Central limit theorems for counting measures in coarse negative curvature

Giulio Tiozzo
University of Toronto

May 27, 2020

Summary

1. Experimental results - geometric length

Summary

1. Experimental results - geometric length
2. The CLM conjecture

Summary

1. Experimental results - geometric length
2. The CLM conjecture
3. History

Summary

1. Experimental results - geometric length
2. The CLM conjecture
3. History
4. Main result

Summary

1. Experimental results - geometric length
2. The CLM conjecture
3. History
4. Main result
5. Applications

Summary

1. Experimental results - geometric length
2. The CLM conjecture
3. History
4. Main result
5. Applications
6. Proof techniques

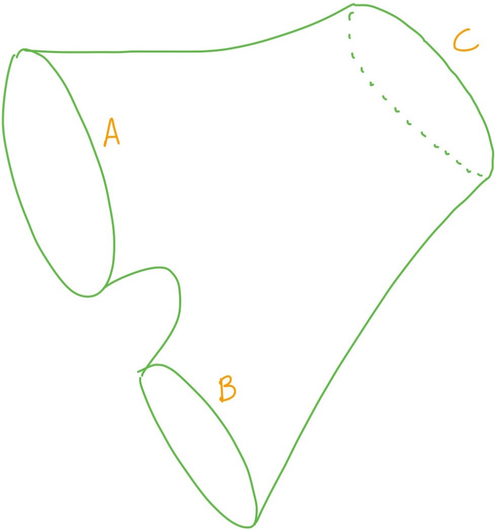
Summary

1. Experimental results - geometric length
2. The CLM conjecture
3. History
4. Main result
5. Applications
6. Proof techniques

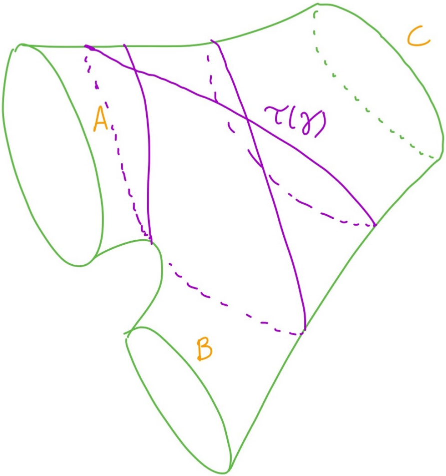
joint with Ilya Gekhtman and Sam Taylor

Distribution of geometric lengths - notation

Let us consider a pair of pants Σ of cuff lengths A, B, C.



Distribution of geometric lengths



Distribution of geometric lengths - notation

- Then $G := \pi_1(\Sigma)$ is a free group, fix a standard generating set.

Distribution of geometric lengths - notation

- Then $G := \pi_1(\Sigma)$ is a free group, fix a standard generating set.
- Each closed geodesic on Σ is represented by a conjugacy class γ.

Distribution of geometric lengths - notation

- Then $G := \pi_1(\Sigma)$ is a free group, fix a standard generating set.
- Each closed geodesic on Σ is represented by a conjugacy class γ.
- The <u>word length</u> of $g \in G$ is

$$\|g\| := \min\{k \; : \; g = s_1 \dots s_k \; : \; s_i \in S\}$$

Distribution of geometric lengths - notation

- Then $G := \pi_1(\Sigma)$ is a free group, fix a standard generating set.
- Each closed geodesic on Σ is represented by a conjugacy class γ.
- The <u>word length</u> of $g \in G$ is

$$\|g\| := \min\{k \ : \ g = s_1 \dots s_k \ : \ s_i \in S\}$$

and for a conjugacy class γ is $\|\gamma\| := \min_{[g]=\gamma} \|g\|$.

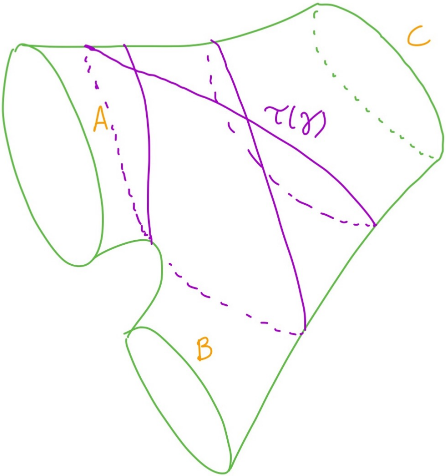
Distribution of geometric lengths - notation

- Then $G := \pi_1(\Sigma)$ is a free group, fix a standard generating set.
- Each closed geodesic on Σ is represented by a conjugacy class γ.
- The word length of $g \in G$ is

$$\|g\| := \min\{k \ : \ g = s_1 \dots s_k \ : \ s_i \in S\}$$

  and for a conjugacy class γ is $\|\gamma\| := \min_{[g]=\gamma} \|g\|$.
- Denote as $\tau(\gamma)$ the hyperbolic length of the closed geodesic in Σ corresponding to γ.

Distribution of geometric lengths



Distribution of geometric lengths (Chas-Li-Maskit, '13)

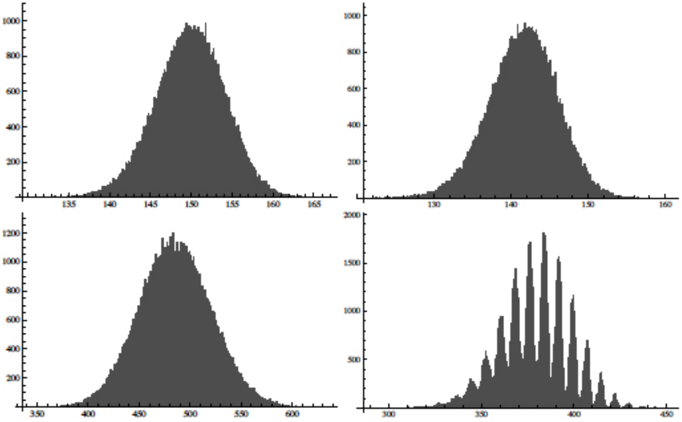


FIGURE 1. Histograms of the geometric length of a sample of 100,000 words of word length 100. The parameters are (A, B, C); $(1, 1, 1)$ top left, $(0.1, 1, 1)$ top right, $(1, 10, 0.1)$ bottom right; $(0.1, 1, 10)$ bottom left

Distribution of geometric lengths (Chas-Li-Maskit, '13)

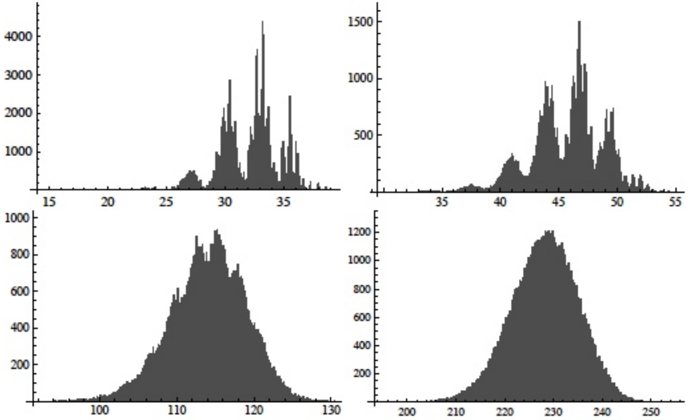


FIGURE 2. Top left: Histogram of all words of word length 14, with metric $(1,1,5)$. Top right, bottom left and bottom right respectively, are histograms of the geometric length of a sample of 100,000 words with parameters $(1,1,5)$ and word length 20, 50 and 100 respectively.

Distribution of self-intersections (Chas-Lalley, 2011)

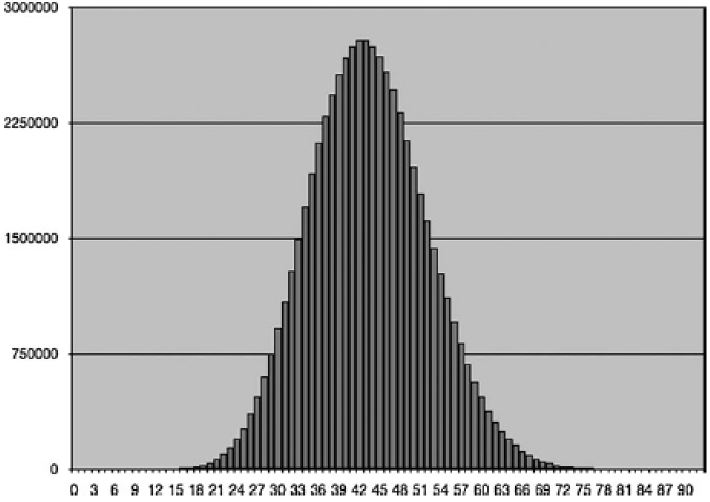


Fig. 2 A *histogram* showing the distribution of self-intersection numbers over all reduced cyclic words of length 19 in the doubly punctured plane. The *horizontal coordinate* shows the self-intersection count k; the *vertical coordinate* shows the number of cyclic reduced words for which the self-intersection number is k

The conjecture

Let us consider a pair of pants Σ of cuff lengths A, B, C.

The conjecture

Let us consider a pair of pants Σ of cuff lengths A, B, C. Fix a standard generating set S for $G := \pi_1(\Sigma)$.

The conjecture

Let us consider a pair of pants Σ of cuff lengths A, B, C. Fix a standard generating set S for $G := \pi_1(\Sigma)$. Each closed geodesic on Σ is represented by a conjugacy class γ.

