Central limit theorems for counting measures in coarse negative curvature

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Summary

1. Experimental results - geometric length
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2. The CLM conjecture
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3. History
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6. Proof techniques
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joint with Ilya Gekhtman and Sam Taylor
Distribution of geometric lengths - notation

Let us consider a pair of pants $\Sigma$ of cuff lengths $A, B, C$. 

![Diagram of a pair of pants with labels A, B, C]
Distribution of geometric lengths
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and for a conjugacy class $\gamma$ is $\|\gamma\| := \min_{[g]=\gamma} \|g\|$. 

Denote as $\tau(\gamma)$ the hyperbolic length of the closed geodesic in $\Sigma$ corresponding to $\gamma$. 

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Distribution of geometric lengths
Distribution of geometric lengths (Chas-Li-Maskit, ’13)

**Figure 1.** Histograms of the geometric length of a sample of 100,000 words of word length 100. The parameters are \((A, B, C)\); \((1, 1, 1)\) top left, \((0.1, 1, 1)\) top right, \((1, 10, 0.1)\) bottom right; \((0.1, 1, 10)\) bottom left.
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**Figure 2.** Top left: Histogram of all words of word length 14, with metric \((1, 1, 5)\). Top right, bottom left and bottom right respectively, are histograms of the geometric length of a sample of 100,000 words with parameters \((1, 1, 5)\) and word length 20, 50 and 100 respectively.
Fig. 2 A histogram showing the distribution of self-intersection numbers over all reduced cyclic words of length 19 in the doubly punctured plane. The horizontal coordinate shows the self-intersection count $k$; the vertical coordinate shows the number of cyclic reduced words for which the self-intersection number is $k$. 

Distribution of self-intersections (Chas-Lalley, 2011)
The conjecture

Let us consider a pair of pants $\Sigma$ of cuff lengths $A, B, C$. 

Fix a standard generating set $S$ for $G := \pi_1(\Sigma)$. Each closed geodesic on $\Sigma$ is represented by a conjugacy class $\gamma$. Denote $\|\gamma\|$ the word length of $\gamma$. $\tau(\gamma)$ the hyperbolic length of the closed geodesic corresponding to $\gamma$.

Conjecture (Chas-Li-Maskit, '13)

Let $\lambda_n$ be the uniform distribution on the set of conjugacy classes of length $n$. Then there exists $L = L(A, B, C) > 0$ and $\sigma = \sigma(A, B, C) > 0$ such that for any $a < b$:

$$\lambda_n(\gamma : a \leq \tau(\gamma) - nL \sqrt{n} \leq b) \rightarrow \sqrt{\frac{2}{\pi}} \int_b^a e^{-t^2/2} dt$$

as $n \to \infty$. 

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\[
\lambda_n(\gamma : a \leq \tau(\gamma) - nL\sigma \sqrt{n} \leq b) \to \frac{1}{\sqrt{2\pi}} \int_b^a e^{-t^2/2} dt \text{ as } n \to \infty.
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Let us consider a pair of pants $\Sigma$ of cuff lengths $A, B, C$. Fix a standard generating set $S$ for $G := \pi_1(\Sigma)$. Each closed geodesic on $\Sigma$ is represented by a conjugacy class $\gamma$. Denote

- $||\gamma||$ the word length of $\gamma$

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$$\lambda_n(\gamma: a \leq \tau(\gamma) - nL \sigma \sqrt{n} \leq b) \to 1$$

as $n \to \infty$. 

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Distribution of closed geodesics

Let us consider a pair of pants $\Sigma$ of cuff lengths $A, B, C$. Fix a standard generating set for $G := \pi_1(\Sigma)$. Each closed geodesic on $\Sigma$ is represented by a conjugacy class $\gamma$. Denote

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**Conjecture (Gekhtman-Taylor-T, ’18)**

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History

- Sinai (1960) - CLT for geodesic flow in constant negative curvature

For the word metric:
- Pollicott-Sharp (1998)

For quasimorphisms:
- Horsham-Sharp (2009)
- Calegari-Fujiwara (2010)
- Björklund-Hartnick (2011)
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Basic definitions

A metric space is $\delta$-hyperbolic if triangles are $\delta$-thin.
Definitions

Let $(X, d)$ be a geodesic, $\delta$-hyperbolic, metric space, $o \in X$ a base point.
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The (stable) [translation length of \(g\)](##) is

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\tau(g) := \lim_{n \to \infty} \frac{d(o, g^n o)}{n}
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An element is loxodromic (hyperbolic) if \(\tau(g) > 0\).
Main result

Theorem (Gekhtman-Taylor-T. ’20)

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Let $(G, S)$ be a finitely generated group admitting a thick bicombing for $S$. Let $G \acts X$ be a non-elementary isometric action on a hyperbolic metric space.
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Let $(G, S)$ be a finitely generated group admitting a thick bicombing for $S$. Let $G \acts X$ be a non-elementary isometric action on a hyperbolic metric space.

