A CENTRAL LIMIT THEOREM FOR THE DEGREE OF A RANDOM PRODUCT OF CREMONA TRANSFORMATIONS

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Abstract. We prove a central limit theorem for the algebraic and dynamical degrees of a random composition of Cremona transformations.

1. Introduction

A rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, defined over the field of complex numbers $\mathbb{C}$, is a function which is given in homogeneous coordinates by

$$f([x:y:z]) = [P_0(x,y,z) : P_1(x,y,z) : P_2(x,y,z)],$$

where P_0, P_1 and P_2 are homogeneous polynomials in $\mathbb{C}[x,y,z]$ of the same degree $d \in \mathbb{N}$ with no common factor. We say that f is **dominant** if its image is not contained in an algebraic curve. The integer d, denoted $\deg(f)$, is called the **degree** (or **algebraic degree**) of f and is in general distinct from the **topological degree** of f, which is the number of preimages by f of a general point. When the topological degree of f is one, we say that f is **birational** or that f defines a **Cremona transformation**.

Unlike the situation on $\mathbb{P}^1$, the algebraic degree of $f \circ g$ where f and g are two rational maps on $\mathbb{P}^2$ is not equal in general to the product $\deg(f)\deg(g)$, but it satisfies a submultiplicative property:

$$\deg(f \circ g) \leqslant \deg(f)\deg(g). \tag{1}$$

Using this fact, one can define the (**first**) **dynamical degree** of f [RS97], denoted $\lambda_1(f)$, given by

$$\lambda_1(f) := \lim_{n\to+\infty} \deg(f^n)^{1/n}. \tag{2}$$

This dynamical quantity is invariant under birational conjugacy [DS05, Tru20, Dan20] and measures the growth rate of the preimages of a generic hyperplane on $\mathbb{P}^2$.

The degree and dynamical degree of an arbitrary composition are quite difficult to predict in general. When f, g are generic maps (i.e. belong to suitable Zariski open subsets of the space of rational maps of degree d), the product satisfies $\deg(f \circ g) = \deg(f)\deg(g)$. However, this does not hold in general due to the presence of points where the rational maps are not defined (called *indeterminacy points*) and their behavior under iteration (see [Sib99, Proposition 1.4.3]).

Let us fix a probability measure μ on the space of Cremona transformations whose support is countable. We consider the **random product**

$$f_n := g_1 g_2 \dots g_n$$

where (g_n) is a sequence of *i.i.d.* random elements of G chosen with distribution μ; thus, the sequence (f_n) describes a **random walk** in the space of birational maps of $\mathbb{P}^2$.

Our aim is to understand the distribution of the sequences of algebraic degrees $(\deg(f_n))$ and dynamical degrees $(\lambda_1(f_n))$.

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For the law of large numbers, observe that because of (1) the sequence $(\log \deg(f_n))$ is subadditive and Kingman's theorem asserts that under a finite moment condition

$$\int_G \log \deg(f)\ d\mu(f) < +\infty, \tag{3}$$

there exists a constant $\ell_\mu \geqslant 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \deg(f_n) = \ell_\mu \tag{4}$$

almost surely. In other words, the sequence $\log \deg(f_n)$ follows a law of large numbers (see also [Hin19]). For birational maps of $\mathbb{P}^2$, it was shown in [MT18] that the limit ℓ_μ is positive if the semigroup generated by the support of μ is non-elementary. Moreover, in that case the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \lambda_1(f_n)$$

exists almost surely (even though $(\log \lambda_1(f_n))$ is *not* subadditive) and also equals ℓ_μ whenever the support of μ is bounded [MT21].

The main result of our paper shows that $\log \deg(f_n)$ for Cremona maps on $\mathbb{P}^2$ satisfies a central limit theorem, in the sense that the difference $\frac{\log \deg(f_n) - n\ell_\mu}{\sqrt{n}}$ converges to either a Gaussian or a "folded" Gaussian. Let μ_n be the distribution of f_n, and let $C_b(\mathbb{R})$ be the space of bounded, continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$. Given $\sigma \geqslant 0$, we denote as $\mathcal{N}_\sigma$ the *Gaussian measure* of variance σ and mean 0, i.e. the probability measure $d\mathcal{N}_\sigma(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} dt$ if $\sigma > 0$, and the δ-mass at 0 if $\sigma = 0$. Given $\sigma > 0$, we define the *folded Gaussian measure* centered at 0 and of variance σ as the probability measure $\mathcal{FN}_\sigma$ on $\mathbb{R}$ defined as the pushforward of a Gaussian measure of mean 0 and variance σ under the map $x \mapsto |x|$.

Theorem A. Let G be a countable semigroup of Cremona transformations and let μ be a measure whose support generates G satisfying

$$\int_G \sqrt{\deg(f)}\ d\mu(f) < +\infty. \tag{5}$$

Then there exists $\ell \geqslant 0$ such that the following two properties hold.

(i) (CLT for algebraic degree) Either, there exists $\sigma \geqslant 0$ such that

$$\lim_{n\to+\infty} \int_G \varphi\left(\frac{\log \deg(f) - n\ell}{\sqrt{n}}\right) d\mu_n(f) = \int_{\mathbb{R}} \varphi(t)\ d\mathcal{N}_\sigma(t)$$

for any function $\varphi \in C_b(\mathbb{R})$, or

$$\lim_{n\to+\infty} \int_G \varphi\left(\frac{\log \deg(f) - n\ell}{\sqrt{n}}\right) d\mu_n(f) = \int_{\mathbb{R}} \varphi(t)\ d\mathcal{FN}_\sigma(t)$$

for any $\varphi \in C_b(\mathbb{R})$.

(ii) (CLT for dynamical degree) A similar limit law holds for the dynamical degree.
There exists $\sigma \geqslant 0$ such that either

$$\lim_{n\to+\infty} \int_G \varphi\left(\frac{\log \lambda_1(f) - n\ell}{\sqrt{n}}\right) d\mu_n(f) = \int_{\mathbb{R}} \varphi(t)\ d\mathcal{N}_\sigma(t)$$

for any $\varphi \in C_b(\mathbb{R})$, or

$$\lim_{n\to+\infty} \int_G \varphi\left(\frac{\log \lambda_1(f) - n\ell}{\sqrt{n}}\right) d\mu_n(f) = \int_{\mathbb{R}} \varphi(t)\ d\mathcal{FN}_\sigma(t)$$

for any $\varphi \in C_b(\mathbb{R})$.

Finally:
(iii) If the semigroup G is non-elementary, then $\sigma > 0$ unless G has arithmetic length spectrum.

Let us remark that condition (5) is only needed if the semigroup generated by the support of μ is parabolic, while the weaker condition (3) is sufficient otherwise. A more refined version of this result, with a classification of all cases, formulas for ℓ and σ, as well as a characterization of the cases where $\sigma = 0$ will be given in § 3.4.

When one considers groups of isometries of finitely dimensional hyperbolic spaces, there are no non-elementary subgroups which have arithmetic length spectrum ([Dal99], [Kim06]). However, we show that such examples *do* exist in the Cremona group:

Proposition 1.1. *There exist non-elementary subgroups in the Cremona group which have arithmetic length spectrum. Moreover, there exist random walks supported on non-elementary semigroups for which $\sigma = 0$ in the central limit theorem.*

Let us now observe that the second case of Theorem A (with the folded normal law) actually occurs. Take a Hénon map $h : \mathbb{C}^2 \to \mathbb{C}^2$ of degree d, of the form:

$$h(x, y) = (y + P(x), x),$$

where $P(x) \in \mathbb{C}[x]$ is a polynomial of degree d and let us take $\mu = \frac{1}{2}\delta_h + \frac{1}{2}\delta_{h^{-1}}$, putting uniform mass on h and h^{-1}. Since $\log \deg(h^p) = \log \lambda_1(h^p) = |p| \log d$ for all $p \in \mathbb{Z}$, the classical central limit theorem yields the convergence:

$$\lim_{n\to+\infty} \frac{1}{\sqrt{n}} \int_G \phi(\log \deg(f)) d\mu_n(f) = \lim_{n\to+\infty} \frac{1}{\sqrt{n}} \int_G \phi(\log \lambda_1(f)) d\mu_n(f) = \int_{\mathbb{R}} \phi(|t|) e^{-t^2/2} dt,$$

for any bounded continuous function $\phi \in L^1(\mathbb{R})$. In this situation, the logarithm of the degree of a random product does not converge to a normal law but to a folded normal law and satisfies the second assertion of Theorem A. This example is an analogue in this setting of Furstenberg-Kesten's [FK60, Example 2] for products of random matrices, where the folded normal law already appears. Further concrete examples of the different asymptotic behaviours are given in Section 2.4.