The conjecture

Let us consider a pair of pants Σ of cuff lengths A, B, C. Fix a standard generating set S for $G := \pi_1(\Sigma)$. Each closed geodesic on Σ is represented by a conjugacy class γ. Denote

- $\|\gamma\|$ the word length of γ

The conjecture

Let us consider a pair of pants Σ of cuff lengths A, B, C. Fix a standard generating set S for $G := \pi_1(\Sigma)$. Each closed geodesic on Σ is represented by a conjugacy class γ. Denote

- $\|\gamma\|$ the word length of γ
- $\tau(\gamma)$ the hyperbolic length of the closed geodesic corresponding to γ

The conjecture

Let us consider a pair of pants Σ of cuff lengths A, B, C. Fix a standard generating set S for $G := \pi_1(\Sigma)$. Each closed geodesic on Σ is represented by a conjugacy class γ. Denote

- $\|\gamma\|$ the word length of γ
- $\tau(\gamma)$ the hyperbolic length of the closed geodesic corresponding to γ

Conjecture (Chas-Li-Maskit, '13)

Let λ_n be the uniform distribution on the set of conjugacy classes of length n.

The conjecture

Let us consider a pair of pants Σ of cuff lengths A, B, C. Fix a standard generating set S for $G := \pi_1(\Sigma)$. Each closed geodesic on Σ is represented by a conjugacy class γ. Denote

- $\|\gamma\|$ the word length of γ
- $\tau(\gamma)$ the hyperbolic length of the closed geodesic corresponding to γ

Conjecture (Chas-Li-Maskit, '13)

Let λ_n be the uniform distribution on the set of conjugacy classes of length n. Then there exists $L = L(A, B, C) > 0$ and $\sigma = \sigma(A, B, C) > 0$ such that for any $a < b$

$$\lambda_n\left(\gamma \ : \ a \leq \frac{\tau(\gamma) - nL}{\sigma\sqrt{n}} \leq b\right) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{t^2}{2}} \, dt$$

as $n \to \infty$.

Distribution of closed geodesics

Let us consider a pair of pants Σ of cuff lengths A, B, C. Fix a standard generating set for $G := \pi_1(\Sigma)$. Each closed geodesic on Σ is represented by a conjugacy class γ. Denote

- $\|\gamma\|$ the word length of γ
- $\tau(\gamma)$ the hyperbolic length of the closed geodesic corresponding to γ

Conjecture (Gekhtman-Taylor-T, '18)

Let λ_n be the uniform distribution on the set of conjugacy classes of length n. Then there exists $L = L(A, B, C) > 0$ and $\sigma = \sigma(A, B, C) > 0$ such that for any $a < b$

$$\lambda_n\left(\gamma \ : \ a \leq \frac{\tau(\gamma) - nL}{\sigma\sqrt{n}} \leq b\right) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{t^2}{2}} \, dt$$

as $n \to \infty$.

History

- Sinai (1960) - CLT for geodesic flow in constant negative curvature

History

- Sinai (1960) - CLT for geodesic flow in constant negative curvature
- Ratner (1973) - CLT for geodesic flow in variable negative curvature

History

- Sinai (1960) - CLT for geodesic flow in constant negative curvature
- Ratner (1973) - CLT for geodesic flow in variable negative curvature
- Bowen (1975)

History

- Sinai (1960) - CLT for geodesic flow in constant negative curvature
- Ratner (1973) - CLT for geodesic flow in variable negative curvature
- Bowen (1975), Lalley (1987)

History

- Sinai (1960) - CLT for geodesic flow in constant negative curvature
- Ratner (1973) - CLT for geodesic flow in variable negative curvature
- Bowen (1975), Lalley (1987), Parry-Pollicott (1990)

History

- Sinai (1960) - CLT for geodesic flow in constant negative curvature
- Ratner (1973) - CLT for geodesic flow in variable negative curvature
- Bowen (1975), Lalley (1987), Parry-Pollicott (1990)

For the word metric:

- Pollicott-Sharp (1998)

History

- Sinai (1960) - CLT for geodesic flow in constant negative curvature
- Ratner (1973) - CLT for geodesic flow in variable negative curvature
- Bowen (1975), Lalley (1987), Parry-Pollicott (1990)

For the word metric:

- Pollicott-Sharp (1998)
- For quasimorphisms: Horsham-Sharp (2009)

History

- Sinai (1960) - CLT for geodesic flow in constant negative curvature
- Ratner (1973) - CLT for geodesic flow in variable negative curvature
- Bowen (1975), Lalley (1987), Parry-Pollicott (1990)

For the word metric:

- Pollicott-Sharp (1998)
- For quasimorphisms: Horsham-Sharp (2009), Calegari-Fujiwara (2010)

History

- Sinai (1960) - CLT for geodesic flow in constant negative curvature
- Ratner (1973) - CLT for geodesic flow in variable negative curvature
- Bowen (1975), Lalley (1987), Parry-Pollicott (1990)

For the word metric:

- Pollicott-Sharp (1998)
- For quasimorphisms: Horsham-Sharp (2009), Calegari-Fujiwara (2010), Björklund-Hartnick (2011)

History

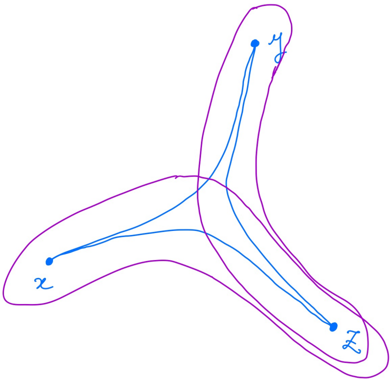
- Sinai (1960) - CLT for geodesic flow in constant negative curvature
- Ratner (1973) - CLT for geodesic flow in variable negative curvature
- Bowen (1975), Lalley (1987), Parry-Pollicott (1990)

For the word metric:

- Pollicott-Sharp (1998)
- For quasimorphisms: Horsham-Sharp (2009), Calegari-Fujiwara (2010), Björklund-Hartnick (2011)
- Cantrell (2019)

Basic definitions

A metric space is δ-hyperbolic if triangles are δ-thin.



Definitions

Let (X, d) be a geodesic, δ-hyperbolic, metric space, $o \in X$ a base point.

Definitions

Let (X, d) be a geodesic, δ-hyperbolic, metric space, $o \in X$ a base point.

Definitions

Let (X, d) be a geodesic, δ-hyperbolic, metric space, $o \in X$ a base point.
Let $G < \text{Isom}(X, d)$. Let S be a finite generating set.

Definitions

Let (X, d) be a geodesic, δ-hyperbolic, metric space, $o \in X$ a base point.
Let $G < \text{Isom}(X, d)$. Let S be a finite generating set. Then the word length is

$$\|g\|_S := \min\{k \; : \; g = s_i \dots s_k, \; s_i \in S\}$$

Definitions

Let (X, d) be a geodesic, δ-hyperbolic, metric space, $o \in X$ a base point.
Let $G < \mathrm{Isom}(X, d)$. Let S be a finite generating set. Then the word length is

$$\|g\|_S := \min\{k \ : \ g = s_i \dots s_k, \ s_i \in S\}$$

and the sphere of radius n is

$$S_n := \{g \in G \ : \ \|g\|_S = n\}$$

Definitions

Let (X, d) be a geodesic, δ-hyperbolic, metric space, $o \in X$ a base point.
Let $G < \text{Isom}(X, d)$. Let S be a finite generating set. Then the word length is

$$\|g\|_S := \min\{k \;:\; g = s_i \dots s_k,\; s_i \in S\}$$

and the sphere of radius n is

$$S_n := \{g \in G \;:\; \|g\|_S = n\}$$

The (stable) translation length of g is

$$\tau(g) := \lim_{n\to\infty} \frac{d(o, g^n o)}{n}$$

Definitions

Let (X, d) be a geodesic, δ-hyperbolic, metric space, $o \in X$ a base point.
Let $G < \text{Isom}(X, d)$. Let S be a finite generating set. Then the word length is

$$\|g\|_S := \min\{k \ : \ g = s_i \dots s_k, \ s_i \in S\}$$

and the sphere of radius n is

$$S_n := \{g \in G \ : \ \|g\|_S = n\}$$

The (stable) translation length of g is

$$\tau(g) := \lim_{n\to\infty} \frac{d(o, g^n o)}{n}$$

An element is loxodromic (hyperbolic) if $\tau(g) > 0$.

Main result

Theorem (Gekhtman-Taylor-T. '20)

Let (G, S) *be a finitely generated group admitting a thick bicombing for* S*.*

Main result

Theorem (Gekhtman-Taylor-T. '20)

Let (G, S) *be a finitely generated group admitting a thick bicombing for* S*. Let* $G \curvearrowright X$ *be a non-elementary isometric action on a hyperbolic metric space.*

Main result

Theorem (Gekhtman-Taylor-T. '20)

Let (G, S) be a finitely generated group admitting a thick bicombing for S. Let $G \curvearrowright X$ be a non-elementary isometric action on a hyperbolic metric space.

