The core entropy of polynomials of higher degree

Giulio Tiozzo
University of Toronto

In memory of Tan Lei
Angers, October 23, 2017
Hi Mr. Giulio Tiozzo,
My name is Tan Lei. I am a chinese mathematician working in France in the field of holomorphic dynamics. Curt McMullen suggested me to contact you for the following questions that you might help.

It seems that one can think of the core entropy as a function on the Mandelbrot set itself. And Milnor had a student who proved entropy is monotone on $M$.
Do you have a copy of this thesis? How to define the core entropy when the Hubbard tree is topologically infinite? Or worse when the critical orbit is dense in $J$? Is the monotonicity proved using puzzles?
Is there a continuity result of the core entropy as a function of the external angle?
Many thanks in advance for your help.
Sincerely yours, Tan Lei
In total: 569 emails
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Consider the real quadratic family

\[ f_c(z) := z^2 + c \quad c \in \left[-2, \frac{1}{4}\right] \]
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How does entropy change with the parameter \( c \)?
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- is \textbf{continuous} and \textbf{monotone} (Milnor-Thurston 1977, Douady-Hubbard).
- $0 \leq h_{\text{top}}(f_c, \mathbb{R}) \leq \log 2$. 
The function $c \rightarrow h_{\text{top}}(f_c, \mathbb{R})$:

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**Question** : Can we extend this theory to complex polynomials?
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**Remark.** If we consider $f : \hat{C} \to \hat{C}$ entropy is **constant** $h_{top}(f, \hat{C}) = \log d$. 
The complex case: Hubbard trees

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The Hubbard tree $T$ of a postcritically finite polynomial $f$ is a forward invariant, connected subset of the filled Julia set which contains the critical orbit. The map $f$ acts on it.
The core entropy

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**Question:** How does $h(f)$ vary with the polynomial $f$?
Primitive majors

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A critical portrait $m$ is said to be a primitive major if moreover the elements of $m$ are pairwise disjoint.
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Define the distance between primitive majors as

$$d(m_1, m_2) := \sup_{x,y} |d(\pi_{m_1}(x), \pi_{m_1}(y)) - d(\pi_{m_2}(x), \pi_{m_2}(y))|$$
Critical markings

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Then

$$\Theta := \{\Theta(c_1), \ldots, \Theta(c_k)\}$$

is a critical marking (Poirier).
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$$PM(2) \cong \partial \mathbb{D}$$
Core entropy for quadratic polynomials
Can you see the Mandelbrot set in this picture?
Question: Can you see the Mandelbrot set in this picture?
The entropy as a function of external angle

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- The core entropy is also proportional to the dimension of the set of biaccessible angles (Zakeri, Smirnov, Zdunik, Bruin-Schleicher ...)

\[ \text{θ is biaccessible if } \exists \eta \neq \theta \text{ s.t. } R(\theta) \text{ and } R(\eta) \text{ land at the same point.} \]

\[ B_c := \{ \theta \in \mathbb{R}/\mathbb{Z} : \theta \text{ is biaccessible} \} \]

\[ H. \dim B_c = h(f_c) \log d \]

Core entropy also proportional to Hausdorff dimension of angles landing on the corresponding vein (T., Jung)
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Any pair of distinct angles $\theta^\pm$ defines four partitions of the circle: $L(\theta^\pm)$ is the circle minus the four points $\theta^\pm/2$, and $\theta^\pm/2 + 1/2$ and $\text{Full}(\theta^\pm)$ is $S^1$ minus the two intervals $[\frac{\theta^-}{2}, \frac{\theta^+}{2}]$ and $[\frac{\theta^-}{2} + \frac{1}{2}, \frac{\theta^+}{2} + \frac{1}{2}]$. 
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Now, rather than, as Douady and Tao Li, looking at angles landing as the Hubbard tree, we look at pairs of angles landing together and pairs of angles landing at the tree.
Tan Lei’s proof of monotonicity

So let
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Once all these are set up cleanly, the result becomes trivial: If you take \( c' \) further than \( c \), than \( \text{Full}(\theta'^\pm) \) contains \( \text{Full}(\theta^\pm) \)
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With pictures the idea would be a lot easier to explain.
All the best, Tan Lei
Continuity in the quadratic case

**Question** (Thurston, Hubbard):
Is $h(\theta)$ a continuous function of $\theta$?
Continuity in the quadratic case

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Is \( h(\theta) \) a continuous function of \( \theta \)?

