperbolic windows  $[\alpha_2, \alpha_1]$  defined in the statement of the theorem, via the translation  $\alpha_1 = \frac{r^-}{2}$  and  $\alpha_2 = \frac{r^+}{2}$ .  $\square$

A similar algorithm is given by Schleicher [Sch] in the context of the theory of *internal addresses*.

### Example

Suppose we want to find all hyperbolic components between the airplane parameter (of period 3) and the basilica parameter (of period 2). The ray landing on the root of the airplane component has angle  $\theta_1 = \frac{3}{7}$ , while the ray landing immediately to the left of the basilica has angle  $\theta_2 = \frac{2}{5}$ . Let us apply the algorithm:

$$\begin{array}{rcl} \theta_2 = \frac{2}{5} & = & 0.011001100110\dots \\ \theta_1 = \frac{3}{7} & = & 0.011011011011\dots \end{array}$$

$$\theta^* = 0.01101$$

hence  $\alpha_1 = 0.\overline{0110} = \frac{2}{5}$  and  $\alpha_2 = 0.\overline{01101001} = \frac{7}{17}$  and we get the component of period 4 which is the doubling of the basilica. Note we do not always get the doubling of the previous component; indeed, the next step would be

$$\begin{array}{rcl} \theta_2 = \frac{7}{17} & = & 0.011010010110\dots \\ \theta_1 = \frac{3}{7} & = & 0.011011011011\dots \end{array}$$

$$\theta^* = 0.011011$$

hence  $\alpha_1 = 0.\overline{01101}$  and we get a component of period 5. Iteration of the algorithm produces all real hyperbolic components. We conjecture that a similar algorithm holds in every vein.

## 9. RENORMALIZATION AND TUNING

The Mandelbrot set has the remarkable property that near every point of its boundary there are infinitely many copies of the whole  $\mathcal{M}$ , called *baby Mandelbrot sets*. A *hyperbolic component*  $W$  of the Mandelbrot set is a connected component of the interior of  $\mathcal{M}$  such that all  $c \in W$ , the orbit of the critical point is attracted to a periodic cycle under iteration of  $f_c$ .

Douady and Hubbard [DH] related the presence of baby copies of  $\mathcal{M}$  to renormalization in the family of quadratic polynomials. More precisely, they associated to any hyperbolic component  $W$  a *tuning map*  $\iota_W : \mathcal{M} \rightarrow \mathcal{M}$  which maps the main cardioid of  $\mathcal{M}$  to  $W$ , and such that the image of the whole  $\mathcal{M}$  under  $\iota_W$  is a homeomorphic copy of  $\mathcal{M}$ .

The tuning map can be described in terms of external angles in the following terms [Do]. Let  $W$  be a hyperbolic component, and  $\eta_0, \eta_1$  the angles of the two external rays which land on the root of  $W$ . Let  $\eta_0 = 0.\overline{\Sigma_0}$  and  $\eta_1 = 0.\overline{\Sigma_1}$  be the (purely periodic) binary expansions of the two angles which land at the root of  $W$ . Let us define the map  $\tau_W : S^1 \rightarrow S^1$  in the following way:

$$\theta = 0.\theta_1\theta_2\theta_3\dots \mapsto \tau_W(\theta) = 0.\Sigma_{\theta_1}\Sigma_{\theta_2}\Sigma_{\theta_3}\dots$$

where  $\theta = 0.\theta_1\theta_2\dots$  is the binary expansion of  $\theta$ , and its image is given by substituting the binary string  $\Sigma_0$  to every occurrence of 0 and  $\Sigma_1$  to every occurrence of 1.

**Proposition 9.1** ([Do3], Proposition 7). *The map  $\tau_W$  has the property that, if  $\theta$  is a characteristic angle of the parameter  $c \in \partial\mathcal{M}$ , then  $\tau_W(\theta)$  is a characteristic angle of the parameter  $\iota_W(c)$ .*

If  $W$  is a real hyperbolic component, then  $\iota_W$  preserves the real axis. We shall now describe its image in terms of lengths.

Let  $c$  be the root of  $W$ ,  $\theta_c$  one of its characteristic angles, and  $\ell_c$  its characteristic length. Moreover, denote  $p$  the (minimal) period of  $\theta_c$  under the doubling map (so that we also have  $T^p(\ell_c) = \ell_c$ ). We shall call *tuning window* associated to  $W$  the half-open interval

$$\Omega_W := (\alpha, \omega]$$

with

$$\begin{aligned} \alpha &:= \ell_c \\ \omega &:= \min\{x > \alpha : T^p(x) = \alpha\}. \end{aligned}$$

The point  $\alpha$  will be called the *root* of the tuning window, and the integer  $p$  its *period*. An angle  $\theta \in \mathcal{R}$  lies in the image of the tuning operator  $\tau_W$  if and only if the length  $\ell_\theta$  belongs to the closure of the tuning window  $\Omega_W$ . Overlapping tuning windows are nested, and we call *maximal tuning window* a tuning window which is not contained in any other tuning window.

**Definition 9.2.** A length  $\ell \in [0, 1]$  is called *renormalizable* if it belongs to some tuning window  $\Omega_W$ , where  $W$  is a hyperbolic component of period  $> 1$ . Otherwise,  $\ell$  is said to be *non-renormalizable*. Moreover, an angle  $\theta \in \mathcal{R}$  is called *renormalizable* if the associated length  $\ell_\theta$  is renormalizable, and a parameter  $c \in [-2, 1/4]$  is called *renormalizable* if so is its characteristic length  $\ell_c$ .

Recall that if  $c$  is renormalizable of period  $p$ , then there exists a closed subinterval  $I$  of the Hubbard tree of  $f_c$  such that the intervals  $I, f_c(I), \dots, f_c^{p-1}(I)$  have disjoint interiors,  $f_c^p(I) \subseteq I$  and the restriction  $f_c^p|_I$  is unimodal.

Note that we take the left-hand side of the interval  $\Omega_W$  to be open, thus roots of maximal tuning windows are non-renormalizable<sup>1</sup>.

Let us describe the behavior of Hausdorff dimension with respect to the tuning operator:

**Proposition 9.3.** Let  $W$  be a hyperbolic component of period  $p$  with root  $r(W)$ , and let  $c \in \mathcal{M}$ . Then we have the equalities

$$\begin{aligned} \text{H.dim } H_{\tau_W(c)} &= \max \{ \text{H.dim } H_{r(W)}, \text{H.dim } \tau_W(H_c) \} \\ \text{H.dim } P_{\tau_W(c)} &= \max \{ \text{H.dim } P_{r(W)}, \text{H.dim } \tau_W(P_c) \}. \end{aligned}$$

Moreover,

$$\text{H.dim } \tau_W(H_c) = \frac{1}{p} \text{H.dim } H_c.$$

*Proof.* Let  $c' := \tau_W(c)$ . The Julia set of  $f_{c'}$  is constructed by taking the Julia set of  $f_{r(W)}$  and inserting a copy of the Julia set of  $f_c$  inside every bounded Fatou component. Hence in particular, the extended Hubbard tree of  $J(f_{c'})$  contains a topological copy  $T_1$  of the extended Hubbard tree of  $f_{r(W)}$  which contains the critical value  $c'$ . The set of angles which land on  $T_1$  are precisely the image  $\tau_W(H_c^{ext})$  via tuning of the set  $H_c^{ext}$  of angles which land on the extended Hubbard tree of  $H_c$ . Let  $\theta \in H_{c'}$  be an angle whose ray lands on the Hubbard tree of  $f_{c'}$ . Then either  $\theta$  also belongs to  $H_{r(W)}$  or it lands on a small copy of the extended Hubbard tree of  $f_{r(W)}$ , hence it eventually maps to  $T_1$ . Hence we have the inclusions

$$H_{r(W)} \cup \tau_W(H_c) \subseteq H_{c'} \subseteq H_{r(W)} \cup \bigcup_{n \geq 0} D^{-n}(\tau_W(H_c^{ext}))$$

from which the claim follows, recalling that  $H_c^{ext} \setminus \{-\beta, \beta\} \subseteq \bigcup_{n \geq 0} D^{-n}(H_c)$ .

<sup>1</sup>This definition is slightly different from some others in the literature: in particular, in many sources the root of a tuning window is considered to be renormalizable, and its renormalization is the polynomial  $f_{1/4}$  at the cusp of  $\mathcal{M}$ .

In parameter space, one notices that the set of rays landing on the vein  $v$  for  $c'$  either land between 0 and  $r(W)$ , or between  $r(W)$  and  $c'$ . In the latter case, they land on the small copy of the Mandelbrot set with root  $r(W)$ , so they are in the image of  $\tau_W$ . Hence

$$P_{c'} = P_{r(W)} \cup \tau_W(P_c)$$

and the claim follows. The last claim follows by looking at the commutative diagram

$$\begin{array}{ccc} \begin{array}{c} \curvearrowright \\ D \\ \downarrow \\ H_c \end{array} & \xrightarrow{\tau_W} & \begin{array}{c} \curvearrowright \\ D^p \\ \downarrow \\ \tau_W(H_c) \end{array} \end{array}$$

Since  $\tau_W$  is injective and continuous restricted to  $H_c$  (because  $H_c$  does not contain dyadic rationals) we have by Proposition 5.3

$$h_{top}(D, H_c) = h_{top}(D^p, \tau_W(H_c))$$

and, since  $H_c$  is forward invariant we can apply Proposition 5.5 and get

$$\text{H.dim } \tau_W(H_c) = \frac{h_{top}(D^p, \tau_W(H_c))}{p \log 2} = \frac{1}{p} \frac{h_{top}(D, H_c)}{\log 2} = \frac{1}{p} \text{H.dim } H_c$$

from which the claim follows.  $\square$

**9.1. Scaling and continuity at the Feigenbaum point.** Among all tuning operators is the operator  $\tau_W$  where  $W$  is the basilica component of period 2 (the associated strings are  $\Sigma_0 = 01$ ,  $\Sigma_1 = 10$ ). We will denote this particular operator simply with  $\tau$ . The fixed point of  $\tau$  is the external angle of the *Feigenbaum point*  $c_{Feig}$ .

Let us explicitly compute the Hausdorff dimension of  $H_{c_{Feig}}$ . Indeed, let  $c_0$  be the airplane parameter of angle  $\theta_0 = 3/7$ , and consider the sequence of parameters of angles  $\theta_n := \tau^n(\theta_0)$  given by successive tuning.

The set  $H_{c_0}$  is given by all angles with binary sequences which do not contain 3 consecutive equal symbols, hence its Hausdorff dimension is easily computable (see example in the introduction):

$$\text{H.dim } H_{\theta_0} = \log_2 \frac{\sqrt{5} + 1}{2}.$$

Now, by repeated application of Proposition 9.3 we have

$$\text{H.dim } H_{\theta_n} = \frac{\text{H.dim } H_{\theta_0}}{2^n}.$$

Note that the angles  $\theta_n$  converge from above to the Feigenbaum angle  $\theta_F$ , also  $\text{H.dim } H_{c_{Feig}} = 0$ ; moreover, since  $\theta_n$  is periodic of period  $2^n$ ,

$$\theta_n - \theta_F \asymp 2^{-2^n}$$

and together with

$$(3) \quad \text{H.dim } H_{\theta_n} - \text{H.dim } H_{\theta_F} = \frac{\text{H.dim } H_{\theta_0}}{2^n}$$

we have proved the

**Proposition 9.4.** *For the Feigenbaum parameter  $c_{Feig}$  we have*

$$\text{H.dim } S_{c_{Feig}} = 0$$

*and moreover, the entropy function  $\theta \mapsto h(\theta)$  is not Hölder-continuous at the Feigenbaum point. Similarly, the Hausdorff dimension of the set  $B_{c_{Feig}}$  of biaccessible angles for the Feigenbaum parameter is 0.*

Note that it also follows that the entropy  $h(c) := h_{\text{top}}(f_c, [-\beta, \beta])$  as a function of the parameter  $c$  has vertical tangent at  $c = c_{\text{Feig}}$ , as shown in Figure 1. Indeed, if  $c_n \rightarrow c_{\text{Feig}}$  is the sequence of period doubling parameters converging to the Feigenbaum point, it is a deep result [Ly2] that  $|c_n - c_{\text{Feig}}| \asymp \lambda^{-n}$ , where  $\lambda \cong 4.6692\dots$  is the Feigenbaum constant; hence, by equation (3), we have

$$\frac{h(c_n) - h(c_{\text{Feig}})}{|c_n - c_{\text{Feig}}|} \asymp \left(\frac{\lambda}{2}\right)^n \rightarrow \infty.$$

## 10. A COPY OF THE HUBBARD TREE INSIDE PARAMETER SPACE

We saw that the set of rays which land on the real axis in parameter space also land in the dynamical plane. In order to establish equality of dimensions, we would like to prove the other inclusion. Unfortunately, in general there is no copy of  $H_c$  inside  $P_c$  (for instance, if  $c$  is the basilica tuned with itself, then the set  $H_c$  of angles of rays landing on the Hubbard tree is an infinite, countable set, while only two pairs of rays land in parameter space to the right of  $c$ ). However, outside of the baby Mandelbrot sets, one can indeed map the combinatorial model for the Hubbard tree into the combinatorial model of parameter space:

**Proposition 10.1.** *Given a non-renormalizable, real parameter  $c \in \partial\mathcal{M}$  and another real parameter  $c' > c$ , there exists a piecewise linear map  $F : S^1 \rightarrow S^1$  such that*

$$F(H_{c'}) \subseteq P_c.$$

The proof is entirely based on the analysis of the dynamics of the tent map

$$T(x) := \begin{cases} 2x & \text{if } x \in [0, 1/2] \\ 2 - 2x & \text{if } x \in [1/2, 1] \end{cases}$$

since, as we have seen in Section 8, the sets  $H_c$  and  $P_c$  can be characterized as

$$(4) \quad H_c := \{\theta \in S^1 : T^n(\ell_\theta) \leq \ell_c \quad \forall n \geq 0\}$$

$$(5) \quad P_c := \{\theta \in S^1 : T^n(\ell_\theta) \leq \ell_\theta \leq \ell_c \quad \forall n \geq 0\}.$$

In view of the above characterizations, we give the following definition.