1. *Then there exists $\ell > 0, \sigma \geq 0$ such that*

$$\lim_{n\to\infty} \frac{1}{\# S_n} \# \left\{ g \in S_n \ : \ \frac{d(o, go) - n\ell}{\sqrt{n}} \in [a, b] \right\} = \int_a^b e^{-\frac{t^2}{2\sigma^2}} \, dt$$

Main result

Theorem (Gekhtman-Taylor-T. '20)

Let (G, S) be a finitely generated group admitting a thick bicombing for S. Let $G \curvearrowright X$ be a non-elementary isometric action on a hyperbolic metric space.

1. *Then there exists $\ell > 0, \sigma \geq 0$ such that*

$$\lim_{n\to\infty} \frac{1}{\#S_n} \# \left\{ g \in S_n \ : \ \frac{d(o, go) - n\ell}{\sqrt{n}} \in [a, b] \right\} = \int_a^b e^{-\frac{t^2}{2\sigma^2}} \, dt$$

2. *Moreover,*

$$\lim_{n\to\infty} \frac{1}{\#S_n} \# \left\{ g \in S_n \ : \ \frac{\tau(g) - n\ell}{\sqrt{n}} \in [a, b] \right\} = \int_a^b e^{-\frac{t^2}{2\sigma^2}} \, dt$$

Main result

Theorem (Gekhtman-Taylor-T. '20)

Let (G, S) be a finitely generated group admitting a thick bicombing for S. Let $G \curvearrowright X$ be a non-elementary isometric action on a hyperbolic metric space.

1. *Then there exists $\ell > 0, \sigma \geq 0$ such that*

$$\lim_{n\to\infty} \frac{1}{\#S_n} \# \left\{ g \in S_n \ : \ \frac{d(o, go) - n\ell}{\sqrt{n}} \in [a, b] \right\} = \int_a^b e^{-\frac{t^2}{2\sigma^2}} \, dt$$

2. *Moreover,*

$$\lim_{n\to\infty} \frac{1}{\#S_n} \# \left\{ g \in S_n \ : \ \frac{\tau(g) - n\ell}{\sqrt{n}} \in [a, b] \right\} = \int_a^b e^{-\frac{t^2}{2\sigma^2}} \, dt$$

3. *Further, $\sigma = 0$ if and only if exists $C > 0$ s.t.*

$$|d(o, go) - \ell \|g\|| \leq C$$

for all $g \in G$.

Applications (I) Geodesic lengths in geometrically finite manifolds

Theorem

Let $M = \mathbb{H}^n/\Gamma$ *be a geometrically finite hyperbolic manifold.*

Applications (I) Geodesic lengths in geometrically finite manifolds

Theorem

Let $M = \mathbb{H}^n/\Gamma$ be a geometrically finite hyperbolic manifold. For $\gamma \in \Gamma = \pi_1(M)$, let $\ell(\gamma)$ be the length of the geodesic in the free homotopy class of γ.

Applications (I) Geodesic lengths in geometrically finite manifolds

Theorem

Let $M = \mathbb{H}^n/\Gamma$ be a geometrically finite hyperbolic manifold. For $\gamma \in \Gamma = \pi_1(M)$, let $\ell(\gamma)$ be the length of the geodesic in the free homotopy class of γ. Then for any S there exists $S' \supseteq S$ such that for S' we have

Applications (I) Geodesic lengths in geometrically finite manifolds

Theorem

Let $M = \mathbb{H}^n/\Gamma$ be a geometrically finite hyperbolic manifold. For $\gamma \in \Gamma = \pi_1(M)$, let $\ell(\gamma)$ be the length of the geodesic in the free homotopy class of γ. Then for any S there exists $S' \supseteq S$ such that for S' we have

$$\frac{\ell(\gamma) - n\ell}{\sqrt{n}} \to \mathcal{N}_\sigma$$

where γ is chosen uniformly at random in the sphere of radius n with respect to S'.

Applications (I) Geodesic lengths in geometrically finite manifolds

Theorem

Let $M = \mathbb{H}^n/\Gamma$ be a geometrically finite hyperbolic manifold. For $\gamma \in \Gamma = \pi_1(M)$, let $\ell(\gamma)$ be the length of the geodesic in the free homotopy class of γ. Then for any S there exists $S' \supseteq S$ such that for S' we have

$$\frac{\ell(\gamma) - n\ell}{\sqrt{n}} \to \mathcal{N}_\sigma$$

where γ is chosen uniformly at random in the sphere of radius n with respect to S'.

- If $\pi_1(M)$ is word hyperbolic, then we can take $S' = S$.

Applications (I) Geodesic lengths in geometrically finite manifolds

Theorem

Let $M = \mathbb{H}^n/\Gamma$ be a geometrically finite hyperbolic manifold. For $\gamma \in \Gamma = \pi_1(M)$, let $\ell(\gamma)$ be the length of the geodesic in the free homotopy class of γ. Then for any S there exists $S' \supseteq S$ such that for S' we have

$$\frac{\ell(\gamma) - n\ell}{\sqrt{n}} \to \mathcal{N}_\sigma$$

where γ is chosen uniformly at random in the sphere of radius n with respect to S'.

- If $\pi_1(M)$ is word hyperbolic, then we can take $S' = S$.
- Already new for finite volume surfaces with cusps

Applications (I) Geodesic lengths in geometrically finite manifolds

Theorem

Let $M = \mathbb{H}^n/\Gamma$ be a geometrically finite hyperbolic manifold. For $\gamma \in \Gamma = \pi_1(M)$, let $\ell(\gamma)$ be the length of the geodesic in the free homotopy class of γ. Then for any S there exists $S' \supseteq S$ such that for S' we have

$$\frac{\ell(\gamma) - n\ell}{\sqrt{n}} \to \mathcal{N}_\sigma$$

where γ is chosen uniformly at random in the sphere of radius n with respect to S'.

- If $\pi_1(M)$ is word hyperbolic, then we can take $S' = S$.
- Already new for finite volume surfaces with cusps
- $\ell(\gamma)$ is <u>not</u> Hölder

Applications (I) Geodesic lengths in geometrically finite manifolds

Theorem

Let $M = \mathbb{H}^n/\Gamma$ be a geometrically finite hyperbolic manifold. For $\gamma \in \Gamma = \pi_1(M)$, let $\ell(\gamma)$ be the length of the geodesic in the free homotopy class of γ. Then for any S there exists $S' \supseteq S$ such that for S' we have

$$\frac{\ell(\gamma) - n\ell}{\sqrt{n}} \to \mathcal{N}_\sigma$$

where γ is chosen uniformly at random in the sphere of radius n with respect to S'.

- If $\pi_1(M)$ is word hyperbolic, then we can take $S' = S$.
- Already new for finite volume surfaces with cusps
- $\ell(\gamma)$ is <u>not</u> Hölder
- $\sigma > 0$ (length spectrum is not arithmetic)

Applications (I') Geodesic lengths in geometrically infinite 3-manifolds

Theorem

Let $M = \mathbb{H}^3/\Gamma$ *be a hyperbolic 3-manifold*

Applications (I’) Geodesic lengths in geometrically infinite 3-manifolds

Theorem

Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold (possibly geometrically infinite).

Applications (I') Geodesic lengths in geometrically infinite 3-manifolds

Theorem

Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold (possibly geometrically infinite). For $\gamma \in \Gamma = \pi_1(M)$, let $\ell(\gamma)$ be the length of the geodesic in the free homotopy class of γ.

Applications (I') Geodesic lengths in geometrically infinite 3-manifolds

Theorem

Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold (possibly geometrically infinite). For $\gamma \in \Gamma = \pi_1(M)$, let $\ell(\gamma)$ be the length of the geodesic in the free homotopy class of γ.

- *If M has no rank 2 cusps, for any S we have*

Applications (I') Geodesic lengths in geometrically infinite 3-manifolds

Theorem

Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold (possibly geometrically infinite). For $\gamma \in \Gamma = \pi_1(M)$, let $\ell(\gamma)$ be the length of the geodesic in the free homotopy class of γ.

- *If M has no rank 2 cusps, for any S we have*

$$\frac{\ell(\gamma) - n\ell}{\sqrt{n}} \to \mathcal{N}_\sigma$$

where γ is chosen uniformly at random in the sphere of radius n with respect to S.

Applications (I') Geodesic lengths in geometrically infinite 3-manifolds

Theorem

Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold (possibly geometrically infinite). For $\gamma \in \Gamma = \pi_1(M)$, let $\ell(\gamma)$ be the length of the geodesic in the free homotopy class of γ.

- *If M has no rank 2 cusps, for any S we have*

$$\frac{\ell(\gamma) - n\ell}{\sqrt{n}} \to \mathcal{N}_\sigma$$

  where γ is chosen uniformly at random in the sphere of radius n with respect to S.

- *If M has rank 2 cusps, for any S there is $S' \supseteq S$ such that the CLT holds for S'.*

Applications (I') Geodesic lengths in geometrically infinite 3-manifolds

Theorem

Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold (possibly geometrically infinite). For $\gamma \in \Gamma = \pi_1(M)$, let $\ell(\gamma)$ be the length of the geodesic in the free homotopy class of γ.

- *If M has no rank 2 cusps, for any S we have*

$$\frac{\ell(\gamma) - n\ell}{\sqrt{n}} \to \mathcal{N}_\sigma$$

  where γ is chosen uniformly at random in the sphere of radius n with respect to S.

- *If M has rank 2 cusps, for any S there is $S' \supseteq S$ such that the CLT holds for S'.*

Proof (1): [Tameness] + [Thurston's hyperbolization] $\Rightarrow \pi_1(M)$ hyperbolic

Applications (II) Intersection with submanifolds

Let M be a hyperbolic manifold and let S be any generating set for $\pi_1(M)$.