1. Then there exists $\ell > 0$, $\sigma \geq 0$ such that

$$
\lim_{n \to \infty} \frac{1}{\#S_n} \# \left\{ g \in S_n : \frac{d(o, go) - n\ell}{\sqrt{n}} \in [a, b] \right\} = \int_a^b e^{-\frac{t^2}{2\sigma^2}} \, dt
$$

Moreover,

2. \hspace{1cm} \lim_{n \to \infty} \frac{1}{\#S_n} \# \left\{ g \in S_n : \tau(g) - n\ell \sqrt{n} \in [a, b] \right\} = \int_a^b e^{-\frac{t^2}{2\sigma^2}} \, dt

3. Further, $\sigma = 0$ if and only if there exists $C > 0$ s.t. $|d(o, go) - \ell \| g \| | \leq C$ for all $g \in G.$
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for all \(g \in G\).
Applications (I) Geodesic lengths in geometrically finite manifolds

Theorem

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\[ \ell(\gamma) \sim \sqrt{n} \sigma \]

If $\pi_1(M)$ is word hyperbolic, then we can take $S' = S$.

Already new for finite volume surfaces with cusps

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\[
\frac{\ell(\gamma) - n\ell}{\sqrt{n}} \to \mathcal{N}_\sigma
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where $\gamma$ is chosen uniformly at random in the sphere of radius $n$ with respect to $S'$. 
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Applications (I') Geodesic lengths in geometrically infinite 3-manifolds

**Theorem**

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Proof (1): [Tameness] + [Thurston’s hyperbolization] $\Rightarrow \pi_1(M)$ hyperbolic
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Let \( M = \mathbb{H}^3 / \Gamma \) be a hyperbolic 3-manifold (possibly geometrically infinite). For \( \gamma \in \Gamma = \pi_1(M) \), let \( \ell(\gamma) \) be the length of the geodesic in the free homotopy class of \( \gamma \).

- If \( M \) has no rank 2 cusps, for any \( S \) we have

\[ \ell(\gamma) - n \ell(\gamma) \sqrt{n} \to N \]

where \( \gamma \) is chosen uniformly at random in the sphere of radius \( n \) with respect to \( S \).

- If \( M \) has rank 2 cusps, for any \( S \) there is \( S' \supseteq S \) such that the CLT holds for \( S' \).

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Proof (1): [Tameness] + [Thurston’s hyperbolization] $\Rightarrow \pi_1(M)$ hyperbolic
Applications (II) Intersection with submanifolds

Let $M$ be a hyperbolic manifold and let $S$ be any generating set for $\pi_1(M)$. Let $\Sigma$ be a (smooth, orientable) codimension-1 submanifold, and let $i(\gamma, \Sigma)$ be the intersection number.

**Theorem**
Suppose that $\Sigma \to M$ is $\pi_1$-injective but not fiber-like. Then there are $\ell, \sigma > 0$ such that $i(\gamma, \Sigma) - \ell n^{\sqrt{n}} \to N \sigma$, where $\gamma$ is chosen uniformly at random in the sphere of radius $n$ with respect to $S$.

▶ Example. $M$ compact surface of genus $g \geq 2$, $\Sigma$ an essential simple closed curve.▶ The action is on a non-proper metric space
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Let $M$ be a hyperbolic manifold and let $S$ be any generating set for $\pi_1(M)$. Let $\Sigma$ be a (smooth, orientable) codimension-1 submanifold, and let $i(\gamma, \Sigma)$ be the intersection number.

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\begin{itemize}
  \item locally infinite tree
\end{itemize}
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Fig. 2 A histogram showing the distribution of self-intersection numbers over all reduced cyclic words of length 19 in the doubly punctured plane. The horizontal coordinate shows the self-intersection count $k$; the vertical coordinate shows the number of cyclic reduced words for which the self-intersection number is $k$. 