**Definition 10.2.** *A point  $x \in [0, 1]$  such that  $T^n(x) \leq x$  for each  $n \geq 0$  will be called extremal.*

Recall that the tent map  $T$  is injective on the two intervals  $I_0 = [0, 1/2]$  and  $I_1 = [1/2, 1]$ . In order to study its symbolic dynamics, for each  $n \geq 1$  we define recursively the partition of order  $n$  as follows:

$$\begin{aligned} \mathcal{P}_1 &:= \{I_0, I_1\}; \\ \mathcal{P}_n &:= \{I \cap T^{-1}(J) : I \in \mathcal{P}_1, J \in \mathcal{P}_{n-1}\} \quad \text{for } n \geq 2. \end{aligned}$$

For each  $I \in \mathcal{P}_n$ , the restriction of the map  $T^n$  to  $I$  is a homeomorphism between  $I$  and  $[0, 1]$ . Each element of  $\mathcal{P}_n$  will be called a *cylinder of order  $n$* . Note that each  $\mathcal{P}_n$  is not exactly a partition in the set-theoretic sense, since the boundaries of different intervals of the partition may intersect. However, every point  $x$  which is not a preimage of 0 lies in exactly one element of  $\mathcal{P}_n$  for each  $n$ .

Given a periodic point  $x \neq 0$  for the tent map  $T$  with period  $n$ , we define the *cylinder associated to  $x$*  to be the element of  $\mathcal{P}_{n+1}$  which contains  $x$ , and denote it  $C_x$ . Note that the restriction

$$T^n|_{C_x} : C_x \rightarrow I$$

is a homeomorphism, where  $I$  is the element of  $\mathcal{P}_1$  which contains  $x$ .

Given two intervals  $I, J$  inside  $[0, 1]$ , we shall write  $I < J$  if  $x \leq y$  whenever  $x \in I$  and  $y \in J$  (note that two such intervals may have a common endpoint), and  $I \leq J$  if either  $I < J$  or  $I = J$ .

**Definition 10.3.** *We say that a periodic point  $x$  for the tent map of period  $n$  is positively oriented if the map  $T^n|_{C_x}$  is orientation preserving.*

An equivalent condition is that the derivative  $(T^n)'(x)$  is positive. If  $x = \ell_c$  arises from a periodic parameter  $c$  on the real slice of the Mandelbrot set,  $x$  is positively oriented if and only if each of the two characteristic rays for  $f_c$  is fixed by  $f_c^n$  (whereas the orientation is negative if the two rays are switched by  $f_c^n$ ).

**Definition 10.4.** *Let  $x$  be a periodic point for the tent map  $T$ , of period  $n$ . We say that  $x$  is dominant if it is positively oriented, and moreover one has*

$$(6) \quad T^k(C_x) < C_x$$

for each  $1 \leq k \leq n - 1$ .

Consequently, a (periodic) external angle  $\theta$  will be called dominant if the associated length  $\ell_\theta$  is dominant according to the above definition; moreover, a parameter  $c$  on the real slice of  $\partial\mathcal{M}$  is called dominant if the associated length  $\ell_c = \ell(\theta_c)$  is dominant.

The interpretation of this definition in the dynamical plane of  $f_c$  is the following. Let  $c$  be the center of some hyperbolic component, of period  $n$ , and  $r_c$  the dynamic root of the Fatou component which contains the critical point  $c$  in the dynamical plane for  $f_c$ . Let us now define the intervals  $J_k$  for  $k = 0, \dots, n$  as subintervals of the spine  $[-\beta, \beta]$  of  $f_c$  in the following way. Let  $J_n := [r_c, 0]$ , and for each  $k < n$  define recursively  $J_k$  as the pullback of  $J_{k+1}$  along the critical orbit: namely,  $J_k$  is the unique subinterval of  $[-\beta, \beta]$  such that

- $f_c$  is a homeomorphism between  $J_k$  and  $J_{k+1}$ ;
- $f_c^k(r_c)$  lies on the boundary of  $J_k$ .

Then, the parameter  $c$  is dominant if and only if the intervals  $J_0$  and  $J_k$  have disjoint interiors for each  $k = 1, \dots, n - 1$ . For instance, the airplane parameter of period 3 is dominant, as can be seen from Figure 9.

In general, renormalizable parameters are never dominant; however, the key result is that dominant angles are dense in the set of non-renormalizable angles:

**Proposition 10.5.** *Let  $\theta \in \mathcal{R} \cap (1/3, 1/2]$  be a non-renormalizable, real angle. Then there exists a sequence  $(\theta_n)$  of real, dominant angles with  $\theta_n < \theta$  for each  $n$ , and  $\theta_n \rightarrow \theta$ .*

Assuming this fact, we are now ready to prove Proposition 10.1. The idea of the proof is that, if  $c$  is dominant and  $c' > c$ , then the pullback of  $J_n = [0, r_c]$  using the local inverse of the map  $f_c^n : J_0 \rightarrow J_n$  provides an embedding of the Hubbard tree of  $f_{c'}$  into parameter space (see Figure 10). We shall now see the details of the proof, which is carried out purely in the space of lengths, that is using the dynamics of the tent map.

*Proof of Proposition 10.1.* For any  $c$ , let  $K_c := H_c \cap [1/4, 3/4]$  be the set of external rays which land on the segment  $[0, c]$ . Since the Hubbard tree for  $f_c$  is  $[c, f_c(c)] = f_c([0, c])$ , we have  $D(K_c) = H_c$ , hence it is sufficient to prove the statement for  $K_{c'}$  instead of  $H_{c'}$ . Moreover,  $K_c$  is characterized as

$$(7) \quad K_c := \{\theta \in S^1 : \ell_\theta \geq 1/2 \text{ and } T^n(\ell_\theta) \leq \ell_c \quad \forall n \geq 0\}.$$

By Proposition 10.5, there exists a dominant point  $x$  for the tent map such that

$$(8) \quad \ell_{c'} < x < \ell_c,$$

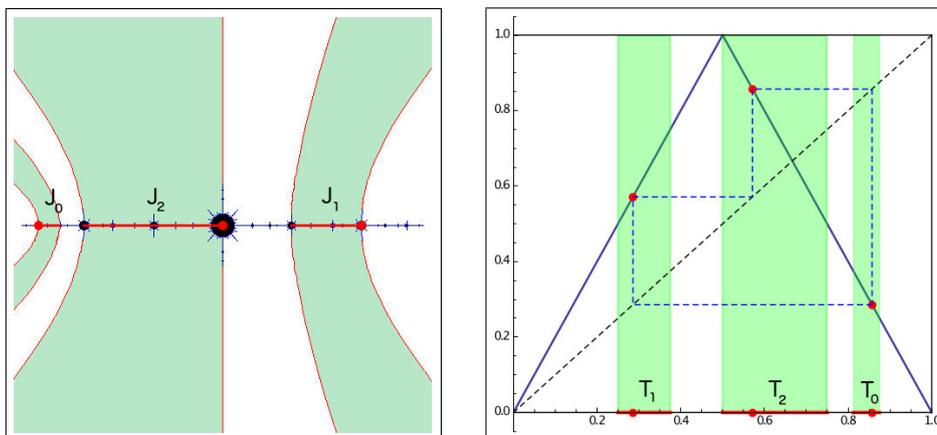


FIGURE 9. The airplane parameter is dominant. To the left: the Julia set for the airplane parameter of period 3. The thickened segments correspond to the intervals  $J_k$  for  $k = 0, 1, 2$  as defined above, while the shaded bands represent the external rays landing on them. The dominant condition holds since the segments, or equivalently the bands, do not overlap. To the right: the same picture in terms of the tent map. The red points are the orbit of  $x = 6/7 = \ell_\theta$ , while the (green) bands here represent the cylinders  $T_k = T^k(C_x)$  in Definition 10.4.

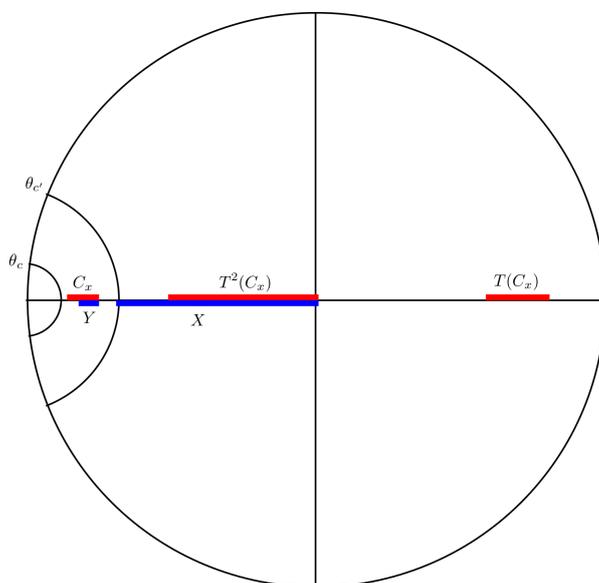


FIGURE 10. The proof of Proposition 10.1, in the case  $f_c$  is the airplane. Rays landing on the segment  $X$  are pulled back to  $Y \subseteq C_x$ , so that  $T^3(Y) = X$ . In particular, if  $\theta$  belongs to  $H_{c'}$  and lands on  $X$ , then its pullback  $\eta$  which lands on  $Y$  belongs to  $P_c$ ; indeed, its forward iterates lie respectively in  $C_x$ , then  $T(C_x)$ , then  $T^2(C_x)$ , and finally in  $X$ . From there on, the iterates lie always to the right of the leaf with endpoint  $\theta_{c'}$ .

and such that the cylinder  $C_x$  associated to  $x$  does not contain  $\ell_{c'}$ . Let  $n$  be the period of  $x$  under  $T$ . Then the map  $T^n : C_x \rightarrow T^n(C_x) = [1/2, 1]$  is a homeomorphism, and we shall consider its local inverse  $U : [1/2, 1] \rightarrow C_x$  such that  $U \circ T^n|_{C_x}$  is the identity. We claim that  $U$  maps  $\ell(K_{c'})$  into  $\ell(P_c)$ , which implies the statement by setting  $F := \ell^{-1} \circ U \circ \ell$ , where  $\ell^{-1}$  is any of the two inverse branches of the ‘‘length function’’  $\ell : S^1 \rightarrow [0, 1]$ .

In order to prove the claim, by virtue of the characterization (5) of  $P_c$ , we need to show that for each  $\theta \in K_{c'}$ , if we set  $u := U(\ell_\theta)$ , we have

$$(9) \quad u \leq \ell_c,$$

and

$$(10) \quad T^k(u) \leq u \quad \text{for all } k \geq 0.$$

In order to prove (10), suppose now  $k < n$ . Then we have

$$T^k(u) \subseteq T^k(C_x) < C_x$$

and since  $u$  belongs to  $C_x$ , the claim (10) is proven. If  $k = n$ , instead, let us note that from (8) and the characterization (7) of  $K_{c'}$  follows

$$\ell_\theta \leq \ell_{c'} < x$$

and, by using that the map  $U$  is contracting and orientation preserving, follows

$$T^n(u) = \ell_\theta \leq U(\ell_\theta) = u < x,$$

which also proves (9) since  $x < \ell_c$ . Now, if  $k > n$  we have, by the characterization of  $K_{c'}$  and the fact that  $C_x$  does not contain  $\ell_{c'}$ ,

$$T^k(u) = T^{k-n}(\ell_\theta) \leq \ell_{c'} < u$$

as required.  $\square$

**10.1. Proof of density of dominant parameters.** We now turn to the proof of Proposition 10.5.

The goal will be to prove that one can approximate extremal points with dominant points; in order to do so, we need to deal with the case when condition (6) in the definition of dominant no longer holds. Indeed, let  $x$  an extremal point which is periodic for the tent map, of period  $n$ . Note that if condition (6) does not hold for some  $k < n$ , then the cylinders  $T^k(C_x)$  and  $C_x$  intersect in their interiors, hence they must be nested, and since  $T^k(C_x)$  has larger diameter one must have  $T^k(C_x) \supseteq C_x$ . For this reason, we define the set of *recurrent times* as

$$\text{Rec}(x) := \{k \in \{1, \dots, n-1\} : \begin{array}{l} T^k(C_x) \supseteq C_x \text{ and} \\ T^k : C_x \rightarrow T^k(C_x) \text{ is orientation reversing} \end{array}\}.$$

By definition, if the point  $x$  is dominant then the set  $\text{Rec}(x)$  is empty.

