Random walks on weakly hyperbolic groups

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Random and Arithmetic Structures in Topology
MSRI - Fall 2020
Random walks on weakly hyperbolic groups - Summary

- **Lecture 1** (Aug 31, 10.30): Introduction to random walks on groups
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Main references:

J. Maher and G. T., *Random walks on weakly hyperbolic groups*, *Random walks, WPD actions, and the Cremona group*
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**Answer.** It depends on the topography (geometry) of the city.
Recurrent random walks

Example 1: Squareville
Recurrent random walks

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In Squareville, blocks form a square grid.
Recurrent random walks

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In Squareville, blocks form a square grid.

What is the probability of coming back to where you started?
Recurrence

Definition
A random walk \((w_n)\) on \(X\) is **recurrent** if for any \(x \in X\), the probability that \(w_n = x\) infinitely often is 1:

\[
P(\text{\(w_n = x\) i.o.}) = 1
\]

Otherwise it is said to be **transient**.

Let \(p_n(x, y) := \text{probability of being at } y \text{ after } n \text{ steps starting from } x\).

**Lemma**
Let \(m = \sum_{n \geq 1} p_n(x, x)\) be the "average number of visits to \(x\)."

Then the random walk is recurrent iff \(m = \infty\).

**Exercise.** Prove the Lemma.
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Exercise. Prove the Lemma.
Recurrent random walks

Let us first consider the easier case where your world is just a line.
Recurrent random walks

Let us first consider the easier case where your world is just a line. What is the probability of going back to where you start after \( N \) steps?

If \( N \) is odd, the probability is zero, but if \( N = 2n \) you get \( p_{2n}(0,0) = \frac{1}{2^n} \binom{2n}{n} \) (choose \( n \) ways to go right).

Is \( \sum_{n \geq 1} \frac{1}{2^n} \binom{2n}{n} \) convergent?

Apply Stirling's Formula:

\[
\frac{n!}{\sqrt{2\pi n} (n/e)^n} \sim \frac{1}{\sqrt{n}} \left( \frac{2n}{e} \right)^n
\]

\[
\therefore \text{our RW is recurrent.}
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\sum_{n \geq 1} \frac{1}{2^n} = \sum_{n \geq 1} \left( \frac{1}{2} \right)^n = \frac{1}{1 - 1/2} = 2
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Apply Stirling's Formula:

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\sqrt{2\pi n} \left( \frac{n}{e} \right)^n \sim \sqrt{n} \left( \frac{e}{n} \right)^n
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$\therefore$ our RW is recurrent.
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\[ p^{2n}(0, 0) = \frac{1}{4^{2n}} \binom{2n}{n}^2 \approx \frac{1}{n} \]
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Theorem (Polya)

The simple random walk on \( \mathbb{Z}^d \) is recurrent iff \( d = 1, 2 \).
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“A drunk man will get back home, but a drunk bird will get lost” (Kakutani).
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Exercise. Prove Polya’s theorem for \( d = 3 \).
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“A drunk man will get back home, but a drunk bird will get lost” (Kakutani).

**Exercise.** Prove Polya’s theorem for \( d = 3 \). Moreover, for the simple random walk on \( \mathbb{Z}^d \), show that \( p^{2n}(0,0) \approx n^{-\frac{d}{2}} \).
Transient random walks

Example 2: Tree City
In Tree City, the map has the shape of a 4-valent tree.
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Theorem

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\[ d_n = \text{“distance of the } n^{th} \text{ step of the RW from the origin”}. \]
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**Theorem**

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\[ d_n = \text{“distance of the } n^{th} \text{ step of the RW from the origin”}. \]

If you give the position of the \( n^{th} \) step, then:

\[ \text{If } d_n > 0 \text{ then } d_{n+1} = \begin{cases} 
  d_n + 1 & \text{with } P = \frac{3}{4} \\
  d_n - 1 & \text{with } P = \frac{1}{4} 
\end{cases} \]

\[ \text{If } d_n = 0 \text{ then } d_{n+1} = d_n + 1 \]

\[ \therefore E(d_{n+1} - d_n) \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2} \]

\[ \therefore E(d_n) \geq \frac{1}{2} \]