Applications (II) Intersection with submanifolds

Let M be a hyperbolic manifold and let S be any generating set for $\pi_1(M)$. Let Σ be a (smooth, orientable) codimension-1 submanifold, and let $i(\gamma, \Sigma)$ be the intersection number.

Applications (II) Intersection with submanifolds

Let M be a hyperbolic manifold and let S be any generating set for $\pi_1(M)$. Let Σ be a (smooth, orientable) codimension-1 submanifold, and let $i(\gamma, \Sigma)$ be the intersection number.

Theorem

Suppose that $\Sigma \to M$ is π_1-injective but not fiber-like.

Applications (II) Intersection with submanifolds

Let M be a hyperbolic manifold and let S be any generating set for $\pi_1(M)$. Let Σ be a (smooth, orientable) codimension-1 submanifold, and let $i(\gamma, \Sigma)$ be the intersection number.

Theorem

Suppose that $\Sigma \to M$ is π_1-injective but not fiber-like. Then there are $\ell, \sigma > 0$ such that

$$\frac{i(\gamma, \Sigma) - \ell n}{\sqrt{n}} \longrightarrow \mathcal{N}_\sigma,$$

where γ is chosen uniformly at random in the sphere of radius n with respect to S.

Applications (II) Intersection with submanifolds

Let M be a hyperbolic manifold and let S be any generating set for $\pi_1(M)$. Let Σ be a (smooth, orientable) codimension-1 submanifold, and let $i(\gamma, \Sigma)$ be the intersection number.

Theorem

Suppose that $\Sigma \to M$ is π_1-injective but not fiber-like. Then there are $\ell, \sigma > 0$ such that

$$\frac{i(\gamma, \Sigma) - \ell n}{\sqrt{n}} \longrightarrow \mathcal{N}_\sigma,$$

where γ is chosen uniformly at random in the sphere of radius n with respect to S.

- **Example.** M compact surface of genus $g \geq 2$, Σ an essential simple closed curve.

Applications (II) Intersection with submanifolds

Let M be a hyperbolic manifold and let S be any generating set for $\pi_1(M)$. Let Σ be a (smooth, orientable) codimension-1 submanifold, and let $i(\gamma, \Sigma)$ be the intersection number.

Theorem

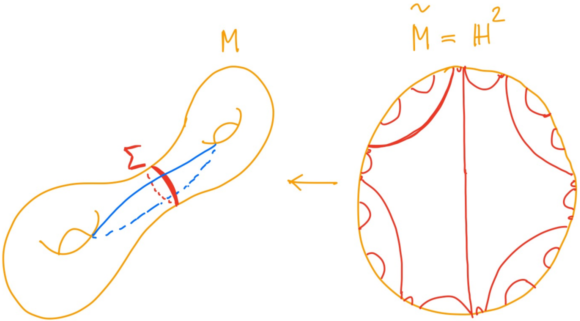
Suppose that $\Sigma \to M$ is π_1-injective but not fiber-like. Then there are $\ell, \sigma > 0$ such that

$$\frac{i(\gamma, \Sigma) - \ell n}{\sqrt{n}} \longrightarrow \mathcal{N}_\sigma,$$

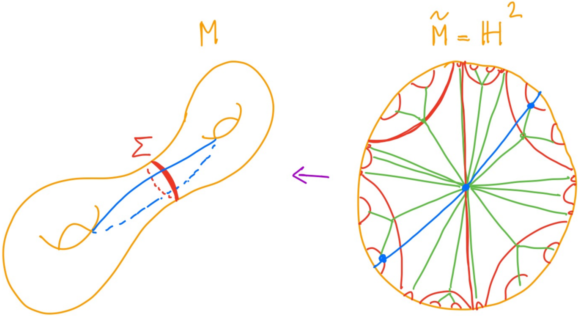
where γ is chosen uniformly at random in the sphere of radius n with respect to S.

- **Example.** M compact surface of genus $g \geq 2$, Σ an essential simple closed curve.
- The action is on a non-proper metric space

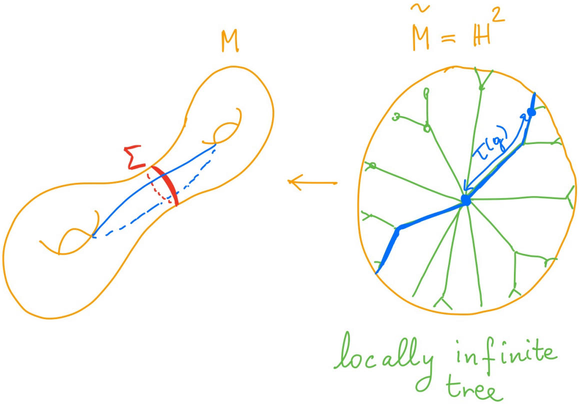
Applications (II) Intersection with submanifolds



Applications (II) Intersection with submanifolds



Applications (II) Intersection with submanifolds



Applications (II) Intersection with submanifolds

Let M be a hyperbolic manifold and let S be any generating set for $\pi_1(M)$. Let Σ be a (smooth, orientable) codimension-1 submanifold, and let $i(\gamma, \Sigma)$ be the intersection number.

Theorem

Suppose that $\Sigma \to M$ is π_1-injective but not fiber-like. Then there are $\ell, \sigma > 0$ such that

$$\frac{i(\gamma, \Sigma) - \ell n}{\sqrt{n}} \longrightarrow \mathcal{N}_\sigma,$$

where γ is chosen uniformly at random in the sphere of radius n with respect to S.

Distribution of self-intersections (Chas-Lalley, 2011)

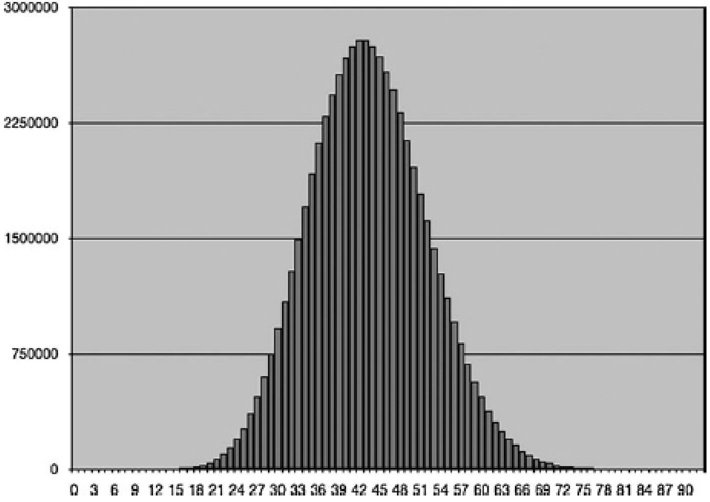


Fig. 2 A *histogram* showing the distribution of self-intersection numbers over all reduced cyclic words of length 19 in the doubly punctured plane. The *horizontal coordinate* shows the self-intersection count k; the *vertical coordinate* shows the number of cyclic reduced words for which the self-intersection number is k

Applications (III) Homomorphisms between hyperbolic groups

Theorem

Let $\phi\colon G \to G'$ be a homomorphism between hyperbolic groups.

Applications (III) Homomorphisms between hyperbolic groups

Theorem

Let $\phi\colon G \to G'$ be a homomorphism between hyperbolic groups. Let S, S' be generating sets.

Applications (III) Homomorphisms between hyperbolic groups

Theorem

Let $\phi\colon G \to G'$ be a homomorphism between hyperbolic groups. Let S, S' be generating sets. Then there exist $\ell > 0, \sigma \geq 0$ such that

$$\frac{\|\phi(g)\|_{S'} - \ell\|g\|_S}{\sqrt{\|g\|_S}} \to \mathcal{N}_\sigma,$$

for $g \in G$ chosen uniformly at random in the sphere of radius n with respect to S.

Applications (III) Homomorphisms between hyperbolic groups

Theorem

Let $\phi\colon G \to G'$ *be a homomorphism between hyperbolic groups. Let* S, S' *be generating sets. Then there exist* $\ell > 0, \sigma \geq 0$ *such that*

$$\frac{\|\phi(g)\|_{S'} - \ell\|g\|_S}{\sqrt{\|g\|_S}} \to \mathcal{N}_\sigma,$$

for $g \in G$ *chosen uniformly at random in the sphere of radius* n *with respect to* S*.*

Applications (III) Homomorphisms between hyperbolic groups

Let $\phi\colon G \to G'$ be a homomorphism between hyperbolic groups. Recall $\partial G := \{\text{geodesic rays based at } o\}/\sim$.

Applications (III) Homomorphisms between hyperbolic groups

Let $\phi \colon G \to G'$ be a homomorphism between hyperbolic groups. Recall $\partial G := \{\text{geodesic rays based at } o\}/\sim$.

Let $v = \limsup_{n\to\infty} \frac{1}{n} \log \#\{g \ : \ d(o, go) \leq n\}$.

Applications (III) Homomorphisms between hyperbolic groups

Let $\phi\colon G \to G'$ be a homomorphism between hyperbolic groups. Recall $\partial G := \{\text{geodesic rays based at } o\}/\sim$.