1.1. **Strategy of proof.** Our proof exploits in a crucial way the relationship between birational maps and a suitable isometric action on an infinite dimensional Gromov-hyperbolic space or Hilbert space, developed by Cantat, Boucksom-Favre-Jonsson, Blanc-Cantat [Can11, BFJ08, BC16]. The construction of this Gromov-hyperbolic space, denoted $\mathbb{H}^\infty$, is of algebraic nature: it is obtained by considering a subspace of divisors on the space of infinite blow-ups of $\mathbb{P}^2$ and taking its completion with respect to a norm induced by the intersection product. The Hodge index theorem guarantees that the intersection product on $\mathbb{H}^\infty$ defines a Lorentzian metric:

$$d(\alpha, \beta) := \cosh^{-1}(\alpha \cdot \beta),$$

where $\alpha, \beta \in \mathbb{H}^\infty$ and $(\alpha \cdot \beta)$ denotes the intersection product of α and β. One advantage in working on the space of divisors over all blow-ups of $\mathbb{P}^2$ is that the pullback action by a birational map f becomes functorial. Namely, if $\alpha \in \mathbb{H}^\infty$, and f, g are birational maps, then:

$$(f \circ g)^* \alpha = g^* f^* \alpha,$$

as if we were working with the action of an endomorphism on the Néron-Severi group of a surface. Cantat exploited the fact that the pullback action on $\mathbb{H}^\infty$ is an isometry to show that the Tits alternative holds for the Cremona group [Can11]. Thus, we obtain a representation $\rho_f := f^*$ from the Cremona group to the group of isometries of $\mathbb{H}^\infty$. Taking as L the class of a line in $\mathbb{P}^2$, a random walk (f_n) on the space of birational maps induces a sample path $\rho_{f_n}(L)$ in the hyperbolic space $\mathbb{H}^\infty$ and we relate the degree to the distance on this space:

$$\cosh d(\rho_{f_n}(L), L) = \deg(f_n).$$

Denote by G the semigroup of birational transformations generated by the support of μ. According to the classification [Gro87] of semigroups of isometries of a hyperbolic space, $\rho(G)$ is either non-elementary or elementary, which is further subdivided into elliptic, parabolic, focal, or lineal (see Section 2.4).

In the non-elementary case, $\rho(G)$ contains two loxodromic elements with different axis; here, we import into our setting the known central limit theorems for the translation length and for the escape rate from [Bjo10, MS20, BQ16b, Gou17, Hor18].

Otherwise, $\rho(G)$ either contains no loxodromic elements (elliptic or parabolic case) or every loxodromic element has a common fixed point on the (Gromov) boundary of $\mathbb{H}^\infty$ (focal or lineal case). Here, we show that G is particularly rigid and we conclude using some techniques from birational geometry by showing that the logarithm of a random product can be reduced to a random walk on the line or on the plane $\mathbb{R}^2$.

The possible limit distributions in the various cases are summarized in the following table. Note that the presence of a folded Gaussian implies that G is lineal.

Type of isometry group	**Limit law for** $\log\deg$	**Limit law for** $\log\lambda_1$
elliptic	Gaussian (possibly trivial)	Gaussian (possibly trivial)
parabolic	Gaussian (possibly trivial)	Gaussian (possibly trivial)
lineal	Gaussian or Folded Gaussian	Gaussian or Folded Gaussian
non-elementary	Gaussian ($\ell > 0$)	Gaussian ($\ell > 0$)

1.2. **Historical remarks.** Theorem A is reminiscent of similar statements, proved in other contexts. When one restricts oneself to monomial maps, this result reduces to the study of a product of random matrices; in that context, the convergence of $\frac{1}{n}\log\deg(f_n)$ follows from Oseledets' theorem, and the central limit theorem from [FK60, GLPR83, GR85, GR86, BQ16a]. The central limit theorem is also known for the translation length and the escape rate for a random composition of isometries on a tree [NW02] or more generally on a Gromov-hyperbolic space [BQ16b, MS20], for quasimorphisms on a random product of elements in a countable hyperbolic group [CF10, BH11], for the distance in the Teichmüller metric of a random product of mapping classes [Hor18].

The growth of the sequence $(\deg(f^n))$ and the dynamical degree of a given rational map f has been the subject of much research, and is known only in certain cases: for endomorphisms of a projective variety, monomial maps [Lin12, FW12], birational surfaces maps [DF01, BC16], polynomial automorphisms and endomorphisms of the affine plane [FM89, Fur99, FJ04, FJ07, FJ11], meromorphic surface maps (under certain assumptions) [BFJ08], birational transformations of hyperkähler manifolds [LB19], certain automorphisms of the affine 3-space [BvS19a, BvS19b] and certain rational maps associated to matrix inversions [AAdBM99, AdMV06, AAdB+99, BK08, BT10]. Starting from dimension 3, the degree sequences are partially known for birational transformations with very slow degree growth [CX20] or for specific examples [D1́8], for a specific group of automorphisms on $SL_2(\mathbb{C})$ [Dan18], while a lower bound on unbounded degree sequences was recently obtained for a large class of birational transformations in [LU20].

Acknowledgements. We thank Jeffrey Diller, Romain Dujardin, Charles Favre, Junyi Xie for useful conversations, the first author's sister Nguyen-Thi Dang for providing us references and Mattias Jonsson for making us meet during our visit to the University of Michigan. G. T. is partially supported by NSERC and the Alfred P. Sloan Foundation.

2. Rational maps, degrees and isometric actions

2.1. **Topological, algebraic and dynamical degrees.** Recall that given a dominant rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, the **topological degree** of f, denoted $\mathrm{d_{top}}(f)$, is the number of preimages, counted with multiplicity, of a generic point of $\mathbb{P}^2$. When the topological degree of f is equal to 1, one says that f is **birational** and its inverse is a rational map which we denote by f^{-1}. In this paper, we will restrict to birational transformations of $\mathbb{P}^2$, which are often referred to as **Cremona transformations** of the plane. Denote by L the divisor on $\mathbb{P}^2$ given by the line at infinity. One can express the topological degree and the degree of f by computing the following intersection products:

$$\mathrm{d_{top}}(f) = 1 = (f^*L \cdot f^*L),$$
$$\deg(f) = (f^*L \cdot L).$$

The dynamical degree and the topological degree are dynamical invariants, hence invariant under cnojugation.

Theorem 2.1. ([RS97], [Tru20, Theorem 1.1], [Dan20, Theorem 1]) *For any birational map $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$,*

$$\lambda_1(f) = \lambda_1(g \circ f \circ g^{-1}).$$

2.2. **The construction of the hyperbolic space.** In this section, we recall the construction of the Picard-Manin space of divisors, following closely the presentation in [BC16]. We start with $X_0 = \mathbb{P}^2$. If $\pi : X \to X_0$ is a birational morphism, we say that X is a **birational model of** X_0. When this happens, the morphism π induces a pullback in the Néron-Severi group

$$\pi^* : \mathrm{NS}(X_0) \to \mathrm{NS}(X).$$

Moreover, for any two birational models X, Y over X_0, there exists a third birational model Z over both X and Y. We thus define the Picard-Manin space as the inductive limit:

$$\mathcal{Z} := \varinjlim \mathrm{NS}(X),$$

where X describes all birational models of X_0. If X is a blow-up of X_0 at one point, we denote by E the exceptional divisor on X and $\mathrm{NS}(X) \simeq \mathrm{NS}(X_0) \oplus \mathbb{Z}E$. If one takes an arbitrary sequence of blow-ups of $\mathbb{P}^2$, we obtain finitely many exceptional divisors which are all inside $\mathcal{Z}$. The Picard-Manin space can be described as:

$$\mathcal{Z} = \mathrm{NS}(\mathbb{P}^2) \oplus \bigoplus \mathbb{Z}E_i \simeq \mathbb{Z}L \oplus \bigoplus \mathbb{Z}E_i,$$

where E_i describes all the exceptional divisors on a birational model of $\mathbb{P}^2$ and where L denotes the class of a line in $\mathbb{P}^2$.

The intersection product on each birational model of $\mathbb{P}^2$ induces a scalar product on $\mathcal{Z}$, denoted $(\alpha \cdot \beta)$, and a norm on $\mathcal{Z} \otimes \mathbb{R}$. We denote by $\overline{\mathcal{Z}}$ the completion of $\mathcal{Z}$ with respect to this norm. Observe that the Hodge index theorem on each birational model of $\mathbb{P}^2$ shows that the metric induced by the intersection product is hyperbolic; as a result, the space $\overline{\mathcal{Z}}$ endowed with the metric induced by the intersection product has the structure of an infinite-dimensional hyperbolic space. For more details on this construction, we shall refer to [Can11].

