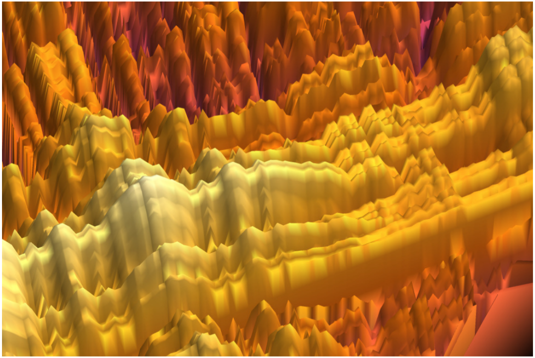

Entropy along the Mandelbrot set

Giulio Tiozzo
University of Toronto

André Aisenstadt Lecture – Montréal, 15^{th} October, 2021

Summary

1. What is... (topological) entropy?

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2. Entropy in dynamical systems

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6. The higher degree case

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Text A

1, 14, 4, 27, 20, 8, 5, 14, 3, 5, 27, 23, 5, 27, 9, 19, 19, 21, 5, 4, 27, 6, 15, 18, 20, 8, 27, 20, 15, 27, 19, 5, 5, 27, 1, 7, 1, 9, 14, 27, 20, 8, 5, 27, 19, 20, 1, 18, 19

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1, 14, 4, 27, 20, 8, 5, 14, 3, 5, 27, 23, 5, 27, 9, 19, 19, 21, 5, 4, 27, 6, 15, 18, 20, 8, 27, 20, 15, 27, 19, 5, 5, 27, 1, 7, 1, 9, 14, 27, 20, 8, 5, 27, 19, 20, 1, 18, 19

Text B

25, 18, 9, 10, 5, 4, 11, 20, 17, 20, 9, 15, 27, 3, 18, 6, 26, 17, 11, 6, 6, 18, 26, 14, 16, 21, 7, 17, 21, 9, 13, 17, 18, 27, 20, 6, 4, 25, 8, 22, 2, 3, 26, 11, 19, 6, 12, 5, 23

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Idea: natural languages have redundancies

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Idea: natural languages have <u>redundancies</u>

Th_ art_st is __e creato_ of be__t_ful th_ng_.
To revea_ a_t and con_ea_ the _rtist _s art'_ a_m.
The cr_t_c is he wh_ _an tra_slat_ into ano_he_ manner
or a n_w mate_ial hi_ impre_sio_ of b_a_tiful _h_ngs.

(Osc__ Wil__, The Picture __ ______ ____)"

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1, 14, 4, 27, 20, 8, 5, 14, 3, 5, 27, 23, 5, 27, 9, 19, 19, 21, 5, 4, 27, 6, 15, 18, 20, 8, 27, 20, 15, 27, 19, 5, 5, 27, 1, 7, 1, 9, 14, 27, 20, 8, 5, 27, 19, 20, 1, 18, 19

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h = 2.52095

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25, 18, 9, 10, 5, 4, 11, 20, 17, 20, 9, 15, 27, 3, 18, 6, 26, 17, 11, 6, 6, 18, 26, 14, 16, 21, 7, 17, 21, 9, 13, 17, 18, 27, 20, 6, 4, 25, 8, 22, 2, 3, 26, 11, 19, 6, 12, 5, 23

h = 3.06246

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h = 3.06246 (Random selection: $\log 27 = 3.29...$)

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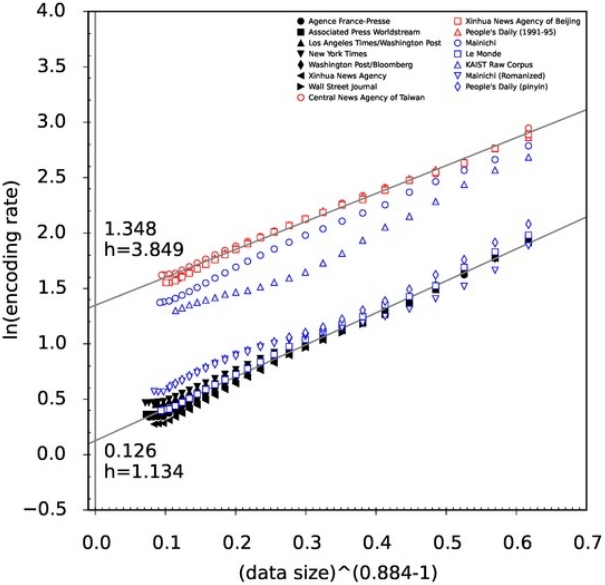

From: R. Takahira, K.Tanaka-Ishii, L. Debowski (2016)

Sequences produced by dynamical systems

Let $f(x) = x^2 + c$.