If  $k$  is a recurrent time, then there exists exactly one fixed point of the restriction  $T^k|_{C_x}$ ; we shall call such a point  $w_k$ . Moreover, we shall denote by  $z_k$  the image of  $w_k$  under  $T^{n-k}$ ; note that we also have  $T^k(z_k) = z_k$ . Moreover, the map  $T^{n-k} : T^k(C_x) \rightarrow T^n(C_x)$  also has a unique fixed point, which will be denoted  $y_k$ . Note that by definition  $z_k$  belongs to  $T^{n-k}(C_x)$ , and  $y_k$  belongs to  $T^k(C_x)$ . Moreover, since both maps  $T^k$  and  $T^{n-k}$  are expanding and orientation reversing, it is not hard to check that  $z_k < y_k < x$ , and  $y_k \notin C_x$ .

A very useful property of dominant points, which we will use in the main proof, is that one can construct dominant points of high period using as building blocks dominant points of lower period. The simplest instance of this phenomenon is that when  $x$  and  $t$  are dominant points, with  $x$  of period  $n$  and  $C_t < C_x$ , then the cylinder  $C := C_x \cap T^{-n}(C_t)$  contains a dominant point. The next Lemma will provide a more general version of this fact.

**Lemma 10.6.** *Let  $x$  be a positively oriented, extremal periodic point, and  $t \in [1/2, 1]$  a dominant point with*

$$C_t < C_x$$

and

$$z_k < t$$

for each recurrent time  $k \in \text{Rec}(x)$ . Then, there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  of dominant points such that  $x_n < x$  and  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

*Proof.* Let  $n$  be the period of  $x$ , and  $m$  the period of  $t$ . If  $x$  is a point of period  $n$ , we define the *generalized cylinder*  $C_{x,k}$  as the cylinder of order  $kn + 1$  which contains  $x$ . The proof works by showing that there exists some integer  $a$  such that for any  $b > 0$  the cylinder

$$C := C_{x,b} \cap T^{-bn}(C_{t,a})$$

contains a dominant point. Intuitively, an element of  $C$  follows for the first  $bn$  iterates the orbit of  $x$ , and for the next  $am$  iterates the orbit of  $t$ .

Since  $T$  is expanding at the periodic point  $t$ , we can choose  $a$  large enough so that  $z_k < C_{t,a}$  for each  $k \in \text{Rec}(x)$ . We need to show that for each  $h = 1, \dots, bn + am - 1$  we have

$$(11) \quad T^h(C) < C.$$

- (1) Let us first consider the case  $h \leq bn$ , and let  $l \in \{0, \dots, n-1\}$  such that  $l \equiv h \pmod{n}$ .

- If  $l \neq 0$  is not a recurrent time for  $x$ , we have one of two cases: either

$$T^l(C_x) < C_x$$

which implies the claim since  $T^h(C) \subseteq T^l(C_x)$  and  $C \subseteq C_x$ ; or, the map  $T^l|_{C_x}$  is orientation preserving (and so is  $T^h|_{C_{x,b}}$ ), in which case we have

$$T^h(C) < T^h(x) < C_x$$

where in the last inequality we used the extremality of  $x$ , together with the fact that no two elements of the orbit of  $x$  lie in the same cylinder of order  $n + 1$ . The claim again follows since  $C \subseteq C_x$ .

- Let us now suppose  $l$  is a recurrent time. Since  $T^n|_{C_x}$  is orientation preserving, we have that eq. (11) is equivalent to

$$T^l(C_1) < C_1$$

where  $C_1 := C_x \cap T^{-n}(C_{t,a})$ . Then, by recalling that  $w_l$  is the fixed point of the orientation reversing map  $T^l$ , it is also equivalent to

$$T^l(C_1) < w_l$$

which, by applying the (orientation reversing) map  $T^{n-l}$  to both sides, becomes

$$T^n(C_1) > T^{n-l}(w_l) = z_l.$$

Now, the claim is satisfied since by hypothesis  $z_l < C_{t,a} = T^n(C_1)$ .

- If  $l = 0$ , let us note that since  $C_{t,a} < C_x$  and the map  $T^n$  is expanding and orientation preserving on  $C_x$ , one has for each  $\alpha = 1, \dots, b$

$$T^{\alpha n}(C) \subseteq C_{x,b-\alpha} \cap T^{-(b-\alpha)n}(C_{t,a}) < C_{x,b-\alpha+1} \supseteq C$$

as required.

- (2) Let now  $h \geq bn$ . Then we have, if  $l \equiv h - bn \pmod{m}$  with  $0 \leq l \leq m - 1$ ,

$$T^h(C) \subseteq T^l(C_t) \leq C_t < C_x$$

which, since  $C_x \supseteq C$ , implies eq. (11).

□

Let us remark that the above lemma also works, with the same proof, in the slightly more general case where  $t$  is any point such that  $T^m(t) = t$ , the map  $T^m$  is orientation preserving at  $t$ , and  $C_t$  is replaced by a cylinder  $C$  of order  $m + 1$  containing  $t$  such that  $T^k(C) \leq C$  for any  $k = 1, \dots, m - 1$ . In particular, it holds for  $t = 2/3$  the fixed point of the tent map, with  $m = 2$ , and taking  $C = [5/8, 3/4]$  the cylinder of order 3 which contains it.

**Lemma 10.7.** *Let  $x$  be an extremal, positively oriented, periodic point for the tent map, of period  $n$ , and  $k$  a recurrent time for  $x$ . Then we have:*

- (i)  $y_k$  is extremal;
- (ii) if  $x$  is non-renormalizable and  $y_k$  belongs to some tuning window  $\Omega_W = (\alpha, \omega]$  for the tent map, then  $z_k$  lies outside the closure of the window: in particular,  $z_k < \alpha$ .

*Proof.* (i) Denote  $h := n - k$ , and let  $a = 1, \dots, h - 1$ . We claim that  $x$  and  $y_k$  belong to the same cylinder of order  $h$ , and the same is true for  $T^{a+k}(x)$  and  $T^a(y_k)$ . This implies that  $T^a(y_k) \leq y_k$  by the following argument: let  $C_1$  be the cylinder of order  $h$  containing  $x$  and  $y_k$ , and  $C_2$  the cylinder of order  $h$  containing  $T^{a+k}(x)$  and  $T^a(y_k)$ . Since  $x$  is extremal, we have  $T^{a+k}(x) \leq x$ , hence  $C_1 \leq C_2$ . Now, if  $C_1 < C_2$ , then  $T^a(y_k) < y_k$  because  $T^a(y_k) \in C_1$  and  $y_k \in C_2$ . Otherwise  $C_1 = C_2$ , so  $T^a(y_k)$  and  $y_k$  are periodic points of period  $h$  which lie in the same cylinder of order  $h$ , hence they must coincide, i.e.  $T^a(y_k) = y_k$ , proving (i).

To prove the claim, note that by definition  $y_k$  and  $T^k(x)$  both belong to  $T^k(C_x)$ , and moreover  $x \in C_x \subseteq T^k(C_x)$ , so the points  $x$ ,  $T^k(x)$  and  $y_k$  all belong to the same cylinder of order  $h + 1$  (namely  $T^k(C_x)$ ), which proves the first half of the claim.

Furthermore, as a consequence, for each  $a$  we have that both  $T^{a+k}(x)$  and  $T^a(y_k)$  belong to  $C_1 := T^{a+k}(C_x)$ , which is a cylinder of order  $b := h - a + 1$ . Moreover, their images under  $T^b$  are  $T^b(T^{a+k}(x)) = T(x)$  and  $T^b(T^a(y_k)) = T(y_k)$ , hence they both belong to  $C_2 := T^{k+1}(C_x)$ , which is a cylinder of order  $h$ . Thus, the points  $T^a(y_k)$  and  $T^{a+k}(x)$  both belong to  $C_1 \cap T^{-b}(C_2)$ , which is a cylinder of order  $h + b \geq h$ , completing the proof.

- (ii) Denote  $\Omega_W = (\alpha, \omega]$  the maximal tuning window which contains  $y_k$ , and let  $p$  its period; recall that by definition  $T^p(\omega) = T^p(\alpha) = \alpha$ , and moreover any extremal periodic point for the tent map inside  $\Omega_W$  has period which is multiple of  $p$ . Since  $x$  is non-renormalizable, we have  $x > \omega$ . If we suppose  $z_k \geq \alpha$ , we then get

$$\alpha \leq z_k < y_k \leq \omega < x,$$

from which we now derive a contradiction. Since  $y_k$  is an extremal point inside  $\Omega_W$  and  $T^h(y_k) = y_k$ , then  $h$  is a multiple of the period  $p$ , hence we also have  $T^h(\omega) = \alpha$ . Now,  $x$  and  $y_k$  belong to  $T^k(C_x)$ , which is a cylinder of order  $h$  (see proof of (i)), and, by definition of recurrent time,  $T^h$  is orientation reversing on it, so we have that

$$T^h(x) < T^h(\omega) = \alpha \leq T^h(y_k) = y_k.$$

Moreover, by definition  $z_k$  and  $T^h(x)$  both belong to  $T^h(C_x)$ , which is a cylinder of order  $k + 1$ , and  $T^k|_{T^h(C_x)}$  is orientation reversing, thus

$$z_k = T^k(z_k) \leq T^k(\alpha) < T^{k+h}(x) = x$$

and since  $z_k > \alpha$  we have

$$T^k(\alpha) \geq z_k \geq \alpha$$

which contradicts the fact that  $\alpha$  is extremal unless  $\alpha = z_k$ . This latter case cannot happen: indeed, by construction  $T^k$  is orientation reversing in a neighborhood of  $z_k$ , while  $\alpha$  is the root of a maximal tuning window, hence it is positively oriented (unless  $\alpha = z_k = 2/3$ , which would imply  $x = 2/3$ ).

□

*Proof of Proposition 10.5.* First note that, since periodic angles are dense in the set of non-renormalizable angles, we can assume  $x = \ell_\theta$  is periodic of period  $p$ , and non-renormalizable. Moreover, since  $x \neq 2/3$ , we have that  $x$  is positively oriented.

First of all, let us suppose that the set  $Rec(x)$  of recurrent times for  $x$  is empty. Then if we take  $t = 2/3$  and  $C_t = [5/8, 3/4] < C_x$ , by Lemma 10.6 (and the remark following it), the sequence of cylinders

$$C_{x,n} \cap T^{-pn}(C_t), \quad n \in \mathbb{N}^+,$$

contains a sequence of dominant points approximating  $x$  from below, as required.

The rest of the proof is by induction on the period of  $x$ ; note that the smallest possible period for  $x$  is 3, and that is the case of the airplane map which we have already checked. Now, let

$$z := \max\{z_k : k \in Rec(x)\}$$

and let  $y$  be the corresponding  $y_k$ . Then, by Lemma 10.7 (i), we have that  $y$  is an extremal point of period strictly smaller than the period of  $x$ . Recall that by construction  $z < y < C_x$ . If  $y$  is non-renormalizable, we can apply the inductive hypothesis to  $y$ , hence choose  $t$  dominant such that  $z < t < y$  (or  $t = 2/3$  in the degenerate case  $y = 2/3$ ); thus, using Lemma 10.6 we can construct a sequence of dominant points arbitrarily close to  $x$ , as required.

If  $y$  is renormalizable, let  $\Omega_W = (\alpha, \omega]$  be the maximal tuning window which contains  $y$ . Then the root  $\alpha$  is non-renormalizable, and it is periodic of period smaller than the period of  $y$ , hence strictly smaller than the period of  $x$ . Moreover, by Lemma 10.7 (ii) we have  $z < \alpha < y$ , hence as before we can find a dominant point  $t$  inside  $(z, \alpha]$  and apply Lemma 10.6, thus producing dominant points which approximate  $x$ , completing the proof. □

**10.2. Proof of Theorem 1.7.** Let us now turn to the proof of equality of dimensions between  $H_c$  and  $P_c$ .

Recall that in Corollary 8.6 we established that  $P_c \subseteq H_c$ , hence we are left with proving that for all real parameters  $c \in \partial\mathcal{M} \cap \mathbb{R}$ , we have

$$(12) \quad \text{H.dim } H_c \leq \text{H.dim } P_c.$$

By Proposition 9.4, the inequality holds for the Feigenbaum point and for all  $c > c_{Feig}$ . Let us now prove that (12) holds for any  $c \in \partial\mathcal{M} \cap \mathbb{R}$  which is non-renormalizable. Indeed, by Proposition 10.1, and the fact that bi-Lipschitz maps preserve Hausdorff dimension, for each  $c' > c$  we have

$$\text{H.dim } H_{c'} \leq \text{H.dim } P_c$$

and, by continuity of entropy ([MT], see also section 12), we have  $\text{H.dim } H_{c'} \rightarrow \text{H.dim } H_c$  as  $c' \rightarrow c$ , proving (12).