Definition 2.2. *The hyperbolic space $\mathbb{H}^\infty$ is the set*

$$\mathbb{H}^\infty := \left\{ \alpha \in \overline{\mathcal{Z}} \ : \ (\alpha \cdot \alpha) = 1, (\alpha \cdot L) > 0 \right\}.$$

It is endowed with a hyperbolic metric $d : \mathbb{H}^\infty \times \mathbb{H}^\infty \to \mathbb{R}^+$ given by the formula:

$$d(\alpha, \beta) := \cosh^{-1}(\alpha \cdot \beta),$$

for any $\alpha, \beta \in \mathbb{H}^\infty$. Its boundary, denoted $\partial\, \mathbb{H}^\infty$, is the projectivization

$$\partial\, \mathbb{H}^\infty := \mathbb{P}\left(\left\{ \alpha \in \overline{\mathcal{Z}} \ : \ (\alpha \cdot \alpha) = 0 \right\} \right).$$

This space corresponds to the choice of a "positive" hyperboloid in the vector space $\overline{\mathcal{Z}}$. We will restrict our action to a smaller subset of $\mathbb{H}^\infty \cup \partial\,\mathbb{H}^\infty$, namely the nef cone. Recall that a class is *nef* if it intersects non-negatively any curve class and that a nef class is also *big* if its self-intersection is positive.

Definition 2.3. *The nef locus* $\mathrm{Nef}_{\mathbb{H}}$ *is the subset of nef classes in* $\mathbb{H}^\infty$ *and we denote by* $\partial\,\mathrm{Nef}_{\mathbb{H}}$ *the set of nef classes on the boundary* $\partial\,\mathbb{H}^\infty$*.*

2.3. **Isometric action of birational maps on the hyperbolic Picard-Manin space.** Let $f:\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map. Its graph Γ_f in $\mathbb{P}^2\times\mathbb{P}^2$ is a natural birational model of $\mathbb{P}^2$ and the maps π_1,π_2 induced by the projection onto the first and second factor, respectively, are regular. If α is a divisor in $\mathbb{P}^2$, then we can take its pullback by π_2, denoted $f^*\alpha$. More generally, we can do the same if α is a class in a birational model X of $\mathbb{P}^2$ by pulling back on the corresponding graph. The latter definition is compatible with the inductive definition and induces a continuous pullback map $f^*:\mathbb{H}^\infty\to\mathbb{H}^\infty$.

We now define a contravariant action by Cremona transformations on $\mathbb{H}^\infty$, namely for any $\alpha\in\mathbb{H}^\infty$ and any birational map f, the element $\rho_f(\alpha)$ is given by the formula

$$\rho_f(\alpha) := f^*\alpha.$$

Note that since $(f\circ g)^* = g^*\circ f^*$, this action reverses the order. Using the fact that $(f^*\alpha\cdot f^*\beta) = (\alpha\cdot\beta)$ for all $\alpha,\beta\in\mathbb{H}^\infty$, one verifies that the above action induces an isometry of $(\mathbb{H}^\infty, d)$ and the pushforward is defined as the pullback by the inverse $f_* = (f^{-1})^*$. These operators are related to one another by the projection formula, for any Cremona transformation f and any classes $\alpha,\beta\in\mathbb{H}^\infty$:

$$(f^*\alpha\cdot\beta) = (f_*f^*\alpha\cdot f_*\beta) = (\alpha\cdot f_*\beta). \tag{6}$$

As a consequence, if the associated isometry is loxodromic on $\mathbb{H}^\infty$ then we relate the dynamical degree of f with the translation distance as follows.

Lemma 2.4. *For any birational map f on $\mathbb{P}^2$, we have*

$$\log\left(f^*L\cdot L\right) \leqslant d(\rho_f(L), L) \leqslant \log\left(2(f^*L\cdot L)\right).$$

Proof. By definition of the hyperbolic metric,

$$\cosh d(\rho_f(L), L) = (f^*L\cdot L)$$

hence, since $\frac{1}{2}e^x \leqslant \cosh x \leqslant e^x$,

$$\frac{e^{d(\rho_f(L),L)}}{2} \leqslant (f^*L\cdot L) \leqslant e^{d(\rho_f(L),L)}$$

which immediately yields the claim. □

Observe that when the action of f is loxodromic, then the two invariant classes on the boundary $\partial\,\mathbb{H}^\infty$ are also in $\partial\,\mathrm{Nef}_{\mathbb{H}}$. We shall use frequently the following observation.

Lemma 2.5. *Let G be a semigroup of birational maps of $\mathbb{P}^2$. Suppose that there exists an element $\alpha\in\mathbb{H}^\infty\cup\partial\,\mathbb{H}^\infty$ which is an eigenvector for every element of G. Then the map $\pi: G\to(\mathbb{R},+)$*

$$\pi(f) := \log(f^*\alpha\cdot L) - \log(\alpha\cdot L)$$

is a morphism of semigroups.

In the following sections, we will use many times the following results. The first one is an estimate of algebraic nature called Siu's inequality stated in [Tra95],[Laz04, Theorem 2.2.15]. For our context, we will need the following formulation:

Proposition 2.6. *(see* [Dan20, Proposition 3.4.1]*) Let $\alpha, \beta \in \mathbb{H}^\infty$ such that α is a nef class and β is big and nef, then one has:*

$$\alpha \leqslant 2\frac{(\alpha \cdot \beta)}{(\beta^2)}\beta, \tag{7}$$

where $\alpha \leqslant \beta$ means that the difference $\beta - \alpha$ lies in the closure of the cone generated by effective curve classes.

The main advantage of the above estimate is that one obtains some inequalities on numbers when one intersects with nef classes, because the cone of nef divisors is dual to the closure of the cone generated by effective divisors via the intersection product.

Lemma 2.7. *If $\alpha \in \mathbb{H}^\infty$ is big and nef then there exists a constant C such that for all $f \in G$, one has*

$$|\log(f^*L \cdot L) - \log(f^*\alpha \cdot L)| \leqslant C. \tag{8}$$

Proof. Since α and L are big and nef and using the relations $(f^*\alpha \cdot f^*L) = (\alpha \cdot L)$, $(f^*\alpha \cdot f^*\alpha) = (\alpha^2)$ and $(f^*L \cdot f^*L) = (L^2)$, Siu's inequality (Proposition 2.6) applied to the pair $(f^*\alpha, f^*L)$ and the reverse pair yields

$$\frac{(L^2)}{2(\alpha \cdot L)} f^*\alpha \leqslant f^*L \leqslant 2\frac{(\alpha \cdot L)}{(\alpha^2)} f^*\alpha.$$

Intersecting with the nef class L thus yields:

$$\frac{(L^2)(f^*\alpha \cdot L)}{2(\alpha \cdot L)} \leqslant \deg(f) = (f^*L \cdot L) \leqslant 2\frac{(\alpha \cdot L)(f^*\alpha \cdot L)}{(\alpha^2)}.$$

which implies the claim. □

2.4. **Classification of semigroups of isometries.** By the classification of semigroups of isometries of hyperbolic spaces (see [DSU17, Theorem 6.2.3 and Proposition 6.2.14]), a semigroup G acting by isometry on a hyperbolic space X satisfies one of the following properties:

(i) G is *elliptic*, i.e. there exists a class $\alpha \in X$ globally fixed by G.
(ii) G is *parabolic*, i.e. there exists a class $\alpha \in \partial X$ globally fixed by G and every element of G is parabolic.
(iii) G is *focal*, i.e. it globally fixes a class $\alpha \in \partial X$ and contains a hyperbolic element.
(iv) G is *non-elementary*, i.e. there exists two hyperbolic elements whose fixed sets at infinity do not intersect.
(v) G is *lineal*, i.e. it contains a hyperbolic element and any other hyperbolic element fixes the same points at infinity.

We call a semigroup G *elementary* if it satisfies condition $(i), (ii), (iii)$ or (v) in the above characterization. In the situation where G is a semigroup of Cremona transformations, the above classification yields.

Proposition 2.8. *Let G be a semigroup of Cremona transformations. Then one of the following properties hold.*

(i) *G is* elliptic*, i.e. there exists a class $\alpha \in \mathbb{H}^\infty$ globally fixed by G.*
(ii) *G is* parabolic*, i.e. there exists a class $\alpha \in \partial\,\mathrm{Nef}_{\mathbb{H}}$ globally fixed by G and every element of G is parabolic.*
(iii) *G is* non-elementary*, i.e. there exists two hyperbolic elements whose fixed sets at infinity do not intersect.*
(iv) *G is* lineal*, i.e. it contains a hyperbolic element and any other hyperbolic element fixes the same points at infinity.*

Proof. By [Ure20, Lemma 7.3], G cannot induce a focal subgroup of isometries on $\mathbb{H}^\infty$. So we are left with the four remaining cases of the classification of isometries. $\square$

We now give some concrete examples of subgroups in each of these classes and discuss the central limit theorem.