Let $v = \limsup_{n\to\infty} \frac{1}{n} \log \#\{g \ : \ d(o, go) \le n\}$.
Moreover, for $s > v$

$$\nu_s := \frac{\sum_g e^{-sd(o,go)} \delta_{go}}{\sum_g e^{-sd(o,go)}}$$

Applications (III) Homomorphisms between hyperbolic groups

Let $\phi\colon G \to G'$ be a homomorphism between hyperbolic groups. Recall $\partial G := \{\text{geodesic rays based at } o\}/\sim$.

Let $v = \limsup_{n\to\infty} \frac{1}{n} \log \#\{g \ : \ d(o, go) \leq n\}$.
Moreover, for $s > v$

$$\nu_s := \frac{\sum_g e^{-sd(o,go)}\delta_{go}}{\sum_g e^{-sd(o,go)}}$$

and the Patterson-Sullivan (PS) measure is

$$\nu_{PS} := \lim_{s\to v} \nu_s$$

Applications (III) Homomorphisms between hyperbolic groups

Let $\phi\colon G \to G'$ be a homomorphism between hyperbolic groups. Recall $\partial G := \{\text{geodesic rays based at } o\}/\sim$.

Let $v = \limsup_{n\to\infty} \frac{1}{n} \log \#\{g \ : \ d(o, go) \le n\}$.
Moreover, for $s > v$

$$\nu_s := \frac{\sum_g e^{-sd(o,go)}\delta_{go}}{\sum_g e^{-sd(o,go)}}$$

and the Patterson-Sullivan (PS) measure is

$$\nu_{PS} := \lim_{s\to v} \nu_s$$

Theorem

In the CLT we have $\sigma = 0$ if and only if ϕ has finite kernel and $\partial\phi\colon \partial G \to \partial G'$ pushes the PS measure class for (G, S) to the PS measure class for $(\phi(G), S')$.

Techniques - Graph structures

A graph structure is $(\Gamma, v_0, \mathrm{ev})$ with:

- Γ is a finite graph

Techniques - Graph structures

A <u>graph structure</u> is $(\Gamma, v_0, \mathrm{ev})$ with:

- Γ is a finite graph
- v_0 is a vertex of Γ (initial vertex)

Techniques - Graph structures

A graph structure is $(\Gamma, v_0, \mathrm{ev})$ with:

- Γ is a finite graph
- v_0 is a vertex of Γ (initial vertex)
- $\mathrm{ev} : E(\Gamma) \to G$ the evaluation map

Techniques - Graph structures

A graph structure is $(\Gamma, v_0, \mathrm{ev})$ with:

- Γ is a finite graph
- v_0 is a vertex of Γ (initial vertex)
- $\mathrm{ev} : E(\Gamma) \to G$ the evaluation map

If v is a vertex, Γ_v is the loop semigroup

Techniques - Graph structures

A graph structure is $(\Gamma, v_0, \mathrm{ev})$ with:

- Γ is a finite graph
- v_0 is a vertex of Γ (initial vertex)
- $\mathrm{ev} : E(\Gamma) \to G$ the evaluation map

If v is a vertex, Γ_v is the loop semigroup

- M: adjacency matrix

Techniques - Graph structures

A graph structure is $(\Gamma, v_0, \mathrm{ev})$ with:

- Γ is a finite graph
- v_0 is a vertex of Γ (initial vertex)
- $\mathrm{ev} : E(\Gamma) \to G$ the evaluation map

If v is a vertex, Γ_v is the loop semigroup

- M: adjacency matrix
- λ: leading eigenvalue

Techniques - Graph structures

A graph structure is $(\Gamma, v_0, \mathrm{ev})$ with:

- Γ is a finite graph
- v_0 is a vertex of Γ (initial vertex)
- $\mathrm{ev} : E(\Gamma) \to G$ the evaluation map

If v is a vertex, Γ_v is the loop semigroup

- M: adjacency matrix
- λ: leading eigenvalue

A component is maximal if its growth rate is λ.

Techniques - Graph structures

A graph structure is $(\Gamma, v_0, \mathrm{ev})$ with:

- Γ is a finite graph
- v_0 is a vertex of Γ (initial vertex)
- $\mathrm{ev} : E(\Gamma) \to G$ the evaluation map

If v is a vertex, Γ_v is the loop semigroup

- M: adjacency matrix
- λ: leading eigenvalue

A component is maximal if its growth rate is λ.

Definition

A graph structure is thick if for any v in a maximal component there exists $B \subseteq G$ finite such that

$$G = B \cdot \mathrm{ev}(\Gamma_v) \cdot B$$

Techniques - Graph structures

A graph structure is $(\Gamma, v_0, \mathrm{ev})$ with:

- Γ is a finite graph
- v_0 is a vertex of Γ (initial vertex)
- $\mathrm{ev} : E(\Gamma) \to G$ the evaluation map

If v is a vertex, Γ_v is the loop semigroup

- M: adjacency matrix
- λ: leading eigenvalue

A component is maximal if its growth rate is λ.

Definition

A graph structure is thick if for any v in a maximal component there exists $B \subseteq G$ finite such that

$$G = B \cdot \mathrm{ev}(\Gamma_v) \cdot B$$

Thick $\Rightarrow$ almost semisimple: for every maximal eigenvalue, its algebraic and geometric multiplicities agree

Techniques - Graph structures

A graph structure is $(\Gamma, v_0, \mathrm{ev})$ with:

- Γ is a finite graph
- v_0 is a vertex of Γ (initial vertex)
- $\mathrm{ev} : E(\Gamma) \to G$ the evaluation map

If v is a vertex, Γ_v is the loop semigroup

- M: adjacency matrix
- λ: leading eigenvalue

A component is maximal if its growth rate is λ.

Definition

A graph structure is thick if for any v in a maximal component there exists $B \subseteq G$ finite such that

$$G = B \cdot \mathrm{ev}(\Gamma_v) \cdot B$$

Thick $\Rightarrow$ almost semisimple: for every maximal eigenvalue, its algebraic and geometric multiplicities agree **Note.** M need not be irreducible and not aperiodic.

Techniques - Graph structures

Definition

A graph structure is biautomatic if

Techniques - Graph structures

Definition

A graph structure is biautomatic if for any $B \subseteq G$ finite $\exists C \geq 0$ such that if g, h are finite length paths with

$$\mathrm{ev}(g) = b_1 \cdot \mathrm{ev}(h) \cdot b_2$$

Techniques - Graph structures

Definition

A graph structure is biautomatic if for any $B \subseteq G$ finite $\exists C \geq 0$ such that if g, h are finite length paths with

$$\mathrm{ev}(g) = b_1 \cdot \mathrm{ev}(h) \cdot b_2$$

then

$$d_G(g_1 \dots g_i, b_1 h_1 \dots h_i) \leq C$$

for all $i \leq \max\{\|g\|, \|h\|\}$.

Techniques - Graph structures

Definition

A graph structure is biautomatic if for any $B \subseteq G$ finite $\exists C \geq 0$ such that if g, h are finite length paths with

$$\mathrm{ev}(g) = b_1 \cdot \mathrm{ev}(h) \cdot b_2$$

then

$$d_G(g_1 \dots g_i, b_1 h_1 \dots h_i) \leq C$$

for all $i \leq \max\{\|g\|, \|h\|\}$.

Definition

A group has a thick bicombing for S if it has a thick, biautomatic graph structure for S such that paths are geodesic for the word length $\|\cdot\|_S$.

Graph structures - examples

1. Hyperbolic groups have thick bicombings for every generating set S (Cannon)

Graph structures - examples

1. Hyperbolic groups have thick bicombings for every generating set S (Cannon)
2. If $G = \pi_1(M)$ with M a geometrically finite hyperbolic manifold, then for any S there exists S' which admits a thick bicombing (Antolin-Ciobanu)

Graph structures - examples

1. Hyperbolic groups have thick bicombings for every generating set S (Cannon)
2. If $G = \pi_1(M)$ with M a geometrically finite hyperbolic manifold, then for any S there exists S' which admits a thick bicombing (Antolin-Ciobanu)
   More generally, relatively hyperbolic groups with virtually abelian parabolic subgroups.

Graph structures - examples

1. Hyperbolic groups have thick bicombings for every generating set S (Cannon)
2. If $G = \pi_1(M)$ with M a geometrically finite hyperbolic manifold, then for any S there exists S' which admits a thick bicombing (Antolin-Ciobanu)
   More generally, relatively hyperbolic groups with virtually abelian parabolic subgroups.
3. Right-angled Artin/Coxeter groups (Hermiller-Meier)

Structure of the proof

1. Central limit theorem for centerable cocycles
   (Benoist-Quint, Horbez)

Structure of the proof

1. Central limit theorem for centerable cocycles (Benoist-Quint, Horbez) + Busemann cocycle for the loop semigroup

Structure of the proof

1. Central limit theorem for centerable cocycles (Benoist-Quint, Horbez) + Busemann cocycle for the loop semigroup $\Rightarrow$ CLT for the random walk on the loop semigroup

Structure of the proof

1. Central limit theorem for centerable cocycles (Benoist-Quint, Horbez) + Busemann cocycle for the loop semigroup $\Rightarrow$ CLT for the random walk on the loop semigroup
2. CLT for suspensions (Melbourne-Török)

Structure of the proof

1. Central limit theorem for centerable cocycles (Benoist-Quint, Horbez) + Busemann cocycle for the loop semigroup $\Rightarrow$ CLT for the random walk on the loop semigroup
2. CLT for suspensions (Melbourne-Török) $\Rightarrow$ CLT for the Markov chain

Structure of the proof

1. Central limit theorem for centerable cocycles (Benoist-Quint, Horbez) + Busemann cocycle for the loop semigroup $\Rightarrow$ CLT for the random walk on the loop semigroup
2. CLT for suspensions (Melbourne-Török) $\Rightarrow$ CLT for the Markov chain
3. Biautomaticity $\Rightarrow$ Uniqueness of drift and variance

Structure of the proof

1. Central limit theorem for centerable cocycles (Benoist-Quint, Horbez) + Busemann cocycle for the loop semigroup $\Rightarrow$ CLT for the random walk on the loop semigroup
2. CLT for suspensions (Melbourne-Török) $\Rightarrow$ CLT for the Markov chain
3. Biautomaticity $\Rightarrow$ Uniqueness of drift and variance
4. For semisimple structures, approximate counting measure by Markov chain measure

Structure of the proof

1. Central limit theorem for centerable cocycles (Benoist-Quint, Horbez) + Busemann cocycle for the loop semigroup $\Rightarrow$ CLT for the random walk on the loop semigroup
2. CLT for suspensions (Melbourne-Török) $\Rightarrow$ CLT for the Markov chain
3. Biautomaticity $\Rightarrow$ Uniqueness of drift and variance
4. For semisimple structures, approximate counting measure by Markov chain measure $\Rightarrow$ CLT for counting measure

Step 1: CLT for cocycles

Let $\mathcal{M}$ a metric space on which G acts continuously.