Distribution of self-intersections (Chas-Lalley, 2011)
Applications (III) Homomorphisms between hyperbolic groups

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Let $\phi : G \to G'$ be a homomorphism between hyperbolic groups.
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Let $\phi: G \to G'$ be a homomorphism between hyperbolic groups. Recall $\partial G := \{\text{geodesic rays based at } o\}/\sim$. 

Moreover, for $s > v$, $\nu_s := \sum_g e^{-sd(o, go)} \delta_{go} \sum_g e^{-sd(o, go)}$. 

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In the CLT we have $\sigma = 0$ if and only if $\phi$ has finite kernel and $\partial \phi : \partial G \to \partial G'$ pushes the PS measure class for $(G, S)$ to the PS measure class for $(\phi(G), S')$. 

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Techniques - Graph structures

A graph structure is \((\Gamma, v_0, ev)\) with:

- \(\Gamma\) is a finite graph
- \(v_0\) is a vertex of \(\Gamma\) (initial vertex)
- \(ev: E(\Gamma) \rightarrow G\) the evaluation map

If \(v\) is a vertex, \(\Gamma_v\) is the loop semigroup

- \(M:\) adjacency matrix
- \(\lambda:\) leading eigenvalue

A component is maximal if its growth rate is \(\lambda\).

Definition

A graph structure is thick if for any \(v\) in a maximal component there exists \(B \subseteq G\) finite such that \(G = B \cdot ev(\Gamma_v) \cdot B\)

Thick \(\Rightarrow\) almost semisimple: for every maximal eigenvalue, its algebraic and geometric multiplicities agree

Note.

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\[ \text{for any } B \subseteq G \text{ finite } \exists C \geq 0 \text{ such that if } g_1, h_1 \text{ are finite length paths with } \text{ev}(g_1) = b_1 \cdot \text{ev}(h_1) \cdot b_2 \text{ then } d_G(g_1, b_1 h_1) \leq C \text{ for all } i \leq \max \{ \|g\|, \|h\| \}. \]
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Definition
A group has a thick bicombing for $S$ if it has a thick, biautomatic graph structure for $S$ such that paths are geodesic for the word length $\| \cdot \|_S$. 
1. **Hyperbolic groups** have thick bicombings for every generating set $S$ (Cannon)
Graph structures - examples

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3. **Right-angled Artin/Coxeter groups** (Hermiller-Meier)
Structure of the proof

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Step 1: CLT for cocycles

Let $\mathcal{M}$ a metric space on which $G$ acts continuously.

A cocycle $\eta$ is $\mu$-stationary if

$$\int g^* \nu \, d\mu(g) = \nu.$$
Step 1: CLT for cocycles

Let $\mathcal{M}$ a metric space on which $G$ acts continuously. A cocycle is $\eta : G \times \mathcal{M} \to \mathbb{R}$ with

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A cocycle $\eta$ is **centerable** if it can be written as

$$\eta(g, x) = \eta_0(g, x) + \psi(x) - \psi(g \cdot x)$$

where $\eta_0$ is a cocycle with constant drift and $\psi : \mathcal{M} \to \mathbb{R}$ a bounded, measurable function.
Theorem (Benoist-Quint + Horbez)

Let $\nu$ be a $\mu$-ergodic, $\mu$-stationary probability measure on $\mathcal{M}$, and let $\eta : G \times \mathcal{M} \to \mathbb{R}$ be a centerable cocycle with drift $\lambda$ and finite second moment. Then there exist $\sigma \geq 0$ such that for any continuous $F : \mathbb{R} \to \mathbb{R}$ with compact support, we have for $\nu$-a.e. $x \in \mathcal{M}$,

$$\lim_{n \to \infty} \int G \left( F\left( \sigma(\cdot, x) - n\lambda \sqrt{n} \right) \right) d\mu^* n(g) = \int \mathbb{R} F(t) dN_{\sigma}(t).$$
Central limit theorem for centerable cocycles

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\[ \lim_{n \to \infty} \int_{G} F(\sigma(g, x) - n\lambda \sqrt{n}) \, d\mu^\ast n(g) = \int_{\mathbb{R}} F(t) \, dN_{\sigma}(t). \]
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The Busemann cocycle

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Then \( \mathcal{M} = \overline{X^h} \) is the closure of \( \rho(X) \) for the topology of pointwise convergence.
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The Busemann cocycle

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\beta(g, \xi) := \lim_{z_n \to \xi} (d(o, z_n) - d(g^{-1}o, z_n))
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The Busemann cocycle
\[ \beta(g, \xi) := \lim_{z_n \to \xi} (d(o, z_n) - d(g^{-1}o, z_n)) \]

**Proposition (Horbez)**
*The Busemann cocycle is centerable.*
Step 1: Random walks on the loop semigroup

Fix a graph structure $\Gamma$ and let $\nu$ be a vertex in a maximal growth component.
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Fix a graph structure $\Gamma$ and let $v$ be a vertex in a maximal growth component. Fix a measure $\mu$ on the set of edges of $\Gamma$. 