**Theorem (T., Dudko-Schleicher)**

*The core entropy function* \( h(\theta) \) *extends to a continuous function from* \( \mathbb{R}/\mathbb{Z} \) *to* \( \mathbb{R} \).
The core entropy for cubic polynomials
Primitive majors

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Primitive majors and polynomials

Let $\mathcal{P}_d$ be the space of monic, centered polynomials of degree $d$. 

Define the potential function of $f$ at $c$ as $G_f(c) := \lim_{n \to \infty} \frac{1}{d^n} \log |f^n(c)|$ which measures the rate of escape of the critical point.

For $r > 0$, define the equipotential locus $Y_d(r) := \{ f \in \mathcal{P}_d : G_f(c) = r \text{ for all } c \in \text{Crit}(f) \}$.

Theorem (Thurston) For each $r > 0$, we have a homeomorphism $Y_d(r) \sim = \text{PM}(d)$. 

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$$Y_d(r) := \{ f \in \mathcal{P}_d : G_f(c) = r \text{ for all } c \in \text{Crit}(f) \}$$

**Theorem (Thurston)**

*For each $r > 0$, we have a homeomorphism*

$$Y_d(r) \cong \text{PM}(d)$$
The space $PM(3)$ of cubic primitive majors

For $d = 3$, 

$$ \frac{a + 1}{3} \leq b \leq a + \frac{2}{3} $$
The space $PM(3)$ of cubic primitive majors

For $d = 3$, generically two leaves: $(a, a + 1/3), (b, b + 1/3)$
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$$\text{PM}(3) = \left\{ (a, b) \in S^1 \times S^1 : a + \frac{1}{3} \leq b \leq a + \frac{1}{2} \right\} / \sim$$
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- $(a, a + 1/2) \sim (a + 1/2, a)$ (wraps 2 times around)
The core entropy for cubic polynomials
The core entropy for cubic polynomials
The core entropy for cubic polynomials
The unicritical slice

\[ f(z) = z^3 + c \]
The symmetric slice

\[ f(z) = z^3 + cz \]
Main theorem, combinatorial version

Theorem (T. - Yan Gao)

Fix \( d \geq 2 \). Then the core entropy extends to a continuous function on the space \( \text{PM}(d) \) of primitive majors.
Main theorem, combinatorial version

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Fix $d \geq 2$. Then the core entropy extends to a continuous function on the space $PM(d)$ of primitive majors.
Main theorem, analytic version

Define $\mathcal{P}_d$ as the space of monic, centered polynomials of degree $d$. 
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Define $\mathcal{P}_d$ as the space of monic, centered polynomials of degree $d$. One says $f_n \to f$ if the coefficients of $f_n$ converge to the coefficients of $f$. 

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**Theorem (T. - Yan Gao)**

Let $d \geq 2$. Then the core entropy is a continuous function on the space of monic, centered, postcritically finite polynomials of degree $d$. 
Define $P_d$ as the space of monic, centered polynomials of degree $d$. One says $f_n \to f$ if the coefficients of $f_n$ converge to the coefficients of $f$.

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Denote $i_\alpha := f^i(c_\alpha)$ the $i^{th}$ iterate of the critical point $c_\alpha$, let

$$P := \{ i_\alpha : i \geq 1, 1 \leq \alpha \leq s \}$$

the postcritical set,
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the set of pairs of postcritical points (= arcs between them)
Computing the entropy: non-separated pair

A pair \((i_\alpha, j_\beta)\) is non-separated if the corresponding arc does not contain critical points.
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\begin{align*}
(1, 2) & \implies (2, 3)
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A pair \((i_\alpha, j_\beta)\) is separated if the corresponding arc contains critical points \(c_{\gamma_1}, \ldots, c_{\gamma_k}\).
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\[(1, 3) \Rightarrow (1, 2) + (1, 4)\]
The algorithm

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Theorem (Thurston; Tan Lei; Gao Yan)

The core entropy of $f$ is given by

\[ h(f) = \log \lambda \]

where $\lambda$ is the leading eigenvalue of $A$. 
The algorithm
Computing entropy: the clique polynomial

Let $\Gamma$ be a finite, directed graph.
Computing entropy: the clique polynomial

Let \( \Gamma \) be a finite, directed graph. Its adjacency matrix is \( A \) such that

\[
A_{ij} := \# \{i \rightarrow j\}
\]

We can consider its spectral determinant

\[
P(t) := \det(I - tA)
\]

Note that \( \lambda - 1 \) is the smallest root of \( P(t) \).