Step 1: CLT for cocycles

Let $\mathcal{M}$ a metric space on which G acts continuously. A cocycle is $\eta : G \times \mathcal{M} \to \mathbb{R}$ with

$$\sigma(gh, x) = \sigma(g, hx) + \sigma(h, x)$$

Step 1: CLT for cocycles

Let $\mathcal{M}$ a metric space on which G acts continuously. A cocycle is $\eta : G \times \mathcal{M} \to \mathbb{R}$ with

$$\sigma(gh, x) = \sigma(g, hx) + \sigma(h, x)$$

A measure ν on $\mathcal{M}$ is μ-stationary if

$$\int g_* \nu \, d\mu(g) = \nu.$$

Step 1: CLT for cocycles

Let $\mathcal{M}$ a metric space on which G acts continuously. A cocycle is $\eta : G \times \mathcal{M} \to \mathbb{R}$ with

$$\sigma(gh, x) = \sigma(g, hx) + \sigma(h, x)$$

A measure ν on $\mathcal{M}$ is μ-stationary if

$$\int g_* \nu \, d\mu(g) = \nu.$$

A cocycle η has constant drift λ if

$$\int \eta(g, x) d\mu(g) = \lambda \qquad \text{for all } x \in \mathcal{M}.$$

Step 1: CLT for cocycles

Let $\mathcal{M}$ a metric space on which G acts continuously. A cocycle is $\eta : G \times \mathcal{M} \to \mathbb{R}$ with

$$\sigma(gh, x) = \sigma(g, hx) + \sigma(h, x)$$

A measure ν on $\mathcal{M}$ is μ-stationary if

$$\int g_* \nu \, d\mu(g) = \nu.$$

A cocycle η has constant drift λ if

$$\int \eta(g, x) d\mu(g) = \lambda \qquad \text{for all } x \in \mathcal{M}.$$

A cocycle η is centerable if it can be written as

$$\eta(g, x) = \eta_0(g, x) + \psi(x) - \psi(g \cdot x)$$

where η_0 is a cocycle with constant drift and $\psi \colon \mathcal{M} \to \mathbb{R}$ a bounded, measurable function.

Central limit theorem for centerable cocycles

Theorem (Benoist-Quint + Horbez)

Let ν be a μ-ergodic, μ-stationary probability measure on $\mathcal{M}$,

Central limit theorem for centerable cocycles

Theorem (Benoist-Quint + Horbez)

Let ν be a μ-ergodic, μ-stationary probability measure on $\mathcal{M}$, and let $\eta\colon G \times \mathcal{M} \to \mathbb{R}$ be a centerable cocycle with drift λ and finite second moment.

Central limit theorem for centerable cocycles

Theorem (Benoist-Quint + Horbez)

Let ν be a μ-ergodic, μ-stationary probability measure on $\mathcal{M}$, and let $\eta\colon G \times \mathcal{M} \to \mathbb{R}$ be a centerable cocycle with drift λ and finite second moment. Then there exist $\sigma \geq 0$ such that for any continuous $F\colon \mathbb{R} \to \mathbb{R}$ with compact support, we have for ν-a.e. $x \in \mathcal{M}$,

$$\lim_{n\to\infty} \int_G F\left(\frac{\sigma(g,x) - n\lambda}{\sqrt{n}}\right)\, d\mu^{*n}(g) = \int_{\mathbb{R}} F(t)\, d\mathcal{N}_\sigma(t).$$

The Busemann cocycle

Horofunction boundary. We have embedding
$\rho : X \to C(X, \mathbb{R})$

The Busemann cocycle

Horofunction boundary. We have embedding
$\rho : X \to C(X, \mathbb{R})$

$$\rho_x(z) := d(x, z) - d(x, o)$$

The Busemann cocycle

Horofunction boundary. We have embedding
$\rho : X \to C(X, \mathbb{R})$

$$\rho_x(z) := d(x, z) - d(x, o)$$

Then $\mathcal{M} = \overline{X}^h$ is the closure of $\rho(X)$ for the topology of pointwise convergence.

The Busemann cocycle

Horofunction boundary. We have embedding
$\rho : X \to C(X, \mathbb{R})$

$$\rho_x(z) := d(x, z) - d(x, o)$$

Then $\mathcal{M} = \overline{X}^h$ is the closure of $\rho(X)$ for the topology of pointwise convergence.

Definition

The Busemann cocycle

$$\beta(g, \xi) := \lim_{z_n \to \xi} (d(o, z_n) - d(g^{-1} o, z_n))$$

The Busemann cocycle

Horofunction boundary. We have embedding $\rho : X \to C(X, \mathbb{R})$

$$\rho_x(z) := d(x, z) - d(x, o)$$

Then $\mathcal{M} = \overline{X}^h$ is the closure of $\rho(X)$ for the topology of pointwise convergence.

Definition

The Busemann cocycle

$$\beta(g, \xi) := \lim_{z_n \to \xi} (d(o, z_n) - d(g^{-1}o, z_n))$$

Proposition (Horbez)

The Busemann cocycle is centerable.

Step 1: Random walks on the loop semigroup

Fix a graph structure Γ and let v be a vertex in a maximal growth component.

Step 1: Random walks on the loop semigroup

Fix a graph structure Γ and let v be a vertex in a maximal growth component. Fix a measure μ on the set of edges of Γ.

Step 1: Random walks on the loop semigroup

Fix a graph structure Γ and let v be a vertex in a maximal growth component. Fix a measure μ on the set of edges of Γ.

Definition

The loop semigroup Γ_v is the set of all loops from v to v.

Step 1: Random walks on the loop semigroup

Fix a graph structure Γ and let v be a vertex in a maximal growth component. Fix a measure μ on the set of edges of Γ.

Definition

The loop semigroup Γ_v is the set of all loops from v to v. The first return measure is

$$\mu_v(l) = \mu(g_1)\dots\mu(g_n)$$

if $l = g_1 \dots g_n$.

Step 1: Random walks on the loop semigroup

Fix a graph structure Γ and let v be a vertex in a maximal growth component. Fix a measure μ on the set of edges of Γ.

Definition

The loop semigroup Γ_v is the set of all loops from v to v. The first return measure is

$$\mu_v(l) = \mu(g_1)\ldots\mu(g_n)$$

if $l = g_1 \ldots g_n$.

Theorem

Let ν_v be a μ_v-ergodic, μ_v-stationary measure on $\overline{X}^h$.

Step 1: Random walks on the loop semigroup

Fix a graph structure Γ and let v be a vertex in a maximal growth component. Fix a measure μ on the set of edges of Γ.

Definition

The loop semigroup Γ_v is the set of all loops from v to v. The first return measure is

$$\mu_v(l) = \mu(g_1) \dots \mu(g_n)$$

if $l = g_1 \dots g_n$.

Theorem

Let ν_v be a μ_v-ergodic, μ_v-stationary measure on $\overline{X}^h$. Then there exist $\ell, \sigma \geq 0$ such that for ν_v-a.e. ξ

Step 1: Random walks on the loop semigroup

Fix a graph structure Γ and let v be a vertex in a maximal growth component. Fix a measure μ on the set of edges of Γ.

Definition

The loop semigroup Γ_v is the set of all loops from v to v. The first return measure is

$$\mu_v(l) = \mu(g_1)\dots\mu(g_n)$$

if $l = g_1 \dots g_n$.

Theorem

Let ν_v be a μ_v-ergodic, μ_v-stationary measure on $\overline{X}^h$. Then there exist $\ell, \sigma \geq 0$ such that for ν_v-a.e. ξ

$$\lim_{n\to\infty} \int_G F\left(\frac{\beta_\xi(o, go) - \ell\|g\|}{\sqrt{n}}\right) \, d\mu_v^{*n}(g) = \int_{\mathbb{R}} F(t) \, d\mathcal{N}_\sigma(t)$$

for any $F \in C_c(\mathbb{R})$.