Example 2.9. If $G \subset \mathrm{PGL}_2(\mathbb{C})$ is a discrete subgroup acting linearly on $\mathbb{P}^2$, then the semigroup induced by G on $\mathbb{H}^\infty$ is elliptic. In this case, the degrees and dynamical degrees are always 1 and the sequence $\log \deg(f_n)$ is the constant random variable equal to zero.

Example 2.10. If G is a family of non-trivial Jonquières transformations, i.e. of the form:

$$(x, y) \mapsto \left(ax + b, \frac{\alpha(x)y + \beta(x)}{\gamma(x)y + \delta(x)} \right),$$

where $a \in \mathbb{C}^*, b \in \mathbb{C}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}(x)$ are non-constant rational functions on x and $\alpha\delta - \beta\gamma$ is a non-zero function. Then the subgroup G induces a parabolic action on $\mathbb{H}^\infty$.

Example 2.11. Take h a Hénon map, i.e. of the form:

$$h : (x, y) \mapsto (y + P(x), x),$$

where $P(x) \in \mathbb{C}[x]$ is a polynomial of degree $d \geqslant 2$. Consider the measure $\mu = \frac{1}{2}\delta_h + \frac{1}{2}\delta_{h^{-1}}$. The subgroup generated by h and its inverse induces a lineal subgroup of isometries on $\mathbb{H}^\infty$. Since $\deg(h^p) = d^{|p|}$ for all $p \in \mathbb{Z}$, we have that $\frac{1}{\sqrt{n}} \log \deg(f_n)$ follows a folded normal law.

2.5. **Characterization of semigroups having a global fixed point on the boundary.** In this section, we study the semigroup of birational transformations whose action on the Picard-Manin space has a global fixed point on the boundary. In many cases, we shall use the following result.

Recall that the *translation length* of an isometry f of a hyperbolic metric space (X, d) is

$$\tau(f) := \lim_{n \to \infty} \frac{d(o, f^n o)}{n},$$

where $o \in X$ is any base point. Moreover the isometry f is *loxodromic* if $\tau(f) > 0$.

In our setting, when one applies it to the action of Cremona transformations on the Picard-Manin space, the translation length is interpreted algebraically as:

$$\tau(\rho_f) = \lim_{n \to +\infty} \frac{1}{n} \cosh^{-1}(L \cdot (f^n)^* L) = \lim_{n \to +\infty} \frac{1}{n} \cosh^{-1}(\deg(f^n)) = \log \lambda_1(f). \tag{9}$$

Proposition 2.12. *Take a semigroup G and suppose that there exists a class $\alpha \in \partial \,\mathrm{Nef}_{\mathbb{H}}$ which is fixed by G and such that $f^*\alpha = \lambda(f)\alpha$ for $\lambda(f) \in \mathbb{R}^*$. Then the following properties are equivalent.*

(i) *One has $\lambda(f) = 1$,*
(ii) *The action of f on $\mathbb{H}^\infty$ is not loxodromic.*

This proposition yields the following corollary:

Corollary 2.13. *Take a semigroup G and suppose that there exists a class $\alpha \in \partial \,\mathrm{Nef}_{\mathbb{H}}$ which is fixed by G and that $f^*\alpha = \lambda(f)\alpha$ for $\lambda(f) \in \mathbb{R}^*$. Then*

$$\lambda_1(f) = \max(\lambda(f), \lambda(f)^{-1}) \tag{10}$$

for all $f \in G$.

Before proving the above statement we will need the following lemma.

Lemma 2.14. *For any f in G, one has*

$$\max\left(\lambda(f), \lambda(f)^{-1}\right) \leqslant 2\deg(f).$$

Proof. Let us prove the inequality $\lambda(f) \leqslant 2\deg(f)$. Since α is nef, we have by Siu's inequality

$$(f^*\alpha \cdot L) \leqslant 2\frac{(\alpha \cdot L)}{(L^2)}(f^*L \cdot L).$$

Hence, since $f^*\alpha = \lambda(f)\alpha$, we obtain $\lambda(f) \leqslant 2\deg(f)$ as required.

For the second inequality, we compute the intersection product $(f_*\alpha \cdot L)$ and obtain:

$$(f_*\alpha \cdot L) = \frac{1}{\lambda(f)}(f_*f^*\alpha \cdot L) = \frac{1}{\lambda(f)}(\alpha \cdot L).$$

Moreover, the projection formula (see (6)) shows that:

$$(f_*\alpha \cdot L) = (\alpha \cdot f^*L).$$

By Siu's inequality, we have:

$$f^*L \leqslant 2\frac{(f^*L \cdot L)}{(L^2)}L,$$

and using the fact that α is nef, we obtain:

$$\frac{1}{\lambda(f)}(\alpha \cdot L) = (\alpha \cdot f^*L) \leqslant 2\deg(f)(\alpha \cdot L).$$

Dividing by $(\alpha \cdot L)$ yields the second inequality. □

One important result is the following lemma, which provides good estimates on the degree.

Lemma 2.15. *Suppose that $\alpha \in \partial\,\mathbb{H}^\infty \cap \partial\,\mathrm{Nef}_{\mathbb{H}}$ and that $f^*\alpha = \alpha$ for any $f \in G$. Then for any $f, g \in G$, one has:*

$$\sqrt{\deg(f \circ g)} \leqslant \sqrt{\deg(f)} + \sqrt{\deg(g)}.$$

Proof of Lemma 2.15. By rescaling α, let us assume $(\alpha \cdot L) = 1$. Given $f, g \in G$, we write:

$$f^*L = \deg(f)\alpha + v_1,$$

and

$$g_*L = \deg(g)\alpha + v_2$$

where $v_1, v_2 \in \mathbb{H}^\infty$ and $(v_i \cdot L) = 0$. Using the projection formula, the fact that $(\alpha^2) = 0$ and our decomposition, we have:

$$1 = (\alpha \cdot L) = (g^*\alpha \cdot L) = (\alpha \cdot g_*L) = (\alpha \cdot v_2). \tag{11}$$

Similarly since $f_*\alpha = \alpha$, we also have:

$$1 = (\alpha \cdot L) = (f_*\alpha \cdot L) = (\alpha \cdot f^*L) = (\alpha \cdot v_1). \tag{12}$$

Let us also compute $(f^*L \cdot f^*L)$ and $(g_*L \cdot g_*L)$,

$$1 = (f^*L \cdot f^*L) = 2\deg(f)(\alpha \cdot v_1) + (v_1^2), \tag{13}$$

and

$$(g_*L \cdot g_*L) = 2\deg(g)(\alpha \cdot v_2) + (v_2^2).$$

Since L is nef and nef classes in $\mathbb{H}^\infty$ are stable by pushforward, we have $(g_*L \cdot g_*L) \geqslant 0$, hence:

$$-(v_2^2) \leqslant 2\deg(g)(\alpha \cdot v_2). \tag{14}$$

We now compute $\deg(f \circ g)$:

$$\deg(f \circ g) = (g^*f^*L \cdot L) = (f^*L \cdot g_*L) = \deg(f)(\alpha \cdot v_2) + \deg(g)(\alpha \cdot v_1) + (v_1 \cdot v_2). \tag{15}$$

Since the intersection form is negative definite on $\{v \in \mathbb{H}^\infty \,|(v \cdot L) = 0\}$, the Cauchy-Schwarz inequality implies that:

$$|(v_1 \cdot v_2)| \leqslant \sqrt{(v_1^2)(v_2^2)}.$$

Applying the above inequality to (15), we get:

$$\deg(f \circ g) \leqslant \deg(f)(\alpha \cdot v_2) + \deg(g)(\alpha \cdot v_1) + \sqrt{(v_1^2)(v_2^2)}.$$

We now apply (13) and (14):

$$\deg(f \circ g) \leqslant \deg(f)(\alpha \cdot v_2) + \deg(g)(\alpha \cdot v_1) + \sqrt{2\deg(g)(\alpha \cdot v_2)\,|1 - 2\deg(f)(\alpha \cdot v_1)|}.$$

This last inequality together with (11) and (12) gives:

$$\deg(f \circ g) \leqslant \deg(f) + \deg(g) + (2\deg(g))^{1/2}\sqrt{|1 - 2\mathrm{deg}(f)|}.$$