Definition

The loop semigroup $\Gamma_v$ is the set of all loops from $v$ to $v$.

The first return measure is $\mu_v(l) = \mu(g_1) \cdots \mu(g_n)$ if $l = g_1 \cdots g_n$.

Theorem

Let $\nu_v$ be a $\mu_v$-ergodic, $\mu_v$-stationary measure on $X$. Then there exist $\ell, \sigma \geq 0$ such that for $\nu_v$-a.e. $\xi$

$$\lim_{n \to \infty} \int_G F(\beta(\xi(o, go)) - \ell \|g\| \sqrt{n}) d\mu^*_{\nu_v}(g) = \int_R F(t) dN_\sigma(t)$$

for any $F \in C_c(R)$. 

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Fix a graph structure \( \Gamma \) and let \( \nu \) be a vertex in a maximal growth component. Fix a measure \( \mu \) on the set of edges of \( \Gamma \).

**Definition**
The loop semigroup \( \Gamma_\nu \) is the set of all loops from \( \nu \) to \( \nu \). The first return measure is

\[
\mu_\nu(l) = \mu(g_1) \cdots \mu(g_n)
\]

if \( l = g_1 \cdots g_n \).

**Theorem**
Let \( \nu_\nu \) be a \( \mu_\nu \)-ergodic, \( \mu_\nu \)-stationary measure on \( X^h \). Then there exist \( \ell, \sigma \geq 0 \) such that for \( \nu_\nu \)-a.e. \( \xi \)

\[
\lim_{n \to \infty} \int_G F \left( \frac{\beta_\xi(o, go) - \ell \|g\|}{\sqrt{n}} \right) d\mu_\nu^n(g) = \int_\mathbb{R} F(t) \, d\mathcal{N}_\sigma(t)
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for any \( F \in C_c(\mathbb{R}) \).
Step 2: Suspension to the Markov chain

Let \( S : (\mathcal{X}, \lambda) \to (\mathcal{X}, \lambda) \), and let \( r : \mathcal{X} \to \mathbb{N} \) be a roof function. Then the discrete suspension flow of \( S \) with roof function \( r \) is \( \hat{S} : \hat{\mathcal{X}} \to \hat{\mathcal{X}} \) where

\[
\hat{\mathcal{X}} := \{(x, n) \in \mathcal{X} \times \mathbb{N} : 0 \leq n \leq r(x) - 1\}
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Then, the map \( \hat{S} \) is defined as

\[
\hat{S}(x, n) = \begin{cases} 
(x, n + 1) & \text{if } n \leq r(x) - 2 \\
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Theorem (Melbourne-Török)

Let \( S: (\mathcal{X}, \lambda) \to (\mathcal{X}, \lambda) \) be ergodic, and let \( \hat{S}: (\hat{\mathcal{X}}, \hat{\lambda}) \to (\hat{\mathcal{X}}, \hat{\lambda}) \) be the suspension flow with roof function \( r \). Let \( \phi: \hat{\mathcal{X}} \to \mathbb{R} \) and define \( \Phi(x) := \sum_{k=0}^{r(x) - 1} \phi(x, k) \).

Suppose that \( \Phi \) and \( r \) satisfy a CLT. Then \( \phi \) satisfies a CLT.
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CLT for Markov chain

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Step 3: Approximating the counting measure

Suppose $M$ is semisimple (there is only one maximal eigenvalue).
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**Lemma**

$$\|\mu_{n-2\log n} - \tilde{\lambda}_n\|_{TV} \to 0$$
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Lemma

$$\|\mu_{n-2\log n} - \tilde{\lambda}_n\|_{TV} \to 0$$
Definition
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$$\|g\| \geq N$$

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Theorem
(\text{CLT for displacement}) \textbf{There exists} $\ell > 0$, $\sigma \geq 0$ \textbf{such that for any} $a < b$ \textbf{we have}
\[
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The end

감사합니다

Thank you!!!