Let now  $\tau$  be the tuning operator whose fixed point is the Feigenbaum point: since the root of its tuning window is the basilica map which has zero entropy, by

Proposition 9.3 we have, for each  $n \geq 0$  and each  $c \in \mathcal{M}$ ,

$$(13) \quad \text{H.dim } H_{\tau^n(c)} = \text{H.dim } \tau^n(H_c) \quad \text{H.dim } P_{\tau^n(c)} = \text{H.dim } \tau^n(P_c).$$

Now, each renormalizable parameter  $c \in \mathcal{M} \cap (-2, c_{Feig})$  is either of the form  $c = \tau^n(c_0)$  with  $c_0$  non-renormalizable, or  $c = \tau^n(\tau_W(c_0))$  with  $W$  a real hyperbolic component such that its root  $r(W)$  is outside the baby Mandelbrot set determined by the image of  $\tau$ .

- (1) In the first case we note that (since tuning operators behave well under the operation of concatenation of binary strings), by applying the operator  $\tau^n$  to both sides of the inclusion of Proposition 10.1 we get for each  $c' > c_0$  a piecewise linear map  $F_0$  such that

$$F_0(\tau^n(H_{c'})) \subseteq \tau^n(P_{c_0})$$

hence, by continuity of entropy and of tuning operators,

$$\text{H.dim } H_c = \sup_{c' > c_0} \text{H.dim } H_{\tau^n(c')} = \text{H.dim } \tau^n(H_{c'}) \leq \text{H.dim } \tau^n(P_{c_0}) = \text{H.dim } P_c.$$

- (2) In the latter case  $c = \tau^n(\tau_W(c_0))$ , by Proposition 9.3 we get

$$\tau^n(P_{\tau_W(c_0)}) = \tau^n(P_{r(W)}) \cup \tau^n(\tau_W(P_{c_0}))$$

and since the period of  $W$  is larger than 2 we have the inequalities

$$\text{H.dim } \tau^n(\tau_W(P_{c_0})) \leq \text{H.dim } \tau^{n+1}(P_{c_0}) \leq \text{H.dim } \tau^{n+1}(\mathcal{R}) \leq \tau^n(P_{r(W)})$$

where in the last inequality we used the fact that the set of rays  $\tau(\mathcal{R})$  land to the right of the root  $r(W)$ . Thus we proved that

$$\text{H.dim } \tau^n(P_{\tau_W(c_0)}) = \text{H.dim } \tau^n(P_{r(W)})$$

and the same reasoning for  $H_c$  yields

$$\text{H.dim } \tau^n(H_{\tau_W(c_0)}) = \text{H.dim } \tau^n(H_{r(W)}).$$

Finally, putting together the previous equalities with eq.(13) and applying the case (1) to  $\tau^n(r(W))$  (recall  $r(W)$  is non-renormalizable), we have the equalities

$$\begin{aligned} \text{H.dim } P_c &= \text{H.dim } \tau^n(P_{\tau_W(c_0)}) = \text{H.dim } \tau^n(P_{r(W)}) = \text{H.dim } P_{\tau^n(r(W))} = \\ &= \text{H.dim } H_{\tau^n(r(W))} = \text{H.dim } \tau^n(H_{r(W)}) = \text{H.dim } \tau^n(H_{\tau_W(c_0)}) = \text{H.dim } H_c. \end{aligned}$$

## 11. THE COMPLEX CASE

In the following sections we will develop in detail the tools needed to prove Theorem 1.3. In particular, in section 12 we prove continuity of entropy along principal veins by developing a generalization of kneading theory to tree maps. Then (section 13) we develop the combinatorial surgery map, which maps the combinatorial model of real Hubbard trees to Hubbard trees along the vein. Finally (section 13.5), we use the surgery to transfer the inclusion of Hubbard tree in parameter space of section 10 from the real vein to the other principal veins.

11.1. **Veins.** A *vein* in the Mandelbrot set is a continuous, injective arc inside  $\mathcal{M}$ . Branner and Douady [BD] showed that there exists a vein joining the parameter at angle  $\theta = 1/4$  to the main cardioid of  $\mathcal{M}$ . In his thesis, J. Riedl [Ri] showed existence of veins connecting any tip at a dyadic angle  $\theta = \frac{p}{2^q}$  to the main cardioid. Other proofs of this fact are due to J. Kahn (see [Do2], Section V.4, and [Sch2], Theorem 5.6) and C. Petersen and P. Roesch [PR]. Riedl also shows that the quasiconformal surgery preserves local connectivity of Julia sets, hence by using the local connectivity of real Julia sets [LvS] one concludes that all Julia sets of maps along the dyadic veins are locally connected ([Ri], Corollary 6.5).

Let us now see how to define veins combinatorially just in terms of laminations. Recall that the quadratic minor lamination  $QML$  is the union of all *minor leaves* of all invariant laminations corresponding to all quadratic polynomials. The degenerate leaf  $\{0\}$  is the natural *root* of  $QML$ . No other leaf of  $QML$  contains the angle 0 as its endpoint. Given a rooted lamination, we define a partial order on the set of leaves by saying that  $\ell_1 < \ell_2$  if  $\ell_1$  separates  $\ell_2$  from the root.

**Definition 11.1.** *Let  $\ell$  be a minor leaf. Then the combinatorial vein defined by  $\ell$  is the set*

$$P(\ell) := \{\ell' \in QML : \{0\} < \ell' \leq \ell\}$$

*of leaves which separate  $\ell$  from the root of the lamination.*

11.2. **Principal veins.** Let  $\frac{p}{q}$  be a rational number, with  $0 < p < q$  and  $p, q$  coprime. The  $\frac{p}{q}$ -limb in the Mandelbrot set is the set of parameters which have rotation number  $\frac{p}{q}$  around the  $\alpha$  fixed point. In each limb, there exists a unique parameter  $c = c_{p/q}$  such that the critical point maps to the  $\beta$  fixed point after exactly  $q$  steps, i.e.  $f_c^q(0) = \beta$ . For instance,  $c_{1/2} = -2$  is the Chebyshev polynomial. These parameters represent the “highest antennas” in the limbs of the Mandelbrot set. The *principal vein*  $v_{p/q}$  is the vein joining  $c_{p/q}$  to the main cardioid. We shall denote by  $\tau_{p/q}$  the external angle of the ray landing at  $c_{p/q}$  in parameter space.

**Proposition 11.2.** *Each parameter  $c \in v_{p/q}$  is topologically finite, and the Hubbard tree  $T_c$  is a  $q$ -pronged star. Moreover, the valence of any point  $x \in T_c$  is at most  $2q$ .*

*Proof.* Let  $\tau$  be the point in the Julia set of  $f_c$  where the ray at angle  $\tau_{p/q}$  lands. Since  $c \in [\alpha, \tau]$ , then  $f^{q-1}(c) \in [\alpha, \beta]$ , hence by Lemma 4.4 the extended Hubbard tree is a  $q$ -pronged star. The unique point with degree larger than 1 is the  $\alpha$  fixed point, which has degree  $q$ , so the second claim follows from Lemma 4.8.  $\square$

Note that, by using combinatorial veins, the statement of Theorem 1.3 can be given in purely combinatorial form as follows. Given a set  $\lambda$  of leaves in the unit disk, let us denote by  $\text{H.dim } \lambda$  the Hausdorff dimension of the set of endpoints of (non-degenerate) leaves of  $\lambda$ . Moreover, if the leaf  $\ell$  belongs to  $QML$  we shall denote as  $\lambda(\ell)$  the invariant quadratic lamination which has  $\ell$  as minor leaf. The statement of the theorem then becomes that, for each  $\ell \in P(\tau_{p/q})$ , the following equality holds:

$$\text{H.dim } P(\ell) = \text{H.dim } \lambda(\ell).$$

We conjecture that the same equality holds for every  $\ell \in QML$ .

11.3. **A combinatorial bifurcation measure.** The approach to the geometry of the Mandelbrot set via entropy of Hubbard trees allows one to define a transverse measure on the quadratic minor lamination  $QML$ . Let  $\ell_1 < \ell_2$  be two ordered leaves of  $QML$ , corresponding to two parameters  $c_1$  and  $c_2$ , and let  $\gamma$  be a transverse

arc connecting  $\ell_1$  and  $\ell_2$ . Then one can assign the measure of the arc  $\gamma$  to be the difference between the entropy of the two Hubbard trees:

$$\mu(\gamma) := h(f_{c_2} |_{T_{c_2}}) - h(f_{c_1} |_{T_{c_1}}).$$

By the monotonicity result of [Li], such a measure can be interpreted as a *transverse bifurcation measure*: in fact, as one crosses more and more leaves from the center of the Mandelbrot set to the periphery, i.e. as the map  $f_c$  undergoes more and more bifurcations, one picks up more and more measure. The measure can also be interpreted as the derivative of the entropy in the direction transverse to the leaves: note also that, since period doubling bifurcations do not change the entropy,  $\mu$  is non-atomic.

The dual to the lamination is an  $\mathbb{R}$ -tree (i.e., a topological space such that any two points on it are connected by a unique continuous path), and the transverse measure  $\mu$  defines a metric on such a tree. By pushing it forward to the actual Mandelbrot set, one endows the union of all veins in  $\mathcal{M}$  with the structure of a metric  $\mathbb{R}$ -tree. It would be very interesting to analyze the properties of such transverse measure, and also comparing it to the other existing notions of bifurcation measure.

In the following sections we will work out the proof of Theorem 1.3.

## 12. KNEADING THEORY FOR HUBBARD TREES

In this section we will analyze the symbolic dynamics of some continuous maps of trees, in order to compute their entropy as zeros of some power series. As a consequence, we will see that the entropy of Hubbard trees varies continuously along principal veins. Our work is a generalization to tree maps of Milnor and Thurston's *kneading theory* [MT] for interval maps. The general strategy is similar to [BdC], but our view is towards application to Hubbard trees. Moreover, since we are mostly interested in principal veins, we will treat in detail only the case of trees with particular topological types. An alternative, independent approach to continuity is in [BS].

**12.1. Counting laps and entropy.** Let  $f : T \rightarrow T$  be a continuous map of a finite tree  $T$ . We will assume  $f$  is a local homeomorphism onto its image except at one point, which we call the *critical point*. At the critical point, the map is a branched cover of degree 2. Let us moreover assume  $T$  is a *rooted tree*, i.e. it has a distinguished end  $\beta$ . The choice of a root defines a partial ordering on the tree; namely,  $x < y$  if  $x$  disconnects  $y$  from the root.

Let  $C_f$  be a finite set of points of  $T$  such that  $T \setminus C_f$  is a union of disjoint open intervals  $I_k$ , and the map  $f$  is monotone on each  $I_k$  with respect to the above-mentioned ordering. The critical point and the branch points of the tree are included in  $C_f$ .

For each subtree  $J \subseteq T$ , the *number of laps* of the restriction of  $f^n$  to  $J$  is defined as  $\ell(f^n |_J) := \#(J \cap \bigcup_{i=0}^{n-1} f^{-i}(C_f)) + \#\text{Ends}(J) - 1$ , in analogy with the real case. Denote  $\ell(f^n) := \ell(f^n |_T)$ . The *growth number*  $s$  of the map  $f : T \rightarrow T$  is the exponential growth rate of the number of laps:

$$(14) \quad s := \lim_{n \rightarrow \infty} \sqrt[n]{\ell(f^n)}.$$

**Lemma 12.1** ([ALM]). *The limit in eq. (14) exists, and it is related to the topological entropy  $h_{\text{top}}(f |_T)$  in the following way:*

$$s = e^{h_{\text{top}}(f|_T)}.$$

The proof is the same as in the analogous result of Misiurewicz and Szlenk for interval maps ([dMvS], Theorem II.7.2). In order to compute the entropy of  $f$ , let us define the generating function

$$\mathcal{L}(t) := 1 + \sum_{n=1}^{\infty} \ell(f^n) t^n$$

where  $\ell(f^n)$  is the number of laps of  $f^n$  on all  $T$ . Moreover, for  $a, b \in T$ , let us denote as  $\ell(f^n |_{[a,b]})$  the number of laps of the restriction of  $f^n$  to the interval  $[a, b]$ . Thus we can construct for each  $x \in T$  the function

$$\mathcal{L}(x, t) := 1 + \sum_{n=1}^{\infty} \ell(f^n |_{[\beta, x]}) t^n$$

and for each  $n$  we shall denote  $L_{n,x} := \ell(f^n |_{[\beta, x]})$ . Let us now relate the generating function  $\mathcal{L}$  to the kneading sequence.