Since $\deg(f) \geqslant 1$, we obtain $\sqrt{2\deg(f) - 1} \leqslant \sqrt{2\deg(f)}$, hence:

$$\deg(f \circ g) \leqslant \left(\sqrt{\deg(f)} + \sqrt{\deg(g)}\right)^2.$$

We conclude that:

$$\sqrt{\deg(f \circ g)} \leqslant \sqrt{\deg(f)} + \sqrt{\deg(g)},$$

as required. □

Proof of Proposition 2.12. We now prove the implication $(i) \Rightarrow (ii)$, by showing that if f is loxodromic, then $\lambda(f) \neq 1$. If f is loxodromic, there exists a nef class $\beta \in \partial\,\mathrm{Nef}_{\mathbb{H}}$ fixed by f and which is not proportional to α. Suppose that $f^*\beta = \mu(f)\beta$ where $\mu \in \mathbb{R}^*$. By the Hodge index theorem, the product $(\alpha \cdot \beta)$ is non-zero and using the projection formula (6), we obtain:

$$(\alpha \cdot \beta) = (f^*\alpha \cdot f^*\beta) = \lambda(f)\mu(f)(\alpha \cdot \beta).$$

We thus obtain that $\mu(f) = \frac{1}{\lambda(f)}$. We now compute:

$$((f^n)^*(\alpha + \beta) \cdot L) = \left(\lambda(f)^n(\alpha \cdot L) + \frac{1}{\lambda(f)^n}(\beta \cdot L)\right) \tag{16}$$

hence

$$((f^n)^*(\alpha + \beta) \cdot L) = (\lambda(f))^n\,(\alpha \cdot L) + \left(\frac{1}{\lambda(f)}\right)^n(\beta \cdot L).$$

Since f is loxodromic and since $\alpha + \beta$ is big and nef, Lemma 2.7 shows that the above sequence must diverge to infinity, hence $\lambda(f) \neq 1$.

We finally show that $(ii) \Rightarrow (i)$. By contradiction, suppose that $\lambda(f) \neq 1$; then Lemma 2.14 implies

$$2\mathrm{deg}(f^n) \geqslant \max\left((\lambda(f))^n, \left(\frac{1}{\lambda(f)}\right)^n\right).$$

We thus conclude that $\lim_{n\to+\infty} \frac{1}{n}\log\deg(f^n) = \tau(\rho_f) > 0$, hence f is loxodromic, which contradicts our assumption. □

Proof of Corollary 2.13. If $f \in G$ is not loxodromic, then $\lambda_1(f) = \lambda(f) = 1$ by Proposition 2.12. Otherwise, f is loxodromic, and by Lemma 2.7 and equation (16) we have

$$\begin{aligned}\log\deg(f^n) &= \log\,((f^n)^*(\alpha+\beta)\cdot L) + O(1)\\ &= \log\,(\lambda(f)^n(\alpha\cdot L) + \lambda(f)^{-n}(\beta\cdot L)) + O(1)\end{aligned}$$

Hence, $\lambda_1(f) = \max(\lambda(f), \lambda(f)^{-1})$, as required. □

3. Proof of Theorem A for elementary semigroups

3.1. **Folded normal law.** For our result, we shall need a random variable which is closely related to a normal law.

Definition 3.1. *Fix $\sigma > 0$. The* folded Gaussian *distribution parametrized by σ, denoted $\mathcal{FN}(0,\sigma)$ is the pushforward of the normal distribution $\mathcal{N}(0,\sigma)$ by the map $\varphi(x) = |x|$.*

We will then apply the following consequence of the central limit theorem.

Proposition 3.2. *Consider a sequence (Z_n) of i.i.d. variables of mean m and of variance σ. Then the following holds:*

(1) *If $m \neq 0$, then*
$$\frac{|\sum_{i=1}^n Z_i| - n|m|}{\sqrt{n}} \to \mathcal{N}(0,\sigma).$$

(2) *If $m = 0$, then*
$$\frac{|\sum_{i=1}^n Z_i|}{\sqrt{n}} \to \mathcal{FN}(0,\sigma)$$
where $\mathcal{FN}(0,\sigma)$ is the folded normal distribution.

3.2. **Central limit theorem for the algebraic degree.**

3.2.1. *Degree for elliptic semigroups.* Suppose that G induces an elliptic semigroup action on $\mathrm{Nef}_{\mathbb{H}}$. Observe that we can write
$$\log\deg(f_n) = \log(f_n^*L \cdot L).$$
Since the semigroup G is elliptic and $d(L, \rho_{f_n}(L)) = \log(f_n^*L \cdot L) + O(1)$ by Lemma 2.4, the second term above is bounded, so the law of the sequence
$$\frac{\log\deg(f_n) - n\ell_\mu}{\sqrt{n}}$$
converges to the Dirac mass.

3.2.2. *Lineal semigroups.* Suppose the action of G on the hyperbolic space is lineal. Let θ_+, θ_-, be the two invariant nef classes on the boundary $\partial\,\mathrm{Nef}_{\mathbb{H}}$ which are either globally G-invariant by pullback or pushforward or swapped, normalized so that $(\theta_+ \cdot L) = (\theta_- \cdot L) = 1$. Let $\lambda : G \to \mathbb{R}$ be defined so that $f^*(\theta_+) = \lambda(f)\theta_+$. Since the classes θ_+ and θ_- are invariant classes on the boundary, the random variables
$$\log(f_n^*\theta_+ \cdot L) \qquad \text{and} \qquad \log(f_n^*\theta_- \cdot L)$$
describe a random walk on the real line by Lemma 2.5. We prove the following central limit theorem.

Theorem 3.3. *Suppose that the averages given by:*
$$\Lambda_\mu := \int_G \log\lambda(f)\ d\mu(f)$$
are finite, and moreover the variance
$$\sigma^2 := \int_G (\log\lambda(f) - \Lambda_\mu)^2\ \ d\mu(f)$$
is also finite. Then:

(1) *If $\Lambda_\mu \neq 0$, then*
$$\frac{\log\deg(f_n) - n\Lambda_\mu}{\sqrt{n}} \to \mathcal{N}(0,\sigma).$$

(2) *If* $\Lambda_\mu = 0$, *then*

$$\frac{\log \deg(f_n)}{\sqrt{n}} \to \mathcal{FN}(0,\sigma)$$

where $\mathcal{FN}(0,\sigma)$ *is the folded normal law.*

To prove Theorem 3.3, we first compute the degree as follows.

Lemma 3.4. *For each* $f \in G$, *we have*

$$\log \deg(f) = |\log \lambda(f)| + O(1)$$

where $O(1)$ *is a constant which depends only on the intersection product* $(\theta_+ \cdot \theta_-)$.

Proof. Take θ_+, θ_- the two invariant nef classes on the boundary which are either globally G-invariant or swapped, normalized so that $(\theta_+ \cdot L) = (\theta_- \cdot L) = 1$. Observe that the Hodge index theorem implies that $\theta_+ + \theta_-$ is big and nef. By Lemma 2.7, we have:

$$| \log \deg(f) - \log\left((f^*\theta_+ \cdot L) + (f^*\theta_- \cdot L)\right) | \leqslant C, \tag{17}$$

where C depends only on $(\theta_+ \cdot \theta_-)$. Since $f^*\theta_+ = \lambda(f)\theta_+$, $f_*\theta_- = \lambda(f)\theta_-$, we have $f^*\theta_- = \frac{1}{\lambda(f)}\theta_-$ and

$$\log\left((f^*\theta_+ \cdot L) + (f^*\theta_- \cdot L)\right) = \log\left(\lambda(f) + \frac{1}{\lambda(f)}\right) = \log(2\cosh u(f)) \tag{18}$$

where $u(f) = \log \lambda(f)$ and cosh is the hyperbolic cosine. Hence, we rewrite (17) as follows:

$$|\log \deg(f) - \log \cosh u(f)| \leqslant C' \tag{19}$$

with $C' = C + \log 2$. Observe that the following inequality is satisfied for all $x \in \mathbb{R}$:

$$\frac{e^{|x|}}{2} \leqslant \cosh(x) \leqslant e^{|x|},$$

hence we get:

$$|\log \cosh u(f) - |u(f)|| \leqslant \log 2.$$

In particular, using the above equation and (19), we obtain:

$$|\log \deg(f) - |u(f)|| \leqslant C'', \tag{20}$$

where $C'' = C + 2\log 2 > 0$. Hence, we decompose $\log \deg(f)$ as follows:

$$\log \deg(f) = |\log \lambda(f)| + O(1), \tag{21}$$

completing the proof of the lemma. □

Proof of Theorem 3.3. We have

$$\log \deg(f_n) = | \log \lambda(f_n)| = \left| \sum_{i=1}^{n} \log(g_i) \right|.$$