Step 2: Suspension to the Markov chain

Let $S\colon (\mathcal{X}, \lambda) \to (\mathcal{X}, \lambda)$, and let $r\colon \mathcal{X} \to \mathbb{N}$ be a roof function. Then the discrete suspension flow of S with roof function r is $\widehat{S} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ where

$$\widehat{\mathcal{X}} := \{(x, n) \in \mathcal{X} \times \mathbb{N} \ : \ 0 \le n \le r(x) - 1\}$$

Step 2: Suspension to the Markov chain

Let $S\colon (\mathcal{X},\lambda) \to (\mathcal{X},\lambda)$, and let $r\colon \mathcal{X} \to \mathbb{N}$ be a roof function. Then the discrete suspension flow of S with roof function r is $\widehat{S} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ where

$$\widehat{\mathcal{X}} := \{(x,n) \in \mathcal{X} \times \mathbb{N} \;:\; 0 \le n \le r(x) - 1\}$$

Then, the map $\widehat{S}$ is defined as

$$\widehat{S}(x,n) = \begin{cases} (x, n+1) & \text{if } n \le r(x) - 2 \\ (S(x), 0) & \text{if } n = r(x) - 1. \end{cases}$$

Step 2: Suspension to the Markov chain

Let $S\colon (\mathcal{X}, \lambda) \to (\mathcal{X}, \lambda)$, and let $r\colon \mathcal{X} \to \mathbb{N}$ be a roof function. Then the discrete suspension flow of S with roof function r is $\widehat{S} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ where

$$\widehat{\mathcal{X}} := \{(x, n) \in \mathcal{X} \times \mathbb{N} \ : \ 0 \leq n \leq r(x) - 1\}$$

Then, the map $\widehat{S}$ is defined as

$$\widehat{S}(x, n) = \begin{cases} (x, n+1) & \text{if } n \leq r(x) - 2 \\ (S(x), 0) & \text{if } n = r(x) - 1. \end{cases}$$

Theorem (Melbourne-Török)

Let $S\colon (\mathcal{X}, \lambda) \to (\mathcal{X}, \lambda)$ be ergodic,

Step 2: Suspension to the Markov chain

Let $S\colon (\mathcal{X},\lambda) \to (\mathcal{X},\lambda)$, and let $r\colon \mathcal{X} \to \mathbb{N}$ be a roof function. Then the discrete suspension flow of S with roof function r is $\widehat{S} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ where

$$\widehat{\mathcal{X}} := \{(x,n) \in \mathcal{X} \times \mathbb{N} \;:\; 0 \le n \le r(x) - 1\}$$

Then, the map $\widehat{S}$ is defined as

$$\widehat{S}(x,n) = \begin{cases} (x, n+1) & \text{if } n \le r(x) - 2 \\ (S(x), 0) & \text{if } n = r(x) - 1. \end{cases}$$

Theorem (Melbourne-Törok)

Let $S\colon (\mathcal{X},\lambda) \to (\mathcal{X},\lambda)$ be ergodic, and let $\widehat{S}\colon (\widehat{\mathcal{X}},\widehat{\lambda}) \to (\widehat{\mathcal{X}},\widehat{\lambda})$ be the suspension flow with roof function r.

Step 2: Suspension to the Markov chain

Let $S\colon (\mathcal{X},\lambda) \to (\mathcal{X},\lambda)$, and let $r\colon \mathcal{X} \to \mathbb{N}$ be a roof function. Then the discrete suspension flow of S with roof function r is $\widehat{S} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ where

$$\widehat{\mathcal{X}} := \{(x,n) \in \mathcal{X} \times \mathbb{N} \ : \ 0 \leq n \leq r(x) - 1\}$$

Then, the map $\widehat{S}$ is defined as

$$\widehat{S}(x,n) = \begin{cases} (x, n+1) & \text{if } n \leq r(x) - 2 \\ (S(x), 0) & \text{if } n = r(x) - 1. \end{cases}$$

Theorem (Melbourne-Törok)

Let $S\colon (\mathcal{X},\lambda) \to (\mathcal{X},\lambda)$ be ergodic, and let $\widehat{S}\colon (\widehat{\mathcal{X}},\widehat{\lambda}) \to (\widehat{\mathcal{X}},\widehat{\lambda})$ be the suspension flow with roof function r.
Let $\phi\colon \widehat{\mathcal{X}} \to \mathbb{R}$ and define $\Phi(x) := \sum_{k=0}^{r(x)-1} \phi(x,k)$.

Step 2: Suspension to the Markov chain

Let $S\colon (\mathcal{X},\lambda)\to(\mathcal{X},\lambda)$, and let $r\colon \mathcal{X}\to\mathbb{N}$ be a roof function. Then the discrete suspension flow of S with roof function r is $\widehat{S}:\widehat{\mathcal{X}}\to\widehat{\mathcal{X}}$ where

$$\widehat{\mathcal{X}} := \{(x,n)\in\mathcal{X}\times\mathbb{N} \ : \ 0\le n\le r(x)-1\}$$

Then, the map $\widehat{S}$ is defined as

$$\widehat{S}(x,n) = \begin{cases} (x,n+1) & \text{if } n\le r(x)-2 \\ (S(x),0) & \text{if } n = r(x)-1. \end{cases}$$

Theorem (Melbourne-Török)

Let $S\colon (\mathcal{X},\lambda)\to(\mathcal{X},\lambda)$ be ergodic, and let $\widehat{S}\colon (\widehat{\mathcal{X}},\widehat{\lambda})\to(\widehat{\mathcal{X}},\widehat{\lambda})$ be the suspension flow with roof function r.
Let $\phi\colon \widehat{\mathcal{X}}\to\mathbb{R}$ and define $\Phi(x) := \sum_{k=0}^{r(x)-1}\phi(x,k)$. Suppose that Φ and r satisfy a CLT.

Step 2: Suspension to the Markov chain

Let $S\colon (\mathcal{X},\lambda)\to(\mathcal{X},\lambda)$, and let $r\colon \mathcal{X}\to\mathbb{N}$ be a roof function. Then the discrete suspension flow of S with roof function r is $\widehat{S}:\widehat{\mathcal{X}}\to\widehat{\mathcal{X}}$ where

$$\widehat{\mathcal{X}} := \{(x,n)\in\mathcal{X}\times\mathbb{N} \ : \ 0\le n\le r(x)-1\}$$

Then, the map $\widehat{S}$ is defined as

$$\widehat{S}(x,n) = \begin{cases} (x,n+1) & \text{if } n\le r(x)-2 \\ (S(x),0) & \text{if } n = r(x)-1. \end{cases}$$

Theorem (Melbourne-Törok)

Let $S\colon (\mathcal{X},\lambda)\to(\mathcal{X},\lambda)$ be ergodic, and let $\widehat{S}\colon (\widehat{\mathcal{X}},\widehat{\lambda})\to(\widehat{\mathcal{X}},\widehat{\lambda})$ be the suspension flow with roof function r.
Let $\phi\colon \widehat{\mathcal{X}}\to\mathbb{R}$ and define $\Phi(x) := \sum_{k=0}^{r(x)-1}\phi(x,k)$. Suppose that Φ and r satisfy a CLT. Then ϕ satisfies a CLT.

CLT for Markov chain

Recall that if M is primitive

CLT for Markov chain

Recall that if M is primitive (i.e. $M^n > 0$)

CLT for Markov chain

Recall that if M is primitive (i.e. $M^n > 0$)

$$\lim_n \frac{M^n}{\lambda^n} = \rho u^T$$

CLT for Markov chain

Recall that if M is primitive (i.e. $M^n > 0$)

$$\lim_n \frac{M^n}{\lambda^n} = \rho u^T$$

Define Markov chain

- Starting probabilities

$$\pi_i := \rho_i u_i$$

CLT for Markov chain

Recall that if M is primitive (i.e. $M^n > 0$)

$$\lim_n \frac{M^n}{\lambda^n} = \rho u^T$$

Define <u>Markov chain</u>

- Starting probabilities

$$\pi_i := \rho_i u_i$$

- Transition probabilities

$$p_{ij} := \frac{\rho_j}{\lambda \rho_i}$$

CLT for Markov chain

Recall that if M is primitive (i.e. $M^n > 0$)

$$\lim_n \frac{M^n}{\lambda^n} = \rho u^T$$

Define Markov chain

- Starting probabilities

$$\pi_i := \rho_i u_i$$

- Transition probabilities

$$p_{ij} := \frac{\rho_j}{\lambda \rho_i}$$

Let $\Omega = G^{\mathbb{N}}$, $\Omega_v = \Gamma_v^{\mathbb{N}}$.

Let $\Omega = G^{\mathbb{N}}$, $\Omega_v = \Gamma_v^{\mathbb{N}}$. Consider skew product $T : \Omega \times \mathcal{M} \to \mathbb{R}$

Let $\Omega = G^{\mathbb{N}}$, $\Omega_v = \Gamma_v^{\mathbb{N}}$. Consider skew product $T : \Omega \times \mathcal{M} \to \mathbb{R}$

$$T(\omega, \xi) := (\sigma(\omega), g_1^{-1}\xi)$$

Let $\Omega = G^{\mathbb{N}}$, $\Omega_v = \Gamma_v^{\mathbb{N}}$. Consider skew product $T : \Omega \times \mathcal{M} \to \mathbb{R}$

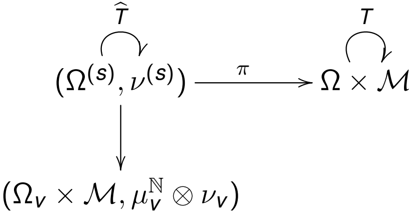
$$T(\omega, \xi) := (\sigma(\omega), g_1^{-1}\xi)$$

Two versions: $T : \Omega \times \mathcal{M} \to \mathbb{R}$ and $T_v : \Omega_v \times \mathcal{M} \to \mathbb{R}$.