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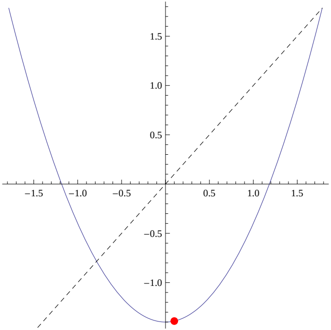

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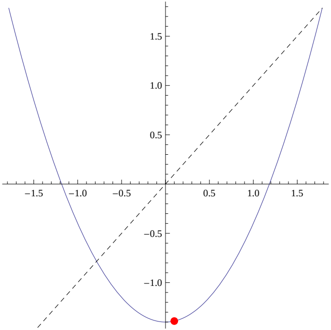

$\Sigma = 1$

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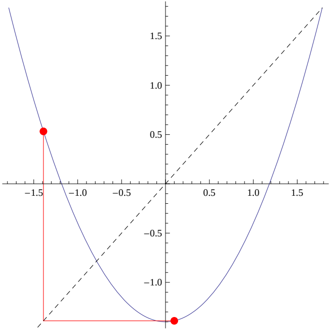

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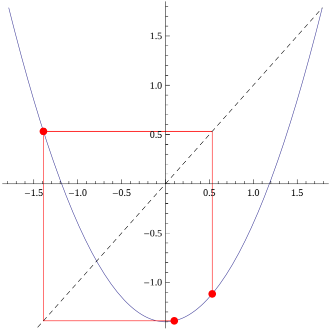

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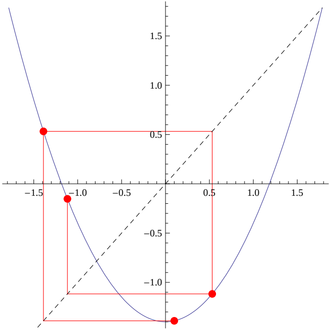

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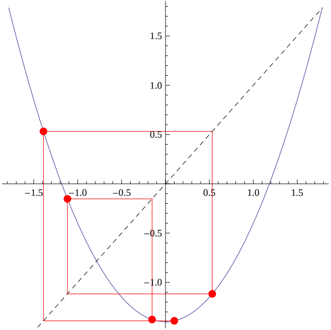

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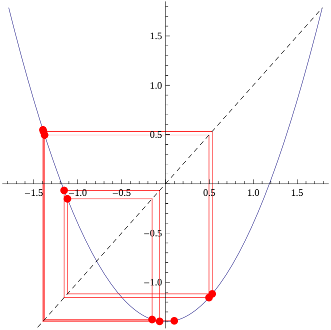

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Topological entropy of real interval maps

Thus, we have a map $\Sigma : I \to \{0, 1\}^{\mathbb{N}}$

starting point $x \mapsto \Sigma(x)$ infinite binary code.

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Note: For quadratic maps $h_{top}(f) \leq \log 2$.

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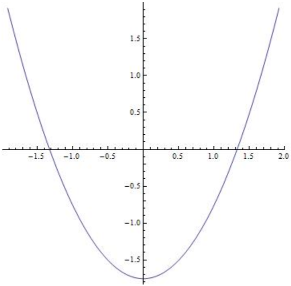

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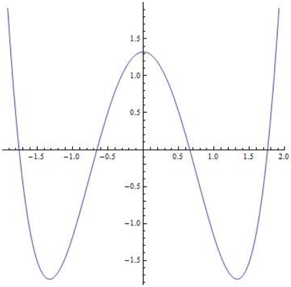

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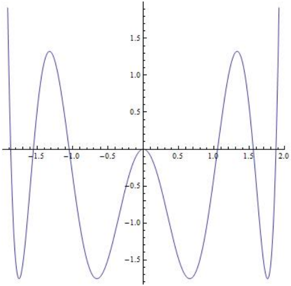