\( P(t) \) can be obtained as the clique polynomial

\[
P(t) = \sum_{\gamma \text{ simple multicycle}} (-1)^{C(\gamma)} t^{\ell(\gamma)}
\]

where:

- A simple multicycle is a disjoint union of (vertex)-disjoint cycles
- \( C(\gamma) \) is the number of connected components of \( \gamma \)
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The clique polynomial: example

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\[ P(t) = 1 - 2t^2 - t^3 + t^5 \]
Computing entropy: the infinite clique polynomial

Let $\Gamma$ be a countable, directed graph.
Computing entropy: the infinite clique polynomial

Let $\Gamma$ be a countable, directed graph. Let us suppose that:

\begin{itemize}
  \item $\Gamma$ has bounded outgoing degree;
  \item $\Gamma$ has bounded cycles: for every $n$ there exists finitely many simple cycles of length $n$.
\end{itemize}

Then we define the growth rate of $\Gamma$ as:

$$r(\Gamma) := \limsup_n \sqrt[n]{C(\Gamma, n)}$$

where $C(\Gamma, n)$ is the number of closed paths of length $n$. 
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**Computing entropy: the infinite clique polynomial**
Computing entropy: the infinite clique polynomial

Let $\Gamma$ with bounded outgoing degree and bounded cycles.
Computing entropy: the infinite clique polynomial

Let $\Gamma$ with bounded outgoing degree and bounded cycles. Then one can define as a formal power series

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Let now $\sigma := \lim \sup \sqrt[n]{S(n)}$ where $S(n)$ is the number of simple multi-cycles of length $n$. 
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**Theorem**

Let $\sigma \leq 1$. Then $P(t)$ defines a holomorphic function in the unit disk, and its root of minimum modulus is $r^{-1}$. 
Wedges

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(1, 2) (1, 3) (1, 4) (1, 5)  
(2, 3) (2, 4) (2, 5)  
(3, 4) (3, 5)  
(4, 5)  

...
Labeled wedges

Label all pairs as either separated or non-separated
Labeled wedges

Label all pairs as either separated or non-separated

(3, 4)  …

(2, 3)  (2, 4)  …

(1, 2)  (1, 3)  (1, 4)  …

(The boxed pairs are the separated ones.)
From wedges to graphs

Define a graph associated to the wedge as follows:

- If \((i, j)\) is non-separated, then \((i, j) \rightarrow (i + 1, j + 1)\).
- If \((i, j)\) is separated, then \((i, j) \rightarrow (1, i + 1)\) and \((i, j) \rightarrow (1, j + 1)\).
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\[
\begin{align*}
(3, 4) & \rightarrow \cdots \\
(2, 3) & \rightarrow (2, 4) \rightarrow \cdots \\
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“This sounds like climbing a mountain; you go up step by step, but you chute all the way to the bottom, and in two broken pieces” (August 25, 2014)
Continuity: sketch of proof

Suppose $\theta_n \to \theta$ (primitive majors)

Then $W_{\theta_n} \to W_{\theta}$ (wedges)

so $P_{\theta_n}(t) \to P_{\theta}(t)$ (spectral determinants)

and $r_{\theta_n} \to r_{\theta}$ (growth rates)
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and \( r(\theta_n) \to r(\theta) \) \hspace{1cm} \text{(growth rates)}
Further directions / questions

1. **Conjecture:** In each stratum the maximum of the core entropy equals

   \[
   \max_{m \in \Pi} h(m) = \log(\text{Depth}(\Pi) + 1)
   \]

   where the Depth of a stratum is the maximum length of a chain of nested leaves in the primitive major.

2. The level sets of the function \( h(\theta) \) determines lamination for the Mandelbrot set:
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4. Derivative of core entropy yields measure on lamination for \( \text{M.set} \)
Further directions / questions

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\[ \max_{m \in \Pi} h(m) = \log(\text{Depth}(\Pi) + 1) \]

where the **Depth** of a stratum is the maximum length of a chain of nested leaves in the primitive major.

2. The level sets of the function \( h(\theta) \) determines lamination for the Mandelbrot set: use \( h \) to define **vein structure** in higher degree?

3. The local Hölder exponent of \( h \) at \( m \) equals \( \frac{h(m)}{\log d} \) (Fels)
   Also true for invariant sets of the doubling map on the circle, (Carminati-T., Bandtlow-Rugh)

4. Derivative of core entropy yields measure on lamination for M.set
Merci!