Using Proposition 3.2, we conclude that Theorem 3.3 holds. □

3.2.3. *Parabolic semigroups.* We now suppose that the action of G on $\mathbb{H}^\infty$ is parabolic. Take $\alpha \in \partial\,\mathbb{H}^\infty \cap \partial\,\mathrm{Nef}_{\mathbb{H}}$ a globally G-invariant class, we shall choose α so that $(\alpha \cdot L) = 1$. For each $f \in G$, we have $f^*\alpha = \alpha$ by Proposition 2.12. Note that $\lambda_1(f) = 1$ and by hypothesis

$$\int_G \sqrt{\deg(f)}\; d\mu(f) < +\infty. \tag{22}$$

By Lemma 2.15, we have:

$$\sqrt{\deg(f \circ g)} \leqslant \sqrt{\deg(f)} + \sqrt{\deg(g)}. \tag{23}$$

Moreover, equation (22) proves that the cocycle $\sqrt{\deg(\cdot)}$ belongs to $\mathrm{L}^1(\mu)$, hence Kingman's subadditive ergodic theorem yields the almost sure convergence

$$\lim_{n\to+\infty} \frac{1}{n}\sqrt{\deg(f_n)} = C,$$

where $C \in [0, \infty)$. This proves that almost surely

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log\deg(f_n) \leqslant 0.$$

Note also that

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log\deg(f_n) \geqslant 0,$$

hence the sequence of random variables

$$\frac{1}{\sqrt{n}}(\log\deg(f_n))$$

converges to zero almost surely, hence in probability and the central limit theorem also holds for $\log\deg(f_n)$.

3.3. **Central limit theorem for the first dynamical degree.** We now prove that $\log\lambda_1(f_n)$ satisfies a central limit theorem. Since there is an invariant class on the boundary, Corollary 2.13 holds and $\lambda_1(f) = \max(\lambda(f), \lambda(f)^{-1})$ for all f in G. This proves that

$$\log\lambda_1(f_n) = \left|\log\lambda(f_n)\right|.$$

Thus, as in the proof of Theorem 3.3, we obtain that the sequence

$$\frac{\log\lambda_1(f_n) - n\ell_\mu}{\sqrt{n}} \tag{24}$$

converges to

$$\mathcal{N}(0,\sigma)$$

if $\Lambda_\mu \neq 0$ and to

$$\mathcal{F}\mathcal{N}(0,\sigma)$$

if $\Lambda_\mu = 0$. This completes the proof of Theorem A for elementary semigroups.

3.4. **Summary of the Central limit theorem in the elementary case.** Let us set

$$\Lambda_\mu = \int \log\lambda(f)\; d\mu(f),$$

where $f^*\alpha = \lambda(f)\alpha$ for all $f \in G$, with $\alpha \in \partial\,\mathbb{H}^\infty \cap \mathrm{Nef}_{\mathbb{H}}$. The following table summarizes all possible limit behaviours. Note that, as a consequence:

Corollary 3.5. *If the semigroup G is elementary, then $\sigma = 0$ if and only if G is elliptic, parabolic, or if G is lineal with $\lambda(f)$ constant on the support of μ.*

Type of group	Mean	Limit law for $\log\deg$	Limit law for $\log\lambda_1$
elliptic	$\ell=0$	Dirac mass at zero	Dirac mass at zero
parabolic	$\ell=0$	Dirac mass at zero	Dirac mass at zero
lineal, $\Lambda_\mu=0$	$\ell=0$	Folded Gaussian $\sigma^2=\int(\log\lambda(f))^2 d\mu$	Folded Gaussian $\sigma^2=\int(\log\lambda(f))^2 d\mu$
lineal, $\Lambda_\mu<0$	$\ell=-\Lambda_\mu>0$	Gaussian $\sigma^2=\int\left(-\log\lambda(f)+\Lambda_\mu\right)^2 d\mu$	Gaussian $\sigma^2=\int\left(-\log\lambda(f)+\Lambda_\mu\right)^2 d\mu$
lineal, $\Lambda_\mu>0$	$\ell=\Lambda_\mu>0$	Gaussian $\sigma^2=\int\left(\log\lambda(f)-\Lambda_\mu\right)^2 d\mu$	Gaussian $\sigma^2=\int\left(\log\lambda(f)-\Lambda_\mu\right)^2 d\mu$
non-elementary (see next sections)	$\ell>0$	Gaussian	Gaussian

4. Non-elementary semigroups

Let us now assume that the semigroup G generated by the support of μ is non-elementary. We recall the following results due to Maher-Tiozzo.

Theorem 4.1 (Maher-Tiozzo [MT18])**.** *Let G be a non-elementary, countable semigroup of isometries of a δ-hyperbolic space X, with hyperbolic (Gromov) boundary ∂X. Let μ be a measure whose support generates G, and let $o\in X$ be a base point. Then for almost every sample path $f_n=g_1\cdot\ldots\cdot g_n$, the sequence $(f_n o)$ converges to a point $\xi\in\partial X$. Moreover, the resulting hitting measure is non-atomic and is the unique μ-stationary measure on the boundary.*

Recall that a measure ν on a G-space M is *μ-stationary* if $\int_G g_\star\nu\ d\mu(g)=\nu$. Moreover, it is *$\mu$-ergodic* if it is not a non-trivial convex combination of μ-stationary probability measures on M.

Theorem 4.2. *Let μ be an atomic non-elementary probability measure on* $\mathrm{Bir}(\mathbb{P}^2)$ *with finite first moment. Then:*

(1) ([MT18]) *there exists $\ell_\mu>0$ such that for a.e. random product $f_n=g_1\cdot\ldots\cdot g_n$, we have:*

$$\lim_{n\to+\infty}\frac{1}{n}\log\deg(f_n)=\ell_\mu.$$

(2) ([MT21]) *Moreover, if* $\deg(f)$ *is bounded on the support of μ, then for almost every sample path, one has:*

$$\lim_{n\to+\infty}\frac{1}{n}\log\lambda_1(f_n)=\ell_\mu.$$

4.1. **The horofunction boundary.** Let us recall the construction of the horofunction compactification of a non-proper hyperbolic space, as developed in [MT18].

Let (X,d) be a metric space and let $o\in X$ be a base point. Then we define for each $x\in X$ the map $\rho_x:X\to\mathbb{R}$

$$\rho_x(z):=d(x,z)-d(x,o)\qquad\text{for } z\in X.$$

The function ρ_x is 1-Lipschitz, and $\rho_x(o)=0$. The assignment $x\mapsto\rho_x$ defines a map $\Phi:X\to\mathrm{Lip}^1(X)$ into the space of 1-Lipschitz functions on X. The *horofunction compactification* $\overline{X}^h$ of X is defined as the closure of $\Phi(X)$ in $\mathrm{Lip}^1(X)$, with respect to the topology of pointwise convergence. If X is separable, then $\overline{X}^h$ is compact and metrizable. Elements of $\overline{X}^h$ are called *horofunctions*, and there are two types of them: *finite horofunctions*, if $\inf_{x\in X}h(x)\in\mathbb{R}$, and *infinite horofunctions* if $\inf_{x\in X}h(x)=-\infty$. We denote as X^h_∞ the space of infinite horofunctions.

Moreover, there is a *local minimum map* $\pi:\overline{X}^h_\infty\to X\cup\partial X$ defined as follows. If $h\in X^h_\infty$, then there exists a sequence $(x_n)\subseteq X$ such that $h(x_n)\to-\infty$. It turns out that such a sequence must converge in the Gromov topology to a point in the Gromov boundary ∂X, and the limit point does not depend on the particular choice of (x_n). Hence, one defines a G-equivariant map $\pi : X^h_\infty \to \partial X$ as

$$\pi(h_x) := \lim_{n\to\infty} x_n \in \partial X.$$

In fact, the local minimum map can also be defined for finite horofunctions, but we do not need it here. By [MT18, Proposition 4.4], any μ-stationary probability measure ν on $\overline{X}^h$ only charges infinite horofunctions, i.e. $\nu(X^h_\infty) = 1$.