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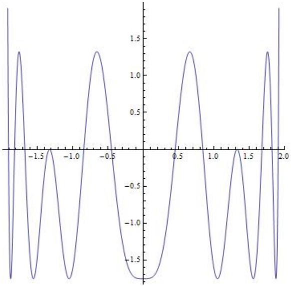

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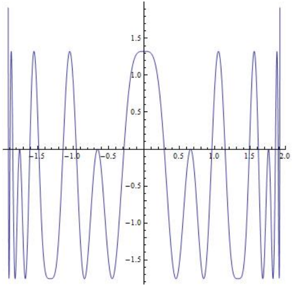

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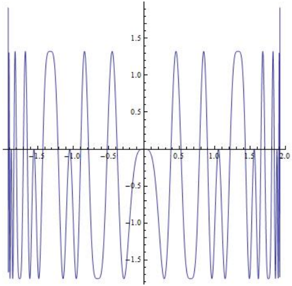

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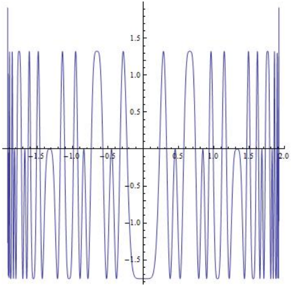

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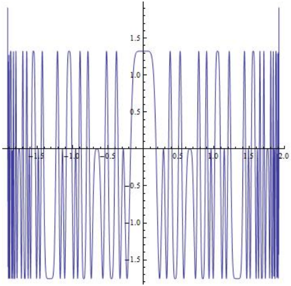

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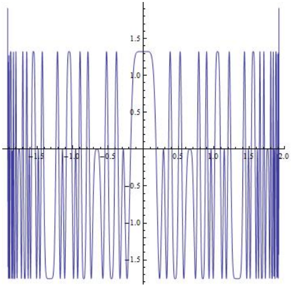

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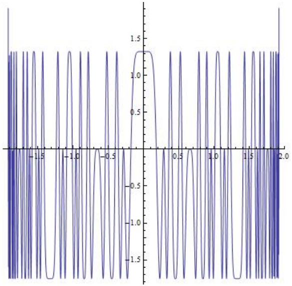

Agrees with general definition for maps on compact spaces using open covers (Misiurewicz-Szlenk)

Example: the airplane map

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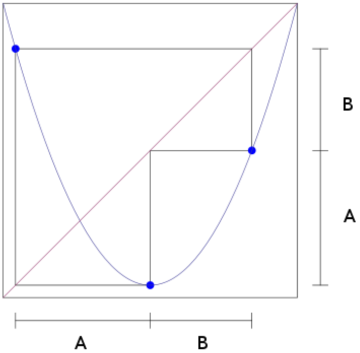

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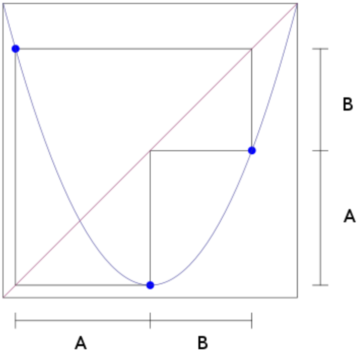

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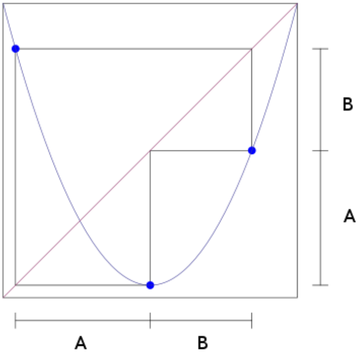

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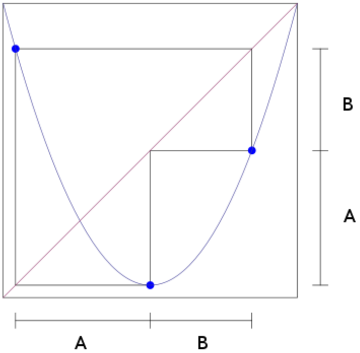