Before doing so, let us introduce some notation; for  $x \notin C_f$ , the sign  $\epsilon(x) \in \{\pm 1\}$  is defined according as to whether  $f$  preserves or reverses the orientation of a neighbourhood of  $x$ . Finally, let us define

$$\eta_k(x) := \epsilon(x) \cdots \epsilon(f^{k-1}(x))$$

for  $k \geq 1$ , and  $\eta_0(x) := 1$ . Moreover, let us introduce the notation

$$\chi_k(x) := \begin{cases} 1 & \text{if } f(x) \in I_k \\ 0 & \text{if } f(x) \notin I_k \end{cases}$$

and  $\hat{\chi}_k(x) := 1 - \chi_k(x)$ .

Let us now focus on the case when  $T$  is the Hubbard tree of a quadratic polynomial along the principal vein  $v_{p/q}$ . Then we can set  $C_f := \{\alpha, 0\}$  the union of the  $\alpha$  fixed point and the critical point, so that

$$T \setminus C_f = I_0 \cup I_1 \cup \cdots \cup I_q$$

where the critical point separates  $I_0$  and  $I_1$ , and the  $\alpha$  fixed point separates  $I_1, I_2, \dots, I_q$ . The dynamics is the following:

- $f : I_k \mapsto I_{k+1}$  homeomorphically, for  $1 \leq k \leq q-1$ ;
- $f : I_q \mapsto I_0 \cup I_1$  homeomorphically;
- $f(I_0) \subseteq I_0 \cup I_1 \cup I_2$ .

We shall now write a formula to compute the entropy of  $f$  on the tree as a function of the itinerary of the critical value.

**Proposition 12.2.** *Suppose the critical point for  $f$  is not periodic. Then we have the equality*

$$\mathcal{L}(c, t) \left[ 1 - 2t\Theta_1(t) + \frac{4t^2}{1+t}\Theta_2(t) \right] = \Theta_3(t)$$

as formal power series, where

$$\Theta_1(t) := \sum_{k=0}^{\infty} \eta_k(c) \hat{\chi}_0(f^k(c)) t^k$$

$$\Theta_2(t) := \sum_{k=0}^{\infty} \eta_k(c) \chi_2(f^k(c)) t^k$$

depend only on the itinerary of the critical value  $c$ , and  $\Theta_3(t)$  is some power series with real, non-negative, bounded coefficients. (Note that, in order to deal with the prefixed case, we extend the definitions of  $\epsilon$ ,  $\hat{\chi}_0$  and  $\chi_2$  by setting  $\epsilon(\alpha) = \hat{\chi}_0(\alpha) = \chi_2(\alpha) = 1$ .)

*Proof.* We can compute the number of laps recursively. Let us suppose  $x \in T$  such that  $f^n(x) \neq 0$  for all  $n \geq 0$ . Then for  $n \geq 2$  we have the following formulas:

$$\ell(f^n |_{[\beta, x]}) = \begin{cases} \ell(f^{n-1} |_{[\beta, f(x)]}) & \text{if } x \in I_0 \cup \{0\} \\ -\ell(f^{n-1} |_{[\beta, f(x)]}) + 2\ell(f^{n-1} |_{[\beta, c]}) + 1 & \text{if } x \in I_1 \\ \ell(f^{n-1} |_{[\beta, f(x)]}) + 2\ell(f^{n-1} |_{[\beta, c]}) - 2\ell(f^{n-1} |_{[\beta, \alpha]}) & \text{if } x \in I_2 \cup \dots \cup I_{q-1} \cup \{\alpha\} \\ -\ell(f^{n-1} |_{[\beta, f(x)]}) + 2\ell(f^{n-1} |_{[\beta, c]}) + 1 & \text{if } x \in I_q \end{cases}$$

Now, recalling the notation  $L_{n,x} := \ell(f^n |_{[\beta, x]})$ , the previous formula can be rewritten as

$$L_{n,x} = \epsilon(x)L_{n-1, f(x)} + 2\hat{\chi}_0(x)L_{n-1, c} - 2\chi_2(x)L_{n-1, \alpha} + \frac{1 - \epsilon(x)}{2}.$$

Moreover, for  $n = 1$  we have

$$L_{1,x} = \epsilon(x) + 2\hat{\chi}_0(x) + \frac{1 - \epsilon(x)}{2} + R(x)$$

where

$$R(x) := \begin{cases} 1 & \text{if } x \in I_q \\ -1 & \text{if } x = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Hence by multiplying every term by  $t^n$  and summing up we get

$$\mathcal{L}(x, t) = t\epsilon(x)\mathcal{L}(f(x), t) + 2t\hat{\chi}_0(x)\mathcal{L}(c, t) - 2t\chi_2(x)\tilde{\mathcal{L}}(\alpha, t) + S(x, t)$$

with  $S(x, t) := \frac{1 - \epsilon(x)}{2} \frac{t}{1-t} + tR(x) + 1$ . If we now apply the formula to  $f^k(x)$  and multiply everything by  $\eta_k(x)t^k$  we have for each  $k \geq 0$

$$\begin{aligned} & \eta_k(x)t^k\mathcal{L}(f^k(x), t) - \eta_k(x)\epsilon(f^k(x))t^{k+1}\mathcal{L}(f^{k+1}(x), t) = \\ & = 2t^{k+1}\eta_k(x)\hat{\chi}_0(f^k(x))\mathcal{L}(c, t) - 2t^{k+1}\eta_k(x)\chi_2(f^k(x))\tilde{\mathcal{L}}(\alpha, t) + \eta_k(x)t^kS(f^k(x), t) \end{aligned}$$

so, by summing over all  $k \geq 0$ , the left hand side is a telescopic series and we are left with

$$(15) \quad \mathcal{L}(x, t) = 2t\Theta_1(x, t)\mathcal{L}(c, t) - 2t\Theta_2(x, t)\tilde{\mathcal{L}}(\alpha, t) + \Theta_3(x, t)$$

where we used the notation  $\tilde{\mathcal{L}}(x, t) := \sum_{n=1}^{\infty} \ell(f^n |_{[\beta, x]})t^n$  and

$$\Theta_3(x, t) := \sum_{k=0}^{\infty} \eta_k(x)S(f^k(x), t)t^k = 1 + \sum_{k=1}^{\infty} \frac{1 + \eta_{k-1}(x)(\epsilon(f^{k-1}(x)) + 2R(f^{k-1}(x)))}{2} t^k$$

is a power series whose coefficients are all real and lie between 0 and 1. The claim now follows by plugging in the value  $x = c$  in eq. (15), and using Lemma 12.3 to write  $\tilde{\mathcal{L}}(\alpha, t)$  in terms of  $\mathcal{L}(c, t)$ .  $\square$

**Lemma 12.3.** *We have the following equalities of formal power series:*

(1)

$$\tilde{\mathcal{L}}(\alpha, t) = \frac{2t\mathcal{L}(c, t)}{1+t}$$

(2)

$$\mathcal{L}(t)t^{q-1} = \frac{(1-t^q)\mathcal{L}(c, t)}{1+t} + P(t)$$

where  $P(t)$  is a polynomial.

*Proof.* (1) We can compute  $\ell(f^n |_{[\beta, \alpha]})$  recursively, since we have for  $n \geq 2$

$$\ell(f^n |_{[\beta, \alpha]}) = 2\ell(f^{n-1} |_{[\beta, c]}) - \ell(f^{n-1} |_{[\beta, \alpha]})$$

while  $\ell(f |_{[\beta, \alpha]}) = 2$ , hence by multiplying each side by  $t^n$  and summing over  $n$  we get

$$\tilde{\mathcal{L}}(\alpha, t) = 2t\mathcal{L}(c, t) - t\tilde{\mathcal{L}}(\alpha, t)$$

and the claim holds.

(2) If we let  $\mathcal{L}_{[\alpha, c]}(t) := 1 + \sum_{n=1}^{\infty} \ell(f^n |_{[\alpha, c]})t^n$ , we have by (1) that

$$\mathcal{L}_{[\alpha, c]}(t) = \frac{(1-t)\mathcal{L}(c, t)}{1+t}.$$

Now, since the Hubbard tree can be written as the union  $T = \bigcup_{i=0}^{q-1} [\alpha, f^i(c)]$ , for each  $n \geq 1$  we have

$$\ell(f^n |_T) = \sum_{i=0}^{q-1} \ell(f^n |_{[\alpha, f^i(c)]}) = \sum_{i=0}^{q-1} \ell(f^{n+i} |_{[\alpha, c]})$$

hence multiplying both sides by  $t^{n+q-1}$  and summing over  $n$  we get

$$\mathcal{L}(t)t^{q-1} = (1+t+\dots+t^{q-1})\mathcal{L}_{[\alpha, c]}(t) + P(t)$$

for some polynomial  $P(t)$ . The claim follows by substituting  $\mathcal{L}_{[\alpha, c]}(t)$  using (1).  $\square$

**Proposition 12.4.** *Let  $s$  be the growth number of the tree map  $f : T \rightarrow T$ . If  $s > 1$ , then the smallest positive, real zero of the function*

$$\Delta(t) := 1 + t - 2t(1+t)\Theta_1(t) + 4t^2\Theta_2(t)$$

*lies at  $t = \frac{1}{s}$ . If  $s = 1$ , then  $\Delta(t)$  has no zeros inside the interval  $(0, 1)$ .*

*Proof.* Recall  $s := \lim_{n \rightarrow \infty} \sqrt[n]{\ell(f^n)}$ , so the convergence radius of the series  $\mathcal{L}(t)$  is precisely  $r = \frac{1}{s}$ . By Proposition 12.2,

$$\mathcal{L}(c, t) = \frac{\Theta_3(t)(1+t)}{\Delta(t)}$$

can be continued to a meromorphic function in the unit disk, and by Lemma 12.3, also  $\mathcal{L}(t)$  can be continued to a meromorphic function in the unit disk, and the set of poles of the two functions inside the unit disk coincide (note both power series expansions begin with 1, hence they do not vanish at 0).

Let us now assume  $s > 1$ . Then  $\mathcal{L}(c, t)$  must have a pole on the circle  $|t| = \frac{1}{s}$ , and since the coefficients of its power series are all positive, it must have a pole on the positive real axis. This implies  $\Delta(1/s) = 0$ . Moreover, since  $\Theta_3(t)$  has real non-negative coefficients, it cannot vanish on the positive real axis, hence  $\Delta(t) \neq 0$  for  $0 < t < 1/s$ .

If instead  $s = 1$ ,  $\mathcal{L}(c, t)$  is holomorphic on the disk, so for the same reason  $\Delta(t)$  cannot vanish inside the interval  $(0, 1)$ .  $\square$

## 12.2. Continuity of entropy along veins.

**Theorem 12.5.** *Let  $v = v_{p/q}$  be the principal vein in the  $p/q$ -limb of the Mandelbrot set. Then the entropy  $h_{\text{top}}(f_c |_{T_c})$  of  $f_c$  restricted to its Hubbard tree depends continuously, as  $c$  moves along the vein, on the angle of the external ray landing at  $c$ .*

*Proof.* Let  $\ell \in P(\tau_{p/q})$  be the minor leaf associated to the parameter  $c \in \partial\mathcal{M}$ ,  $\ell = (\theta^-, \theta^+)$ . Since the entropy does not change under period doubling, we may assume that  $c$  is not the period doubling of some other parameter along the vein; thus, there exist  $\{\ell_n\}_{n \geq 1} \subseteq P(\tau_{p/q})$  a sequence of leaves of  $QML$  which tends to  $\ell$ .