Let $\Omega = G^{\mathbb{N}}$, $\Omega_v = \Gamma_v^{\mathbb{N}}$. Consider skew product $T : \Omega \times \mathcal{M} \to \mathbb{R}$

$$T(\omega, \xi) := (\sigma(\omega), g_1^{-1}\xi)$$

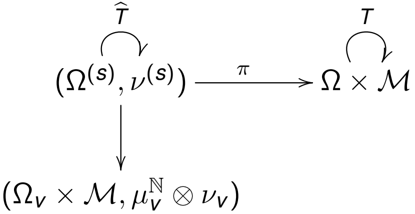
Two versions: $T : \Omega \times \mathcal{M} \to \mathbb{R}$ and $T_v : \Omega_v \times \mathcal{M} \to \mathbb{R}$.



Let $\Omega = G^{\mathbb{N}}$, $\Omega_v = \Gamma_v^{\mathbb{N}}$. Consider skew product $T : \Omega \times \mathcal{M} \to \mathbb{R}$

$$T(\omega, \xi) := (\sigma(\omega), g_1^{-1}\xi)$$

Two versions: $T : \Omega \times \mathcal{M} \to \mathbb{R}$ and $T_v : \Omega_v \times \mathcal{M} \to \mathbb{R}$.



Consider observable

$$f(\omega, \xi) := \beta_\xi(o, g_1 o)$$

Consider observable

$$f(\omega, \xi) := \beta_\xi(o, g_1 o)$$

Consider observable

$$f(\omega, \xi) := \beta_\xi(o, g_1 o)$$

Then

$$\beta_\xi(o, g_1 \ldots g_n o) = \sum_{j=0}^{n-1} f \circ T^j(\omega, \xi)$$

hence

Theorem

Suppose that Γ is primitive and let μ_n be the n-th step distribution of the Markov chain on Γ.

Consider observable

$$f(\omega, \xi) := \beta_\xi(o, g_1 o)$$

Then

$$\beta_\xi(o, g_1 \dots g_n o) = \sum_{j=0}^{n-1} f \circ T^j(\omega, \xi)$$

hence

Theorem

Suppose that Γ *is primitive and let* μ_n *be the n-th step distribution of the Markov chain on* Γ*. Then there are* ℓ *and* σ *such that*

Consider observable

$$f(\omega,\xi) := \beta_\xi(o, g_1 o)$$

Then

$$\beta_\xi(o, g_1 \dots g_n o) = \sum_{j=0}^{n-1} f \circ T^j(\omega,\xi)$$

hence

Theorem

Suppose that Γ is primitive and let μ_n be the n-th step distribution of the Markov chain on Γ. Then there are ℓ and σ such that

$$\lim_{n\to\infty} \int_G F\left(\frac{\beta_\xi(o, go) - n\ell}{\sqrt{n}}\right) \, d\mu_n(g) = \int_{\mathbb{R}} F(t) \, d\mathcal{N}_\sigma(t)$$

for any $F \in C_c(\mathbb{R})$.

Consider observable

$$f(\omega, \xi) := \beta_\xi(o, g_1 o)$$

Then

$$\beta_\xi(o, g_1 \dots g_n o) = \sum_{j=0}^{n-1} f \circ T^j(\omega, \xi)$$

hence

Theorem

Suppose that Γ is primitive and let μ_n be the n-th step distribution of the Markov chain on Γ. Then there are ℓ and σ such that

$$\lim_{n\to\infty} \int_G F\left(\frac{d(o, go) - n\ell}{\sqrt{n}}\right) \, d\mu_n(g) = \int_{\mathbb{R}} F(t) \, d\mathcal{N}_\sigma(t)$$

for any $F \in C_c(\mathbb{R})$.

Step 3: Approximating the counting measure

Suppose M is semisimple (there is only one maximal eigenvalue).

Step 3: Approximating the counting measure

Suppose M is semisimple (there is only one maximal eigenvalue).

Let λ_n be the uniform measure on all paths of length n, and let $\widetilde{\lambda}_n$ be the distribution of the subpath of positions $[\log n, n - \log n]$.

Step 3: Approximating the counting measure

Suppose M is semisimple (there is only one maximal eigenvalue).

Let λ_n be the uniform measure on all paths of length n, and let $\widetilde{\lambda}_n$ be the distribution of the subpath of positions $[\log n, n - \log n]$.

Lemma

$$\|\mu_{n-2\log n} - \widetilde{\lambda}_n\|_{TV} \to 0$$

Step 3: Approximating the counting measure

Suppose M is semisimple (there is only one maximal eigenvalue).

Let λ_n be the uniform measure on all paths of length n, and let $\widetilde{\lambda}_n$ be the distribution of the subpath of positions $[\log n, n - \log n]$.

Lemma

$$\|\mu_{n-2\log n} - \widetilde{\lambda}_n\|_{TV} \to 0$$

Definition

A function $f\colon \Omega^* \to \mathbb{R}$ is uniformly bicontinuous

Definition

A function $f\colon \Omega^* \to \mathbb{R}$ is <u>uniformly bicontinuous</u> if for any finite set $B \subseteq G$ and any $\eta > 0$,

Definition

A function $f\colon \Omega^* \to \mathbb{R}$ is uniformly bicontinuous if for any finite set $B \subseteq G$ and any $\eta > 0$, there exists $N \geq 0$ such that if $\|g\| \geq N$ and

Definition

A function $f \colon \Omega^* \to \mathbb{R}$ is uniformly bicontinuous if for any finite set $B \subseteq G$ and any $\eta > 0$, there exists $N \geq 0$ such that if $\|g\| \geq N$ and

$$b_1 \cdot \text{ev}(g) \cdot b_2 = \text{ev}(h)$$

for $b_1, b_2 \in B$,

Definition

A function $f\colon \Omega^* \to \mathbb{R}$ is uniformly bicontinuous if for any finite set $B \subseteq G$ and any $\eta > 0$, there exists $N \geq 0$ such that if $\|g\| \geq N$ and

$$b_1 \cdot \mathrm{ev}(g) \cdot b_2 = \mathrm{ev}(h)$$

for $b_1, b_2 \in B$, then

$$|f(g) - f(h)| < \eta.$$

Definition

A function $f\colon \Omega^* \to \mathbb{R}$ is uniformly bicontinuous if for any finite set $B \subseteq G$ and any $\eta > 0$, there exists $N \geq 0$ such that if $\|g\| \geq N$ and

$$b_1 \cdot \mathrm{ev}(g) \cdot b_2 = \mathrm{ev}(h)$$

for $b_1, b_2 \in B$, then

$$|f(g) - f(h)| < \eta.$$

Consider

$$\varphi(g) := \frac{d(o, go) - \ell\|g\|}{\sqrt{\|g\|}}$$

Definition

A function $f\colon \Omega^* \to \mathbb{R}$ is uniformly bicontinuous if for any finite set $B \subseteq G$ and any $\eta > 0$, there exists $N \geq 0$ such that if $\|g\| \geq N$ and

$$b_1 \cdot \mathrm{ev}(g) \cdot b_2 = \mathrm{ev}(h)$$

for $b_1, b_2 \in B$, then

$$|f(g) - f(h)| < \eta.$$

Consider

$$\varphi(g) := \frac{d(o, go) - \ell\|g\|}{\sqrt{\|g\|}}$$

Then if $g = g_0 g_1 g_2$ we have

$$|\varphi(g_1) - \varphi(g)| \leq \epsilon$$

Definition

A function $f \colon \Omega^* \to \mathbb{R}$ is uniformly bicontinuous if for any finite set $B \subseteq G$ and any $\eta > 0$, there exists $N \geq 0$ such that if $\|g\| \geq N$ and

$$b_1 \cdot \mathrm{ev}(g) \cdot b_2 = \mathrm{ev}(h)$$

for $b_1, b_2 \in B$, then

$$|f(g) - f(h)| < \eta.$$

Consider

$$\varphi(g) := \frac{d(o, go) - \ell\|g\|}{\sqrt{\|g\|}}$$

Then if $g = g_0 g_1 g_2$ we have

$$|\varphi(g_1) - \varphi(g)| \leq \epsilon$$

$\Rightarrow$ CLT for the counting measure

Definition

A function $f\colon \Omega^* \to \mathbb{R}$ is uniformly bicontinuous if for any finite set $B \subseteq G$ and any $\eta > 0$, there exists $N \geq 0$ such that if $\|g\| \geq N$ and

$$b_1 \cdot \mathrm{ev}(g) \cdot b_2 = \mathrm{ev}(h)$$

for $b_1, b_2 \in B$, then

$$|f(g) - f(h)| < \eta.$$

Theorem

(CLT for displacement) *There exists $\ell > 0$, $\sigma \geq 0$ such that for any $a < b$ we have*

$$\lim_{n\to\infty} \frac{1}{\#S_n} \# \left\{ g \in S_n \ : \ \frac{d(o, go) - n\ell}{\sqrt{n}} \in [a, b] \right\} = \int_a^b d\mathcal{N}_\sigma(t).$$

The end

감사합니다

Thank you!!!