4.2. **Central limit theorems for cocycles.** Fix G a semigroup of birational maps of $\mathbb{P}^2$, and let M be a compact G-space. Recall that a *cocycle* is a function $\sigma : G \times M \to \mathbb{R}$ such that

$$\sigma(gh, x) = \sigma(g, hx) + \sigma(h, x) \qquad\qquad \forall g, h \in G, \forall x \in M.$$

A cocycle $\sigma : G \times M \to \mathbb{R}$ has *constant drift* λ if there exists $\lambda \in \mathbb{R}$ such that

$$\int_G \sigma(g, x)\ d\mu(g) = \lambda$$

for any $x \in M$. A cocycle $\sigma : G \times M \to \mathbb{R}$ is *centerable* if it can be written as

$$\sigma(g, x) = \sigma_0(g, x) + \psi(x) - \psi(g \cdot x)$$

where σ_0 is a cocycle with constant drift and where $\psi : M \to \mathbb{R}$ is a bounded, measurable function. Given a cocycle, we denote by $\sigma_{sup}(g) := \sup_{x\in M} |\sigma(g, x)|$. Finally, a cocycle has *unique covariance* v if

$$v^2 = \int_{G\times M} (\sigma(g, x) - \lambda)^2\ d\mu(g)d\nu(x)$$

for any μ-stationary measure ν. Recall the key ingredient in Benoist-Quint's central limit theorem for cocycles ([BQ16a, Theorem 3.4]).

Theorem 4.3 (Central limit theorem for cocycles, I)**.** *Let G be a discrete group, M be a compact metrizable G-space and μ an atomic measure on G. Assume $\sigma : G \times M \to \mathbb{R}$ is a centerable cocycle with drift λ and unique covariance $v \geqslant 0$ and such that*

$$\int_G \sigma^2_{sup}(g)\ d\mu(g) < +\infty.$$

Then for any bounded continuous function F on $\mathbb{R}$, uniformly in $x \in M$, one has:

$$\lim_{n\to+\infty} \int_G F\left(\frac{\sigma(g, x) - n\lambda}{\sqrt{n}}\right) d\mu_n(g) = \frac{1}{\sqrt{2\pi} v} \int_{\mathbb{R}} F(t) e^{-\frac{t^2}{2v^2}}\ dt.$$

However, a more general version of this theorem does not require the cocycle to have unique covariance. Indeed we have the following. As remarked in [Hor18, Remark 1.7], the proof is exactly the same as the proof of [BQ16b, Theorem 4.7].

Theorem 4.4 (Central limit theorem for cocycles, II)**.** *Let G be a discrete group, M be a compact metrizable G-space and μ an atomic measure on G. Let ν be a μ-ergodic, μ-stationary probability measure on M, and let $\sigma : G \times M \to \mathbb{R}$ be a centerable cocycle with drift λ. Then there exists $v \geqslant 0$ such that for ν-a.e. $x \in M$ we have, for any bounded, continuous function F,*

$$\lim_{n\to\infty} \int_G F\left(\frac{\sigma(g, x) - n\lambda}{\sqrt{n}}\right)\ d\mu_n(g) = \frac{1}{\sqrt{2\pi} v} \int_{\mathbb{R}} F(t) e^{-\frac{t^2}{2v^2}}\ dt.$$

Remark 4.5. Let us note that [MT18] define the random walk as $f_n = g_1 \dots g_n$, while [BQ16b], [Hor18] use the definition $f_n = g_n \dots g_1$. In this paper, we define the random walk as $f_n = g_1 \dots g_n$ on the semigroup of rational maps, which, since the pullback is contravariant, induces the random walk $\rho_{f_n} = \rho_{g_n} \dots \rho_{g_1}$ on the space of isometries. Thus, we can use the results of [BQ16b], [Hor18] verbatim. Note finally that the n-step distributions μ_n of the left and right random walk are equal, hence, as far as convergence in probability is concerned, results on one and the other are equivalent. On the other hand, results on almost sure convergence do *not* automatically translate, but we do not directly use them here.

4.3. **The Busemann cocycle.** Let us now define V as the subset of the Picard-Manin space given by $V := \overline{\mathrm{Span}_{f\in G}(f^\star L)}$ and $X := V \cap \mathbb{H}^\infty$. Then X, with the metric d induced by $\mathbb{H}^\infty$, is a geodesic, δ-hyperbolic, and separable (since G is countable) metric space, hence we can construct its horofunction compactification $M := \overline{X}^h$, which is metrizable. Moreover, G acts by isometries on X and by homeomorphisms on M.

Let us define the **Busemann cocycle**, denoted $\beta : G \times \overline{X}^h \to \mathbb{R}$, as

$$\beta(g, x) := h_x(g_* \cdot L),$$

where h_x is the horofunction associated to x. We use the following properties of the Busemann cocycle.

Proposition 4.6. *Let $\beta : G\times \overline{X}^h \to \mathbb{R}$ be the Busemann cocycle, and let μ be an atomic probability measure on the group of isometries of (X, d) with finite second moment. Then there exists $\lambda \in \mathbb{R}$ such that:*

(1) ([Hor18, Corollary 2.7]) *For any μ-stationary measure ν on $\overline{X}^h$,*

$$\int \beta(g, x)\ d\mu(g)d\nu(x) = \lambda.$$

(2) ([Hor18, Proposition 2.8]) *For all $\epsilon > 0$ there exists a sequence $(C_n) \in \ell^1(\mathbb{N})$ such that*

$$\mu_n(g \in G\ :\ |\beta(g, x) - n\lambda| \geqslant \epsilon n) \leqslant C_n$$

for any $x \in \overline{X}^h$.

Let us recall that there is a G-equivariant map $\pi : X^h_\infty \to \partial X$ from the set of infinite horofunctions to the Gromov boundary. We shall exploit the following result [Hor18, Corollary 2.3].

Lemma 4.7. *For all $x, y \in X^h_\infty$ such that $\pi(x) \neq \pi(y)$, there exists $C > 0$ such that for all $g \in G$*

$$d(o, go) - C \leqslant \max\{\beta(g, x), \beta(g, y)\} \leqslant d(o, go).$$

Recall that the *Gromov product* between y and z based at x is $\langle y, z\rangle_x := \frac{d(x,y)+d(x,z)-d(y,z)}{2}$. We use the following basic fact about the Gromov product (see e.g. [MT18, Proposition 5.8]).

Lemma 4.8. *Let (X, d) be a δ-hyperbolic space, and let $o \in X$ be a base point. Then there exists a constant $C > 0$ such that, for any isometry f of X,*

$$|\tau(f) - d(o, fo) + 2\langle fo, f^{-1}o\rangle_o| \leqslant C.$$

Moreover, we use the fact that the Gromov product decays faster than any given function:

Lemma 4.9 (Taylor-Tiozzo [TT16], Lemma 3.4)**.** *Let μ be a non-elementary probability measure on a countable group G of isometries of a δ-hyperbolic space X, let $o \in X$ be a base point and let (f_n) be a random walk driven by μ. Then for any function $\varphi : \mathbb{N} \to \mathbb{R}$ with $\lim_{n\to\infty} \varphi(n) = +\infty$ and $\limsup_{n\to\infty} \frac{\varphi(n)}{n} = 0$, we have*

$$\mathbb{P}\left(\langle f_n o, f_n^{-1} o\rangle_o \geqslant \varphi(n)\right) \to 0.$$

5. Arithmeticity properties of the length spectrum

5.1. Examples of subgroups with arithmetic length spectrum.

Definition 5.1. *We call the* length spectrum *of a semigroup* $G < \mathrm{Bir}(\mathbb{P}^2)$ *the set*

$$LS(G) := \{\log \lambda_1(g) \ : \ g \in G\}.$$

Then, we say G *has* arithmetic length spectrum *if there exists* $a \in \mathbb{R}$ *such that* $LS(G) \subseteq a\mathbb{N}$*. Otherwise, we say the length spectrum of* G *is* non-arithmetic*.*

Let us note for discrete subgroups of the group of isometries of a finite dimensional hyperbolic space, the length spectrum can never be arithmetic unless the group is elementary, as shown by [Dal99] in dimension 2 and [Kim06] in any dimension. We will now show, however, that in infinite dimension, as in the case of the Cremona group, there exist non-elementary subgroups with arithmetic length spectrum, proving Proposition 1.1 from the introduction.

Proposition 5.2. *There exist non-elementary subgroups of* $\mathrm{Bir}(\mathbb{P}^2)$ *which have arithmetic length spectrum.*

Consider two polynomials $P_1, P_2 \in \mathbb{C}[x]$ of degree d_1, d_2 such that $\deg(P_1 \pm P_2) = \max(\deg(P_1), \deg(P_2))$. Take two elementary maps $e_i := (x, y) \mapsto (x + P_i(y), y)$ where $i = 1, 2$ and take $a := (x, y) \mapsto (2x + y, x + y)$ the cat map. Consider $F := e_1 \circ a$ and $G := a \circ e_2 \circ a^2$.