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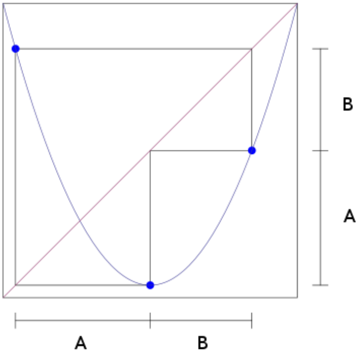

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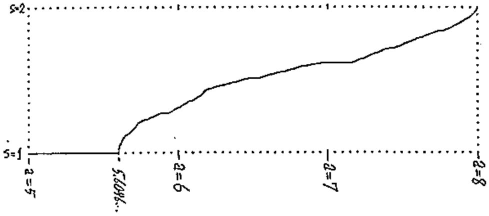

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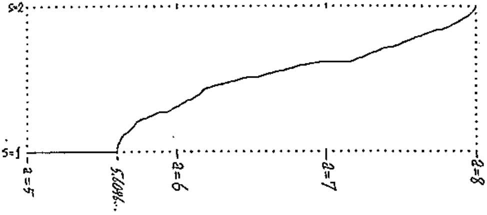

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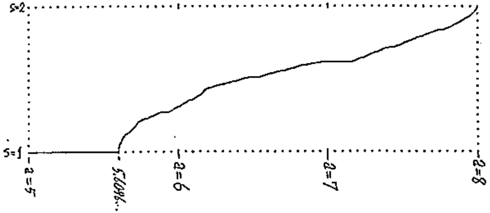

Question : Can we extend this theory to complex polynomials?

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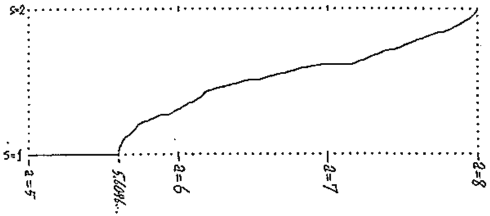
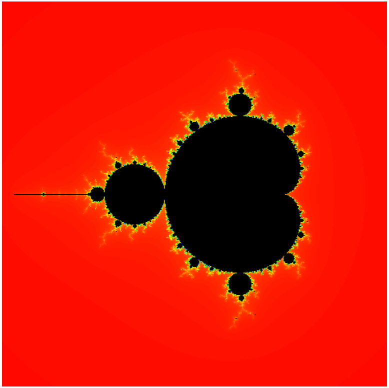


Remark. If we consider $f_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ entropy is constant $h_{top}(f_c, \hat{\mathbb{C}}) = \log 2$. (Lyubich 1980)

Mandelbrot set

The Mandelbrot set $\mathcal{M}$ is the connectedness locus of the quadratic family $f_c(z) := z^2 + c$.

$$\mathcal{M} = \{c \in \mathbb{C} \ : \ f_c^n(0) \not\to \infty\}$$



External rays

Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected, it can be uniformized by the exterior of the unit disk

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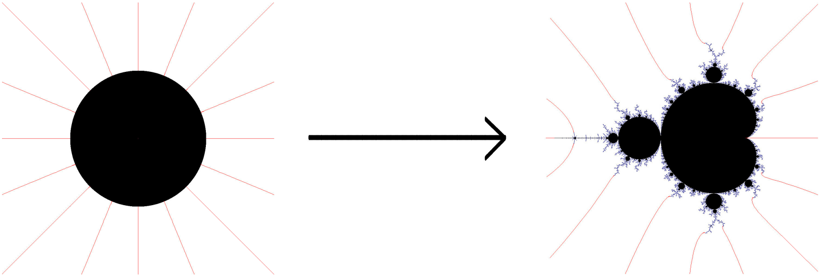

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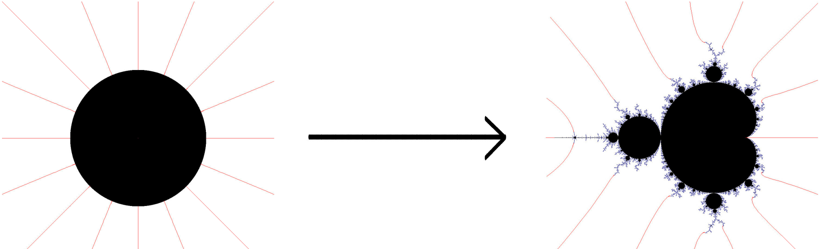

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As a consequence, the Mandelbrot set is homeomorphic to a quotient of the closed disk (hence locally connected).