Since  $c \in \partial\mathcal{M}$ , the orbit  $f_c^n(0)$  never goes back to 0, so we can apply Propositions 12.2 and 12.4. Thus we can write

$$(16) \quad \mathcal{L}(c, t) = \frac{F(t)}{\Delta(t)}$$

and the entropy  $h_{top}(f_c|_{T_c})$  is then  $\log s$ , where  $1/s$  is the smallest real positive root of  $\Delta(t)$ . Finally note that both  $F(t)$  and  $\mathcal{L}(c, t)$  have real non-negative coefficients, and do not vanish at  $t = 0$ . The coefficients of  $\Delta(t)$  and  $F(t)$  depend on the coefficients of  $\Theta_1(t)$ ,  $\Theta_2(t)$  and  $\Theta_3(t)$ , which in turn depend only on the itinerary of the angle  $\theta^-$  with respect to the doubling map  $D$  and the partition given by the complement, in the unit circle, of the set

$$\{\theta_1, \dots, \theta_q, \tau_{p/q}, \tau_{p/q} + 1/2\}$$

where  $\theta_1, \dots, \theta_q$  are the angles of rays landing on the  $\alpha$  fixed point. Let  $\Delta_n(t), F_n(t)$  denote the functions  $\Delta(t), F(t)$  of equation (16) relative to the parameter corresponding to the leaf  $\ell_n$ . If  $f_c^n(0) \neq \alpha$  for all  $n \geq 0$ , then  $D^n(\theta^-)$  always lies in the interior of the partition, so if  $\theta_n^-$  is sufficiently close to  $\theta^-$ , its itinerary will share a large initial part with the itinerary of  $\theta^-$ , hence the power series for  $\Delta(t)$  and  $\Delta_n(t)$  share arbitrarily many initial coefficients and their coefficients are uniformly bounded, so  $\Delta_n(t)$  converges uniformly on compact subsets of the disk to  $\Delta(t)$ , and similarly  $F_n(t) \rightarrow F(t)$ . Let us now suppose, possibly after passing to a subsequence, that  $s_n^{-1} \rightarrow s_*^{-1}$ . Then by uniform convergence on compact subsets of  $\mathbb{D}$ ,  $s_*^{-1}$  is either 1 or a real, non-negative root of  $\Delta(t)$ , so in either case

$$\liminf_{n \rightarrow \infty} s_n^{-1} \geq s^{-1}.$$

Now, if we have  $s_*^{-1} < s^{-1}$ , then by Rouché's theorem  $\Delta_n$  must have a non-real zero  $z_n$  inside the disk of radius  $s_n^{-1}$  with  $z_n \rightarrow s_*^{-1}$ , hence by definition of  $s_n$  and equation (16) one also has  $F_n(z_n) = 0$ , but since  $F$  has real coefficients then also its conjugate  $\bar{z}_n$  is a zero of  $F_n$ , hence in the limit  $s_*^{-1}$  is a real, non-negative zero of  $F$  with multiplicity two, but this is a contradiction because the derivative  $F'(t)$  also has real, non-negative coefficients so it does not vanish on the interval  $[0, 1)$ . This proves the claim

$$\lim_{n \rightarrow \infty} s_n^{-1} = s^{-1}$$

and continuity of entropy follows.

Things get a bit more complicated when some iterate  $f_c^n(0)$  maps to the  $\alpha$  fixed point. In this case, the iterates of  $\theta$  under the doubling map hit the boundary of the partition, hence its itinerary is no longer stable under perturbation. However, a simple check proves that even in this case the coefficients for the function  $\Delta_n(t)$  still converge to the coefficients of  $\Delta(t)$ . Indeed, if  $n$  is the smallest step  $k$  such that  $f_c^k(c) = \alpha$ , then for each  $k \geq n$  we have  $\epsilon(f_c^k(c)) = \hat{\chi}_0(f_c^k(c)) = \chi_2(f_c^k(c)) = 1$ . On the other hand, as  $\theta_n^-$  tends to  $\theta^-$ , the itinerary of the critical value with respect to the partition  $I_0 \cup I_1 \cup \dots \cup I_q$  approaches a preperiodic cycle of period  $q$ , where the period is either  $(\bar{I}_2, \bar{I}_2, \dots, \bar{I}_2, \bar{I}_3, \bar{I}_1)$  or  $(\bar{I}_1, \bar{I}_2, \bar{I}_2, \dots, \bar{I}_2, \bar{I}_3)$ . In both cases one can check by explicit computation that the coefficients in the power series expansion of  $\Delta_n(t)$  converge to the coefficients of  $\Delta(t)$ .  $\square$

### 13. COMBINATORIAL SURGERY

The goal of this section is to transfer the result about the real line to the principal veins  $v_{p/q}$ ; in order to do so, we will define a surgery map (inspired by the construction of Branner-Douady [BD] for the 1/3-limb) which carries the combinatorial principal vein in the real limb to the combinatorial principal vein in the  $p/q$ -limb.

**13.1. Orbit portraits.** Let  $0 < p < q$ , with  $p, q$  coprime. There exists a unique set  $C_{p/q}$  of  $q$  points on the unit circle which is invariant for the doubling map  $D$  and such that the restriction of  $D$  on  $C_{p/q}$  preserves the cyclic order of the elements and acts as a rotation of angle  $p/q$ . That is  $C_{p/q} = \{x_1, \dots, x_q\}$ , where  $0 \leq x_1 < x_2 < \dots < x_q < 1$  are such that  $D(x_i) = x_{i+p}$  (where the indices are computed mod  $q$ ).

The  $p/q$ -limb in the Mandelbrot set is the set of parameters  $c$  for which the set of angles of rays landing on the  $\alpha$  fixed point in the dynamical plane for  $f_c$  is precisely  $C_{p/q}$  (for a reference, see [Mi]). In Milnor's terminology, the set  $C_{p/q}$  is an *orbit portrait*: we shall call it the  $\alpha$  *portrait*.

Given  $p/q$ , there are exactly two rays landing on the intersection of the  $p/q$ -limb with the main cardioid: let us denote these two rays as  $\theta_0$  and  $\theta_1$ . The angle  $\theta_0$  can be found by computing the symbolic coding of the point  $p/q$  with respect to the rotation of angle  $p/q$  on the circle and using the following partition:

$$A_0 := \left(0, 1 - \frac{p}{q}\right] \quad A_1 := \left(1 - \frac{p}{q}, 1\right).$$

For instance, if  $p/q = 2/5$ , we have that the orbit is  $(2/5, 4/5, 1/5, 3/5, 0)$ , hence the itinerary is  $(0, 1, 0, 0, 1)$  and the angle is  $\theta_0 = 0.\overline{01001} = 9/31$ . The other angle  $\theta_1$  is obtained by the same algorithm but using the partition:

$$A_0 := \left[0, 1 - \frac{p}{q}\right) \quad A_1 := \left[1 - \frac{p}{q}, 1\right)$$

(hence if  $p/q = 2/5$ , we have the itinerary  $(0, 1, 0, 1, 0)$  and  $\theta_1 = 0.\overline{01010} = 10/31$ .) Let us denote as  $\Sigma_0$  the first  $q - 1$  binary digits of the expansion of  $\theta_0$ , and  $\Sigma_1$  the first  $q - 1$  digits of the expansion of  $\theta_1$ .

**13.2. The surgery map.** Branner and Douady [BD] constructed a continuous embedding of the  $1/2$ -limb of the Mandelbrot set into the  $1/3$ -limb, by surgery in the dynamical plane. The image of the real line under this surgery map is a continuous arc inside the Mandelbrot set, joining the parameter at angle  $\theta = 1/4$  with the cusp of  $\mathcal{M}$ . Let us now describe, for each  $p/q$ -limb, the surgery map on a combinatorial level.

In order to construct the surgery map, let us first define the following coding for external angles: for each  $\theta \neq \frac{1}{3}, \frac{2}{3}$ , we set

$$A_{p/q}(\theta) := \begin{cases} 0 & \text{if } 0 \leq \theta < \frac{1}{3} \\ \Sigma_0 & \text{if } \frac{1}{3} < \theta < \frac{1}{2} \\ \Sigma_1 & \text{if } \frac{1}{2} \leq \theta < \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} < \theta < 1. \end{cases}$$

Then we can define the following map on the set of external angles:

**Definition 13.1.** *Let  $0 < p < q$ , with  $p, q$  coprime. The combinatorial surgery map  $\Psi_{p/q} : S^1 \rightarrow S^1$  is defined on the set of external angles as follows.*

- *If  $\theta$  does not land on a preimage of the  $\alpha$  fixed point (i.e.  $D^k(\theta) \neq \frac{1}{3}, \frac{2}{3}$  for all  $k \geq 0$ ), we define  $\Psi_{p/q}(\theta)$  as the number with binary expansion*

$$\Psi_{p/q}(\theta) := 0.s_1 s_2 s_3 \dots \quad \text{with } s_k := A_{p/q}(D^k(\theta)).$$

- *Otherwise, let  $h$  be the smallest integer such that  $D^h(\theta) \in \{\frac{1}{3}, \frac{2}{3}\}$ . Then we define*

$$\Psi_{p/q}(\theta) := 0.s_1 s_2 \dots s_{h-1} s_h$$

$$\text{with } s_k := A_{p/q}(D^k(\theta)) \text{ for } k < h \text{ and } s_h := \begin{cases} \overline{\Sigma_0 1} & \text{if } D^h(\theta) = \frac{1}{3} \\ \overline{\Sigma_1 0} & \text{if } D^h(\theta) = \frac{2}{3}. \end{cases}$$

Intuitively, the surgery takes the Hubbard tree of a real map, which is a segment, breaks it into two parts  $[c, \alpha]$  and  $[\alpha, f(c)]$  and maps them to two different branches of a  $q$ -pronged star (see Figure 11).

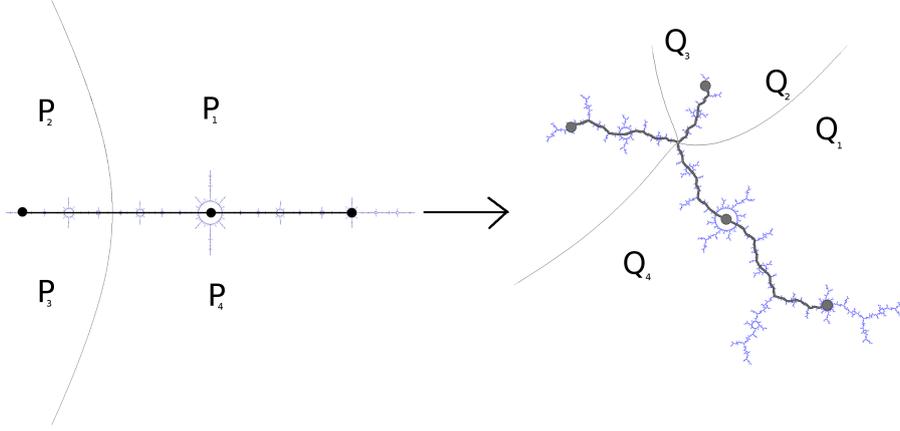


FIGURE 11. The surgery map  $\Psi_{1/3}$ . The original tree (left) is a segment, which gets “broken” at the  $\alpha$  fixed point and a new branch is added so as to form a tripod (right). External rays belonging to the sectors  $P_1, P_2, P_3, P_4$  are mapped to sectors  $Q_1, Q_2, Q_3, Q_4$  respectively.

The image of  $1/2$  under  $\Psi_{p/q}$  is the external angle of the “tip of the highest antenna” inside the  $p/q$ -limb and is denoted as  $\tau_{p/q} := \Psi_{p/q}(1/2) = 0.\Sigma_1$ .

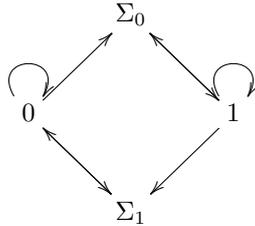
Let us now fix a rotation number  $p/q$  and denote the surgery map  $\Psi_{p/q}$  simply as  $\Psi$ .

**Lemma 13.2.** *The map  $\Psi$  is strictly increasing (hence injective), in the sense that if  $0 \leq \theta < \theta' < 1$ , then  $0 \leq \Psi(\theta) < \Psi(\theta') < 1$ .*

*Proof.* Let us consider the partitions  $P_1 := [0, 1/3)$ ,  $P_2 := [1/3, 1/2)$ ,  $P_3 := [1/2, 2/3)$ ,  $P_4 := [2/3, 1)$  and  $Q_1 := [0, 0.\overline{0\Sigma_1})$ ,  $Q_2 := [0.\overline{\Sigma_0 1}, 0.\Sigma_1)$ ,  $Q_3 := [0.\Sigma_1, 0.\overline{\Sigma_1 0})$ ,  $Q_4 := [0.\overline{1\Sigma_0}, 1)$ . It is elementary (even though a bit tedious) to check that the map  $\Psi$  respects the partitions, in the sense that  $\Psi(P_i) \subseteq Q_i$  for each  $i = 1, 2, 3, 4$ . Indeed, we know

$$\begin{aligned} D(P_1) &\subseteq P_1 \cup P_2 \cup P_3 \\ D(P_2) &= P_4 \\ D(P_3) &= P_1 \\ D(P_4) &\subseteq P_2 \cup P_3 \cup P_4 \end{aligned}$$

so the binary expansion of any element  $\Psi(\theta)$  is represented by an infinite path in the graph



Let us now check for instance that  $\Psi(P_1) \subseteq Q_1$ . Indeed, if  $\theta \in P_1$  then in the above graph the coding of  $\varphi(\theta)$  starts from 0 and hence by looking at the graph can be

either of the form

$$\Psi(\theta) = 0.(0\Sigma_1)^k 0^n \Sigma_0 \cdots < 0.\overline{0\Sigma_1} \quad k \geq 0, n \geq 1$$

or

$$\Psi(\theta) = 0.(0\Sigma_1)^k 0^n \Sigma_1 \cdots < 0.\overline{0\Sigma_1} \quad k \geq 0, n \geq 2$$

so in both cases  $0 \leq \Psi(\theta) < 0.\overline{0\Sigma}$  and the claim is proven.