Lemma 5.3. *The length spectrum of the group* $\Gamma := \langle F, G\rangle$ *is* $\mathbb{N} \log d_1 + \mathbb{N} \log d_2$*. Moreover, when* $d_1 = d_2 = d$*, any word* g *of length* N *in* F, G *(without inverses) satisfies* $\log \lambda_1(g) = N \log d$*.*

Proof. Denote by E the subgroup of elements of the form $(\alpha x + P(y), \beta y + \gamma)$ where $\alpha, \beta \in \mathbb{C}^*, \gamma \in \mathbb{C}$ and let us denote by A the group of affine transformations. Any non-trivial element g of Γ can be conjugated to $g' = F^{i_1} G^{j_1} \dots F^{i_n} G^{j_n}$, where $i_1, j_n \in \mathbb{Z}$, $i_2, \dots, i_n, j_1, \dots, j_{n-1} \in \mathbb{Z}^*$ and so that the word is cyclically reduced (i.e., the last letter is not the inverse of the first one). Since $a \notin A \cap E$, this element can be decomposed into an alternating product of elements in $\{e_1, e_2, e_1^{-1}, e_2^{-1}, e_2e_1, e_1^{-1}e_2^{-1}\} \subset E \setminus (A \cap E)$ and elements in $\{a^k\}_{k\in\mathbb{Z}^*} \subset A \setminus (A \cap E)$. Using [FM89, Theorem 2.1], we deduce that the degree of g' is of the form $d_1^k d_2^l$ where $k, l \in \mathbb{N}$. Moreover, since g' is cyclically reduced, $\deg((g')^n) = (\deg(g'))^n$ for any $n \geqslant 1$; hence, $\lambda_1(g) = \lambda_1(g')$ is also of the form $d_1^k d_2^l$, as required.

Let us now assume that g is a word of length N in F, G; this means that the indices $i_1, \dots, i_n, j_1, \dots, j_n$ are non-negative integers with $N = \sum_{k=1}^n (i_k + j_k)$. Since g is already in reduced form, by [FM89, Theorem 2.1] we obtain

$$\log \deg(g) = \sum_{k=1}^n (i_k + j_k) \log d = N \log d,$$

and the fact that $\log \deg(g^p) = p \log \deg(g)$ for all $p \geqslant 1$ implies that

$$\log \lambda_1(g) = N \log d.$$

□

We say a polynomial automorphism of $\mathbb{C}^2$ is of *Hénon type* if it is conjugate via a polynomial automorphism of $\mathbb{C}^2$ to a product $g_1 \cdot \dots \cdot g_m$ where g_i are of the form $(x, y) \mapsto (y, a_i x + P_i(y))$ with $a_i \in \mathbb{C}^*, P_i \in \mathbb{C}[y]$ of degree $\geqslant 2$. The following is due to Lamy:

Lemma 5.4 ([Lam01], Proposition 4.3 and Corollary 4.5)**.** *Let* f, g *be two automorphisms of Hénon type such that* $\lambda_1(fgf^{-1}g^{-1}) \geqslant 2$*. Then the group* $\Gamma = \langle f, g\rangle$ *contains a free subgroup* Γ' *of rank* 2 *such that every element* $g \in \Gamma' \setminus \{\mathrm{Id}\}$ *satisfies* $\lambda_1(g) \geqslant 2$*.*

Lemma 5.5. *Any group $\Gamma = \langle f, g\rangle$ constructed as in Lemma 5.4 has a non-elementary action on $\mathbb{H}^\infty$.*

Proof. By Lemma 5.4, Γ contains a free subgroup Γ' of rank 2 such that every element $f \in \Gamma'$ except the identity has dynamical degree $\lambda_1(f) > 1$. Assume that the action of Γ on $\mathbb{H}^\infty$ is elementary: then $f^*\alpha = \lambda(f)\alpha$ for all $f \in \Gamma$, where $\lambda(f) \in \mathbb{R}^*$ and $\alpha \in \partial\mathbb{H}^\infty \cup \mathbb{H}^\infty$ is an invariant class. Then $\psi(f) := \log \lambda(f)$ is a morphism of groups $\psi\colon \Gamma' \to (\mathbb{R}, +)$. Since all non-trivial elements of Γ' are loxodromic on $\mathbb{H}^\infty$, the kernel of ψ is reduced to the identity, hence ψ is injective, and this contradicts the fact that Γ' is a free group. □

Proof of Proposition 5.2. Let us consider two polynomials $P_1(x), P_2(x)$ in one variable of the same degree $d \geqslant 2$, so that both $P_1(x) + P_2(x)$ and $P_1(x) - P_2(x)$ still have degree d. Then if we define F, G as above, by Lemma 5.4 the group Γ generated by F, G contains a free group. Using Lemma 5.5, the action of Γ on $\mathbb{H}^\infty$ is non-elementary. Moreover, Lemma 5.3 shows that Γ has arithmetic length spectrum, contained in $\mathbb{N}\log d$. □

Remark 5.6. Note that if one is only interested in producing a non-elementary *semigroup* with arithmetic length spectrum, it is enough to consider *any* semigroup generated by two automorphisms of Hénon type with distinct geodesic axes. Our example satisfies the stronger requirement that the *group* generated by F, G has arithmetic length spectrum.

5.2. **Proof of Theorem A for non-elementary subgroups.** We are now ready to complete the proof of our main theorem.

Proof of Theorem A; non-elementary case. Let us first prove the CLT for $\log\deg(f_n)$. By [Hor18, Proposition 1.5] the cocycle $\beta(g, x)$ is centerable, as a consequence of Proposition 4.6. Now, since M is compact, there exists a μ-stationary measure ν on M, and by taking its ergodic components we can assume that ν is μ-ergodic. By [MT18, Proposition 4.4], stationarity implies $\nu(\overline{X}^h_\infty) = 1$.

Moreover, by Theorem 4.4 we obtain that for ν-almost every $x \in M$, the sequence $(\beta(f_n, x))$ satisfies a central limit theorem. Then, by taking a generic $x \in \overline{X}^h_\infty$ and applying Lemma 4.7, we obtain a central limit theorem for

$$d(L, \rho_{f_n}(L)).$$

We have by definition

$$\log\deg(f_n) = \log(f_n^* L \cdot L)$$

thus, since

$$(f_n^* L \cdot L) \geqslant 1,$$

we get:

$$\begin{aligned}\log\deg(f_n) &= \cosh^{-1}(f_n^* L \cdot L) + O(1)\\ &= d(L, \rho_{f_n}(L)) + O(1),\end{aligned}$$

so we also obtain the central limit theorem for $(\log\deg f_n)$.

Let us now prove it for $(\log\lambda_1(f_n))$. As we just proved, the sequence

$$\frac{d(L, \rho_{f_n}(L)) - n\ell}{\sqrt{n}} \tag{25}$$

converges in distribution to a Gaussian. Note that

$$\log\lambda_1(f) = \tau(f)$$

and by Lemma 4.8,

$$\log\lambda_1(f) = d(L, \rho_f(L)) - 2\langle\rho_f(L), \rho_{f^{-1}}(L)\rangle_L + O(1).$$

Now, by Lemma 4.9 applied to $\varphi(n) = \sqrt{n}$,

$$\mathbb{P}\left(\frac{\langle \rho_{f_n}(L), (\rho_{f_n})^{-1}(L)\rangle_L}{\sqrt{n}} \geqslant \epsilon\right) \to 0. \tag{26}$$

Hence, by combining (25) and (26), we get a CLT for $\log \lambda_1(f_n)$, as required.

Finally, if $\sigma = 0$, then, as in [BQ16b, Proof of Theorem 4.7], there exists λ such that

$$\tau(g) = n\lambda \tag{27}$$

for any g in the support of μ_n. This implies that G has arithmetic length spectrum, completing the proof. □

Proof of Proposition 1.1. Let F, G as in Proposition 5.2, and let μ be a probability measure supported on $\{F, G\}$. Then for any sample path w_n is a word of length n in the semigroup generated by F, G, hence by Lemma 5.3 we have

$$\tau(w_n) = n \log d$$

which implies $\sigma = 0$. □

Finally, let us show that such counterexamples cannot exist if we assume the measure μ to be symmetric.

Lemma 5.7. *If the support of μ is symmetric and contains at least one loxodromic element, then $\sigma > 0$.*

Proof. If $\sigma = 0$, by (27) there exists λ such that

$$\tau(g) = n\lambda$$

for any n and any g in the support of μ_n. If g is a loxodromic element in the support of μ, then

$$\tau(g) = \lambda > 0.$$

On the other hand, $e = gg^{-1}$ belongs to the support of μ_2, hence

$$\tau(e) = \tau(gg^{-1}) = 2\lambda.$$

But $\tau(e) = 0$, which implies $\lambda = 0$, contradicting the fact that g is loxodromic. □

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