Then \( E(d_n) \geq \frac{1}{2} \Rightarrow \text{RW is transient} \] (do we know \( \lim_{n \to \infty} d_n \) exist?)
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\[ \therefore E(d_n + 1 - d_n) \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2} \]

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\[ \therefore \mathbb{E}(d_{n+1} - d_n) \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2} \therefore \mathbb{E}\left(\frac{d_n}{n}\right) \geq \frac{1}{2} \]
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Then \( E \left( \frac{d_n}{n} \right) \geq \frac{1}{2} \Rightarrow \text{RW is transient} \]
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Then \( \mathbb{E}\left(\frac{d_n}{n}\right) \geq \frac{1}{2} \Rightarrow \text{RW is transient} \)

(Do we know \( \lim_{n \to \infty} \frac{d_n}{n} \) exist?)
A radially symmetric tree of valence \((a_1, a_2, \ldots)\) is a tree where all vertices at distance \(n\) from the base point have exactly \(a_{n-1}\) children.

Prove that the simple random walk on a radially symmetric tree \((a_1, a_2, \ldots)\) is transient iff
\[
\sum_{n \geq 1} a_1 \cdot a_2 \cdots a_n < \infty
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Exercise (P. Lessa)
A radially symmetric tree of valence \((a_1, a_2, \ldots)\) is a tree where all vertices at distance \(n\) from the base point have exactly \(a_{n-1}\) children.
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Let $G$ be a group and $(X, d)$ a metric space.
General setup

Let $G$ be a group and $(X, d)$ a metric space. The isometry group of $X$ is the group of elements which preserve distance:
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Let $G$ be a group and $(X, d)$ a metric space. The **isometry group** of $X$ is the group of elements which preserve distance:

$$\text{Isom}(X) = \{ f : X \to X : d(x, y) = d(f(x), f(y)) \text{ for all } x, y \in X \}$$
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**Definition**

A group action of $G$ on $X$ is a homomorphism

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Example: the group of reals acting on itself by translations:
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Definition

A group action of $G$ on $X$ is a homomorphism

$$\rho : G \rightarrow \text{Isom}(X).$$

Example: the group of reals acting on itself by translations: $X = \mathbb{R}$, $G = \mathbb{R}$ and the action $\rho : \mathbb{R} \rightarrow \text{Isom}(\mathbb{R})$ is given by $\rho(t) : x \mapsto x + t$. 
General setup

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The sequence $(g_n)$ is the sequence of increments, and we are interested in the products $w_n := g_1 \cdots g_n$. The sequence $(w_n)$ is called a sample path for the random walk. More formally, the space of increments (or step space) is the product space $(G^N, \mu^N)$. Consider the map $\Phi : G^N \to G^N$ $\Phi : (g_n) \mapsto (w_n)$ where $w_n = g_1 \cdots g_n$ and define the sample space as the space $(\Omega, P)$ where $\Omega = G^N$ and $P = \Phi \star \mu^N$ is the pushforward. If you fix a basepoint $x \in X$ you can look at the sequence $(w_n \cdot x) \subseteq X$. 

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2. The same holds for $G = \mathbb{R}^d$ or $G = \mathbb{Z}^d$ acting by translations on
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3. $X = 4$-valent tree $G = F_2 = \{\text{reduced words in the alphabet } \{a, b, a^{-1}, b^{-1}\}\}$

   Reduced := there are no redundant pairs, i.e. there is no
   $a$ after $a^{-1}$, no $a^{-1}$ after $a$, no $b$ after $b^{-1}$, and no
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   $\mu = \frac{\delta_a + \delta_{a^{-1}} + \delta_b + \delta_{b^{-1}}}{2}$ ⇒ RW in Tree City
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**Cayley graphs**

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$$\|g\| := \min \{ k : g = s_1 s_2 \ldots s_k, s_i \in S \cup S^{-1} \}.$$ 

Moreover, we define the word metric or word distance between $g, h \in G$ as

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▶ If \( G = \mathbb{Z}^2 \), \( S = \{(1, 0), (0, 1)\} \) then \( \text{Cay}({\mathbb{Z}^2}, S) \) is the square grid.
Example: the hyperbolic plane

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The disc has a natural topological boundary, i.e. the circle. This RW converges a.s. to the boundary (Furstenberg).
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Questions

1. Does a typical sample path escape to $\infty$ or it comes back to the origin infinitely often?

Definition

We define the drift or speed or rate of escape of the random walk to be the limit

$$L := \lim_{n \to \infty} d(w^n x, x^n)$$

if it exists.