Then, given  $0 \leq \theta < \theta' < 1$ , let  $k$  the smallest integer such that  $D^k(\theta)$  and  $D^k(\theta')$  lie in two different elements of the partition  $\bigcup_i P_i$ . Since the map  $D^k$  is increasing and the preimage of 0 lies on the boundary of the partition, we have  $D^k(\theta) \in P_i$  and  $D^k(\theta') \in P_j$  with  $i < j$ , so  $\Psi(D^k(\theta)) < \Psi(D^k(\theta'))$  because the first one belongs to  $Q_i$  and the second one to  $Q_j$ , hence we have

$$\Psi(\theta) = 0.s_1 s_2 \dots s_{k-1} \Psi(D^k(\theta)) < 0.s_1 s_2 \dots s_{k-1} \Psi(D^k(\theta')) = \Psi(\theta').$$

□

We can also define the map  $\Psi$  on the set of real leaves by defining the image of a leaf to be the leaf joining the two images (if  $\ell = (\theta_1, \theta_2)$ , we set  $\Psi(\ell) := (\Psi(\theta_1), \Psi(\theta_2))$ ). The previous lemma implies monotonicity on the set of leaves:

**Lemma 13.3.** *The surgery map  $\Psi = \Psi_{p/q}$  is strictly increasing on the set of leaves. Indeed, if  $\{0\} \leq \ell_1 < \ell_2 \leq \{1/2\}$ , then  $\{0\} \leq \Psi(\ell_1) < \Psi(\ell_2) \leq \{\tau_{p/q}\}$ .*

Let us now denote by  $\Theta_0 := 0.1\overline{\Sigma_0}$  and  $\Theta_1 := 0.0\overline{\Sigma_1}$  the two preimages of  $\theta_0$  and  $\theta_1$  which lie in the portrait  $C_{p/q}$ . Note that  $D(\Theta_i) = \theta_i$  for  $i = 0, 1$ .

**13.3. Forbidden intervals.** The leaves  $(\theta_0, \theta_1)$  and  $(\Theta_0, \Theta_1)$  divide the circle in three parts. Let us denote by  $\Delta_0$  the part containing 0, and as  $\Delta_1$  the part containing  $\tau_{p/q}$ . Moreover, for  $2 \leq i \leq q-1$ , let us denote  $\Delta_i := D^{i-1}(\Delta_1)$ . With this choice, the intervals  $\Delta_0, \Delta_1, \dots, \Delta_{q-1}$  are the connected components of the complement of the  $\alpha$  portrait  $C_{p/q}$ .

Let us also denote by  $\hat{C}_{p/q} := C_{p/q} + \frac{1}{2}$  the set of angles of rays landing on the preimage of the  $\alpha$  fixed point, and  $\hat{\Delta}_i := \Delta_i + \frac{1}{2}$  for  $0 \leq i \leq q-1$ , so that  $\hat{\Delta}_0, \hat{\Delta}_1, \dots, \hat{\Delta}_{q-1}$  are the connected components of the complement of  $\hat{C}_{p/q}$ .

The *forbidden interval*  $I_{p/q}$  is then defined as

$$I_{p/q} := \bigcup_{i=1}^{q-2} \hat{\Delta}_i.$$

The name ‘‘forbidden interval’’ arises from the fact that this interval is avoided by the trajectory of an angle landing on the Hubbard tree of some parameter on the vein  $v_{p/q}$ . Indeed, the following characterization is true:

**Proposition 13.4.** *Let  $\ell \in P(\tau_{p/q})$  be the characteristic leaf of a parameter  $c$  on the principal vein  $v_{p/q}$ , with  $\ell = (\theta^-, \theta^+)$ , and let  $J := (D^{q-1}(\theta^-), D^{q-1}(\theta^+))$  the interval delimited by  $D^{q-1}(\ell)$  and containing 0. Then the set of rays landing on the Hubbard tree of  $c$  is characterized as*

$$H_c := \{\theta \in S^1 : D^n(\theta) \notin I_{p/q} \cup J \quad \forall n \geq 0\}.$$

*Proof.* It follows from the description of  $H_c$  in Proposition 8.3 together with the fact that the Hubbard tree is a  $q$ -pronged star. □

The explicit characterization also immediately implies that the sets  $H_c$  are increasing along principal veins:

**Proposition 13.5.** *Let  $\ell < \ell'$  be the characteristic leaves of parameters  $c, c'$  which belong to the principal vein  $v_{p/q}$ .*

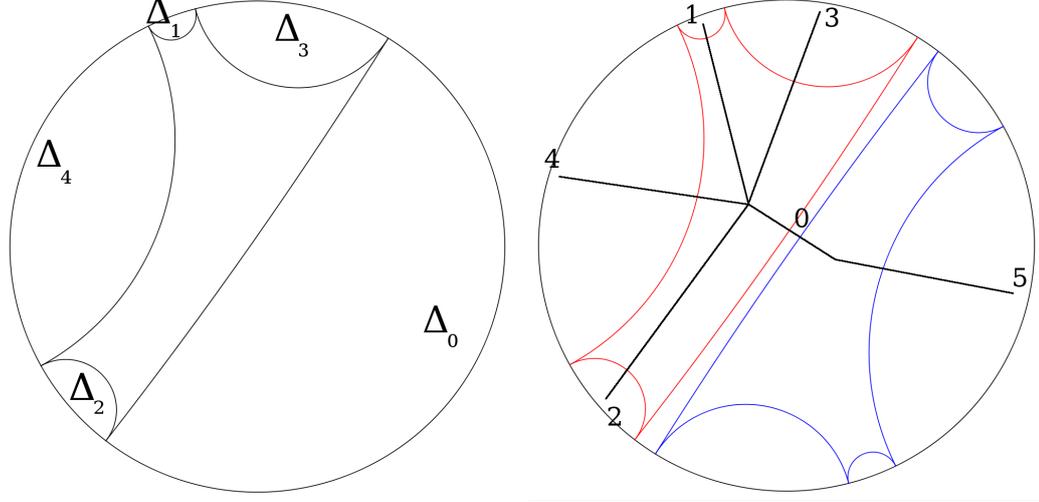


FIGURE 12. Left: the  $\alpha$  fixed portrait  $C_{p/q}$  when  $p/q = 2/5$ , with the complementary intervals  $\Delta_i$ . Right: The portraits  $C_{p/q}$  and  $\hat{C}_{p/q}$ , with the Hubbard tree drawn as dual to the lamination. The numbers indicate the position of the iterates of the critical value.

(1) Then we have the inclusion

$$H_c \subseteq H_{c'};$$

(2) if  $T_c$  and  $T_{c'}$  are the respective Hubbard trees, we have

$$h_{top}(f_c |_{T_c}) \leq h_{top}(f_{c'} |_{T_{c'}}).$$

*Proof.* (1) Let  $J$  be the interval containing 0 delimited by  $D^{q-1}(\ell)$ , and  $J'$  the interval delimited by  $D^{q-1}(\ell')$ . Since  $\ell < \ell' < \{\tau_{p/q}\}$ , one has  $\{0\} < D^{q-1}(\ell') < D^{q-1}(\ell)$ , so  $J' \subseteq J$ . If  $\theta \in H_c$ , then by Proposition 13.4 its orbit avoids  $I_{p/q} \cup J$ , hence it also avoids  $I_{p/q} \cup J'$  so it must belong to  $H_{c'}$ .

(2) From (1) and Theorem 7.1,

$$h_{top}(f_c |_{T_c}) = \text{H.dim } H_c \cdot \log 2 \leq \text{H.dim } H_{c'} \cdot \log 2 = h_{top}(f_{c'} |_{T_{c'}}).$$

□

Monotonicity of entropy along arbitrary veins is proven, for postcritically finite parameters, in Li Tao's thesis [Li].

**13.4. Surgery in the dynamical and parameter planes.** The usefulness of the surgery map comes from the fact that it maps the real vein in parameter space to the other principal veins, and also the Hubbard trees of parameters along the real vein to Hubbard trees along the principal veins. As we will see in this subsection, the correspondence is almost bijective.

Let  $Z$  denote the set of angles which never map to the endpoints of fixed leaf  $\ell_0 = (1/3, 2/3)$ :

$$Z := \{\theta \in S^1 : D^n(\theta) \neq 1/3, 2/3 \ \forall n \geq 0\}.$$

Moreover, we denote by  $\Omega$  the set of angles which never map to either the forbidden interval  $I_{p/q}$  or the  $\alpha$  portrait  $C_{p/q}$ :

$$\Omega := \{\theta \in \Delta_0 \cup \Delta_1 : D^n(\theta) \notin I_{p/q} \cup C_{p/q} \ \forall n \geq 0\}.$$

It is easy to check the following

**Lemma 13.6.** *The map  $\Psi$  is continuous on  $Z$ , and the image  $\Psi(Z)$  is contained in  $\Omega$ . Given  $\theta \in \Omega$ , let  $0 = n_0 < n_1 < n_2 < \dots$  be the return times of  $\theta$  to  $\Delta_0 \cup \Delta_1$ . Then the map*

$$\Phi(\theta) := 0.s_0s_1s_2\dots \quad \text{with } s_k = \begin{cases} 0 & \text{if } D^{n_k}(\theta) \in [0, \Theta_1) \cup (\theta_0, \tau_{p/q}) \\ 1 & \text{if } D^{n_k}(\theta) \in [\tau_{p/q}, \theta_1) \cup (\Theta_0, 1) \end{cases}$$

*defined on  $\Omega$  is an inverse of  $\Psi$ , in the sense that  $\Phi \circ \Psi(\theta) = \theta$  for all  $\theta \in Z$ .*

**Proposition 13.7.** *The surgery map  $\Psi = \Psi_{p/q}$  maps the real combinatorial vein bijectively onto the principal combinatorial vein  $P(\tau_{p/q})$  in the  $p/q$ -limb, up to a countable set of prefixed parameters; indeed, one has the inclusions*

$$P(\tau_{p/q}) \setminus \bigcup_{n \geq 0} D^{-n}(C_{p/q}) \subseteq \Psi(P(1/2)) \subseteq P(\tau_{p/q}).$$

*Proof.* Let  $m \in P(1/2)$  be a minor leaf, and  $M_1, M_2$  its major leaves. By the criterion of Proposition 3.3, all the elements of the forward orbit of  $m$  have disjoint interior, and their interior is also disjoint from  $m, M_1$  and  $M_2$ , so the set of leaves  $\{D^n(m) : n \geq 0\} \cup \{M_1, M_2\}$  (which may be finite or infinite) is totally ordered, and they all lie between  $\{0\}$  and  $\{1/2\}$ . Indeed, they are all smaller than  $m$ , which is also the shortest leaf of the set. Now, by Lemma 13.3, the set

$$\{\Psi(D^n(m)) : n \geq 0\} \cup \{\Psi(M_1), \Psi(M_2)\}$$

is also totally ordered, and all its elements have disjoint interiors and lie between  $\{0\}$  and  $\Psi(m)$ . Note that all leaves smaller than  $\ell_0 := (1/3, 2/3)$  map under  $\Psi$  to leaves smaller than  $(\Theta_0, \Theta_1)$ , and all leaves larger than  $\ell_0$  map to leaves larger than  $\Psi(\ell_0) = (\theta_0, \theta_1)$ . Note moreover that if a leaf  $\mathcal{L}$  is larger than  $(\theta_0, \theta_1)$ , then its length increases under the first  $q - 1$  iterates (i.e. until it comes back to  $\Delta_0$ ):

$$\ell(D^k(\mathcal{L})) = 2^k \ell(\mathcal{L}) \quad 0 \leq k \leq q - 1.$$

As a consequence, the shortest leaf in the set

$$S := \{D^n(\Psi(m)) : n \geq 0\} \cup \{\Psi(M_1), \Psi(M_2)\}$$

is  $\Psi(m)$ , and its images all have disjoint interiors, hence by Proposition 3.3 we have that  $\Psi(m)$  belongs to  $QML$ , and it is smaller than  $\tau_{p/q}$  by monotonicity of  $\Psi$ . Conversely, any leaf  $\ell$  of  $P(\tau_{p/q})$  whose endpoints never map to the fixed orbit portrait  $C_{p/q}$  belongs to  $\Omega$ , hence  $\Psi(\ell)$  is well-defined and, since  $\Psi$  preserves the ordering, it belongs to  $P(1/2)$  by Proposition 3.3.  $\square$

**Proposition 13.8.** *Let  $c \in [-2, 1/4]$  be a real parameter, with characteristic leaf  $\ell$ , and let  $c'$  be a parameter with characteristic leaf  $\ell' = \Psi(\ell)$ . Moreover, let us set  $\tilde{H}_{c'} := H_{c'} \cap (\Delta_0 \cup \Delta_1) \setminus \bigcup_n D^{-n}(C_{p/q})$ . Then the inclusions*

$$\tilde{H}_{c'} \subseteq \Psi(H_c) \subseteq H_{c'}$$

*hold. As a consequence,*

$$\text{H.dim } \Psi(H_c) = \text{H.dim } H_{c'}.$$

*Proof.* Let  $\theta \in H_c$  and  $\ell := (\theta, 1 - \theta)$  be its associated real leaf and let  $\ell_c$  the postcharacteristic leaf for  $f_c$ . Let us first assume  $D^n(\theta) \neq 1/3, 2/3$  for all  $n$ . Then by Lemma 13.6  $\Psi(\theta)$  lies in  $\Omega$ , so its orbit always avoids  $I_{p/q}$ . Moreover, by Proposition 8.3

$$D^n(\ell) \geq \ell_c \quad \text{for all } n \geq 0.$$

Then, by monotonicity of the surgery map (Lemma 13.3)

$$\Psi(D^n(\ell)) \geq \Psi(\ell_c) \quad \text{for all } n \geq 0.$$

Moreover, given  $N \geq 0$  either

$$D^N(\Psi(\ell)) \notin \Delta_0 \cup \Delta_1$$

or one can write

$$D^N(\Psi(\ell)) = \Psi(D^n(\ell))$$

for some integer  $n$ , so the orbit of  $\Psi(\theta)$  always avoids the interval delimited by the leaf  $\Psi(\ell_c)$ , hence by Proposition 8.3 we have  $\Psi(\theta) \in H_{c'}$ . The case when  $D^n(\theta)$  hits  $\{1/3, 2/3\}$  is analogous, except that the leaf  $\ell$  is eventually mapped to the leaf  $(\theta_0, \theta_1)$  which belongs to the  $\alpha$  portrait.