A measure $\mu$ on $G$ has finite first moment on $X$ if for some (equivalently, any) $x \in X$

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Proof.

For any $x \in X$, the function $a(n, \omega) := d(x, w_n(\omega)x)$ is a subadditive cocycle,
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If so, define the hitting measure $\nu$ on $\partial X$ as

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for any $A \subset \partial X$. 
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6. Is $(\partial X, \nu)$ a model for the Poisson boundary of $(G, \mu)$? That is, do you have a representation formula for bounded harmonic functions?
Hyperbolic metric spaces

Let \((X, d)\) be a geodesic, metric space, and let \(x_0 \in X\) be a basepoint.
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The following are $\delta$-hyperbolic spaces:
- $X = \mathbb{R}$ (NOT $\mathbb{R}^2$!)
- $X =$ tree

Recall a space is proper if metric balls $\{z \in X : d(x, z) \leq R\}$ are compact.
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$$\tau(g) := \lim_{n \to \infty} \frac{d(g^n x, x)}{n}$$

Exercise: the limit exists and is independent of the choice of $x$.

Lemma (Classification of isometries of hyperbolic spaces)
Let $g$ be an isometry of a $\delta$-hyperbolic metric space $X$ (not necessarily proper).

Then either:
1. $g$ has bounded orbits. Then $g$ is called **elliptic**.
2. $g$ has unbounded orbits and $\tau(g) = 0$. Then $g$ is called **parabolic**.
3. $\tau(g) > 0$. Then $g$ is called **hyperbolic or loxodromic**, and has precisely two fixed points on $\partial X$, one attracting and one repelling.
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Weakly hyperbolic groups

Definition
Two loxodromic elements are independent if their fixed point sets are disjoint.

Example
- $G = F_2$, $X = \text{Cay}(F_2, S)$
- $G$ a word hyperbolic group, $X = \text{Cay}(G, S)$
- $G$ a relatively hyperbolic group, $X = \text{coned-off space}$
- $G$ a mapping class group, $X = \text{curve complex}$
- $G$ a right-angled Artin group, $X = \text{extension graph}$
- $G = \text{Out}(F_n)$, $X = \text{free splitting/free factor complex}$
- $G$ the Cremona group, $X = \text{Picard-Manin hyperboloid}$
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**Definition**
Two loxodromic elements are *independent* if their fixed point sets are disjoint. A semigroup of isometries of $X$ is *non-elementary* if it contains 2 independent hyperbolic elements.
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A group is weakly hyperbolic if it admits a non-elementary action on a (possibly non-proper) hyperbolic metric space.
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- $G = \mathbb{F}_2$, $X = \text{Cay}(\mathbb{F}_2, S)$
- $G$ a word hyperbolic group, $X = \text{Cay}(G, S)$
- $G$ a relatively hyperbolic group, $X = \text{coned-off space}$
- $G$ a mapping class group, $X = \text{curve complex}$
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Statement of results

Theorem (Maher-T. '18)

Let $G$ be a countable group of isometries of a $\delta$-hyperbolic metric space $X$, such that the semigroup generated by the support of $\mu$ is non-elementary.

Then:

1. (Boundary convergence)
   For a.e. $(w_n)$ and every $x \in X$
   \[ \lim_{n \to \infty} w_n x = \xi \in \partial X \exists. \]

2. (Positive drift)
   \[ \exists L > 0 \text{ s.t.} \lim \inf_{n \to \infty} d(w_n x, x) = L > 0. \]

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