Conversely, let  $\theta' \in \tilde{H}_{c'}$  and  $\ell'$  be its corresponding leaf. Then by Proposition 13.4 it never maps to  $I_{p/q}$ , so by Lemma 13.6 there exists  $\theta \in Z$  such that  $\theta' = \Psi(\theta)$ . Let  $\ell := (\theta, 1 - \theta)$  be its corresponding real leaf. Moreover, also by Proposition 13.4 all iterates of  $\ell'$  are larger than  $\Psi(\ell_c)$ , so by monotonicity of the surgery map all iterates of  $\ell$  are larger than  $\ell_c$ , so, by Proposition 8.3,  $\theta$  lies in  $H_c$ . The equality of dimensions arises from the fact that for  $2 \leq i \leq q - 1$  one has

$$H_{c'} \cap \Delta_i = D^{q-1}(H_c \cap \Delta_1)$$

and the doubling map preserves Hausdorff dimension.  $\square$

Finally, we need to check that the surgery map behaves well under renormalization. Indeed we have the

**Lemma 13.9.** *Let  $W$  be a real hyperbolic component, and  $\Psi$  the surgery map. Then for each  $\theta \in \mathcal{R}$ ,*

$$\Psi(\tau_W(\theta)) = \tau_{\Psi(W)}(\theta)$$

where  $\Psi(W)$  is the hyperbolic component whose endpoints are the images via surgery of the endpoints of  $W$ .

*Proof.* Let  $\theta = 0.\theta_1\theta_2\dots$  be the binary expansion of  $\theta$ . Denote as  $\theta^- = 0.\overline{S_0}$ ,  $\theta^+ = 0.\overline{S_1}$  the angles of parameter rays landing at the root of  $W$ , and as  $\Theta^- := \Psi(\theta^-) = 0.\overline{T_0}$  and  $\Theta^+ := \Psi(\theta^+) = 0.\overline{T_1}$  the angles landing at the root of  $\Psi(W)$ . Finally, let  $p := |S_0|$  denote the the period of  $W$ . Then  $\tau_W(\theta)$  has binary expansion

$$\tau_W(\theta) = 0.S_{\theta_1}S_{\theta_2}\dots$$

By using the fact that either  $\theta^- \leq \theta^+ < 1/3$  or  $2/3 < \theta^- \leq \theta^+$ , one checks that for each  $0 \leq k < p$ , the points

$$D^k(0.S_{\theta_1}S_{\theta_2}\dots)$$

and

$$D^k(0.\overline{S_{\theta_1}})$$

lie in the same element of the partition  $\bigcup_{i=1}^4 P_i$ . As a consequence, by definition of the surgery map  $\Psi$ , we get that

$$\Psi(\tau_W(\theta)) = 0.T_{\theta_1}T_{\theta_2}\dots$$

and the claim follows.  $\square$

### 13.5. Proof of Theorem 1.3.

**Definition 13.10.** *The set  $\mathcal{D}_{p/q}$  of dominant parameters along  $v_{p/q}$  is the image of the set of (real) dominant parameters  $\mathcal{D}$  under the surgery map:*

$$\mathcal{D}_{p/q} := \Psi_{p/q}(\mathcal{D}).$$

We can now use the surgery map to transfer the inclusion of the Hubbard trees of real maps in the real slice of the Mandelbrot set to an inclusion of the Hubbard trees in the set of angles landing on the vein in parameter space.

**Proposition 13.11.** *Let  $c \in v_{p/q}$  be a parameter along the vein with non-renormalizable combinatorics, and  $c'$  another parameter along the vein which separates  $c$  from the main cardioid (i.e. if  $\ell$  and  $\ell'$  are the characteristic leaves,  $\ell' < \ell \leq \{\tau_{p/q}\}$ ). Then there exists a piecewise linear map  $F : S^1 \rightarrow S^1$  such that*

$$F(\tilde{H}_{c'}) \subseteq P_c.$$

*Proof.* Let  $\theta \in [0, \tau_{p/q}]$  be a characteristic angle for  $c$ . Let us first assume that the forward orbit of  $\theta$  never hits  $C_{p/q}$ . Then by Proposition 13.7 there exists an angle  $\theta_R \in [0, 1/2] \cap \mathcal{R}$  such that  $\theta = \Psi(\theta_R)$ , and by Lemma 13.9  $\theta_R$  is not renormalizable. Then, by Proposition 10.1, there exist a  $\theta'_R < \theta_R$  arbitrarily close to  $\theta_R$  (and by continuity of  $\Psi$  we can choose it so that  $\Psi(\theta'_R)$  lands on the vein closer to  $c$  than  $c'$ ) and a piecewise linear map  $F_R$  of the circle such that

$$(17) \quad F_R(H_{\theta'_R}) \subseteq P_{\theta_R}.$$

We claim that the map  $F := \Psi \circ F_R \circ \Psi^{-1}$  satisfies the claim. Indeed, if  $\xi \in [0, 1/2]$  recall that the map  $F_R$  constructed in Proposition 10.1 has the form

$$F_R(\xi) = s + \xi \cdot 2^{-N}$$

where  $s$  is a dyadic rational number and  $N$  is some positive integer. Thus,  $D^N(F_R(\xi)) = \xi$ , so also

$$\Psi(\xi) = \Psi(D^N(F_R(\xi))) = D^M(\Psi(F_R(\xi)))$$

for some integer  $M$ . Thus we can write for  $\xi \in H_{\theta'_R} \cap Z$

$$\Psi(F_R(\xi)) = t + \Psi(\xi) \cdot 2^{-M}$$

where  $t$  is a dyadic rational number, and  $t$  and  $M$  only depend on  $s$  and the element of the partition  $\bigcup P_i$  to which  $\xi$  belongs. Thus we have proven that  $F = \Psi \circ F_R \circ \Psi^{-1}$  is piecewise linear. Now, by Proposition 13.8, eq. (17), and Proposition 13.7 we have the chain of inclusions

$$\Psi \circ F_R \circ \Psi^{-1}(\tilde{H}_{c'}) \subseteq \Psi \circ F_R(H_{\theta'_R}) \subseteq \Psi(P_{\theta_R}) \subseteq P_c.$$

Finally, if the forward orbit of  $\theta$  hits  $C_{p/q}$ , then by density one can find an angle  $\tilde{\theta} \in (\theta', \theta)$  such that its forward orbit does not hit  $C_{p/q}$ , and apply the previous argument to the parameter  $\tilde{c}$  with characteristic angle  $\tilde{\theta}$ , thus getting the inclusion

$$F(\tilde{H}_{c'}) \subseteq P_{\tilde{c}} \subseteq P_c.$$

□

*Proof of Theorem 1.3.* Let  $c$  be a parameter along the vein  $v_{p/q}$ . Then by Theorem 5.5

$$\frac{h_{top}(f_c |_{T_c})}{\log 2} = \text{H.dim } H_c.$$

We shall prove that the right hand side equals  $\text{H.dim } P_c$ . Now, since  $P_c \subseteq H_c$ , it is immediate that

$$\text{H.dim } P_c \leq \text{H.dim } H_c$$

hence we just have to prove the converse inequality. Let us now assume  $c \in v_{p/q}$  non-renormalizable. Then by Proposition 13.11 for each  $c' \in [0, c]$  we have the inclusion

$$F(\tilde{H}_{c'}) \subseteq P_c$$

so, since  $F$  is linear hence it preserves Hausdorff dimension, we have

$$\text{H.dim } H_{c'} = \text{H.dim } \tilde{H}_{c'} \leq \text{H.dim } P_c$$

and as a consequence

$$\text{H.dim } P_c \geq \sup_{c' \in [0, c]} \text{H.dim } H_{c'}$$

where  $[0, c]$  is the segment of the vein  $v_{p/q}$  joining 0 with  $c$ . Now by continuity of entropy (Theorem 12.5)

$$\sup_{c' \in [0, c]} \text{H.dim } H_{c'} = \text{H.dim } H_c$$

hence the claim is proven for all non-renormalizable parameters along the vein. Now, the general case follows as in the proof of Theorem 1.7 by successively renormalizing and using the formulas of Proposition 9.3.  $\square$

So far we have worked with the combinatorial model for the veins, which conjecturally coincides with the set of angles of rays which actually land on the vein. Finally, the following proposition proves that the vein and its combinatorial model actually have the same dimension, independently of the MLC conjecture.

**Proposition 13.12.** *Let  $c \in v_{p/q} \cap \partial\mathcal{M}$  and  $\ell$  its characteristic leaf. Let*

$$M_c := \{\theta \in S^1 : R_M(\theta) \text{ lands on } v_{p/q} \cap [0, c]\}$$

*be the set of angles of rays landing on the vein  $v$  closer than  $c$  to the main cardioid, and*

$$P_c := \{\theta \in S^1 : \theta \text{ is endpoint of some } \ell' \in QML, \ell' \leq \ell\}$$

*its combinatorial model. Then the two sets have equal dimension:*

$$\text{H.dim } M_c = \text{H.dim } P_c.$$

*Proof.* Fix a principal vein  $v_{p/q}$ , and let  $\tau_W$  be the tuning operator relative to the hyperbolic component of period  $q$  in  $v_{p/q}$ ; moreover, denote as  $\tau$  the tuning operator relative to the hyperbolic component of period 2. Let  $P_c^{fr}$  the set of angles which belong to the  $P_c$  with finitely renormalizable combinatorics; then Proposition 3.2 yields the inclusions

$$\text{H.dim } P_c^{fr} \subseteq \text{H.dim } M_c \subseteq \text{H.dim } P_c$$

hence to prove the proposition it is sufficient to prove the equality

$$\text{H.dim } P_c^{fr} = \text{H.dim } P_c.$$

Let now  $c_n := \tau_W(\tau^n(-2))$  the tips of the chain of nested baby Mandelbrot sets which converge to the Feigenbaum parameter in the  $p/q$ -limb, and let  $\ell_n$  be the characteristic leaf of  $c_n$ . Then if  $\text{H.dim } P_c > 0$ , there exists a unique  $n \geq 1$  such that  $\ell_n < \ell \leq \ell_{n-1}$ , hence by monotonicity and by Theorem 1.3 we know

$$\text{H.dim } P_c \geq \text{H.dim } P_{c_n} = \frac{1}{2^n q}.$$

Now, each element of  $P_c$  is either of the form  $\tau_W \tau^{n-1}(c')$  with  $c'$  non-renormalizable, or of the form  $\tau_W(\tau^{n-1}(\tau_V(c')))$  where  $V$  is some hyperbolic window of period larger than 2. However, we know by Proposition 9.3 that the image of  $\tau_W \circ \tau^{n-1} \circ \tau_V$  has Hausdorff dimension at most  $\frac{1}{q \cdot 2^{n-1} \cdot 3} < \text{H.dim } P_c$ , hence one must have

$$\text{H.dim } P_c = \text{H.dim } \{\theta \in M_c : \theta = \tau_W \tau^{n-1}(\theta'), \theta' \text{ non-renormalizable}\} \leq \text{H.dim } P_c^{fr}$$

which yields the claim.  $\square$

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