Thurston's quadratic minor lamination (QML)

Define $\theta_1 \sim_M \theta_2$ on $\mathbb{Q}/\mathbb{Z}$ if $R_M(\theta_1)$ and $R_M(\theta_2)$ land together.

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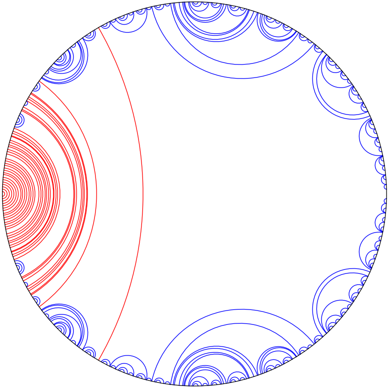

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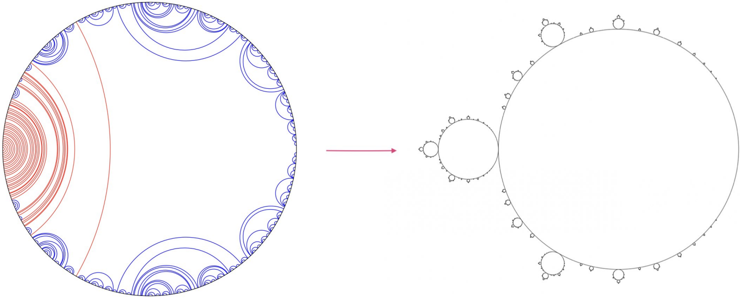

The quotient $\mathcal{M}_{abs}$ of the disk by the lamination is a (locally connected) model for the Mandelbrot set, and homeomorphic to it if MLC holds.

Julia sets

Let $f_c(z) = z^2 + c$. Then the filled Julia set of f_c is the set of points which do not escape to infinity under forward iteration:

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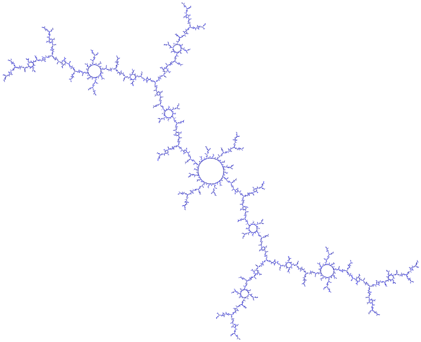
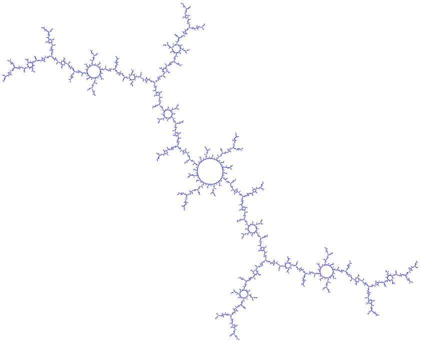


The complex case: Hubbard trees

The Hubbard tree T_c of a quadratic polynomial is a forward invariant, connected subset of the filled Julia set which contains the critical orbit.



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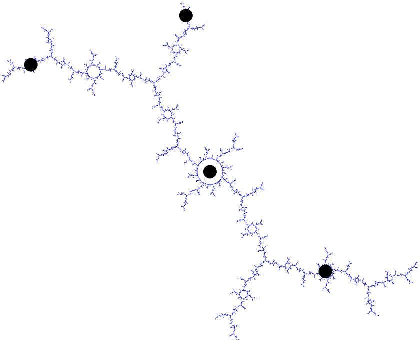

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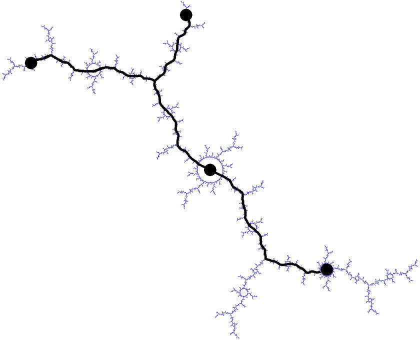

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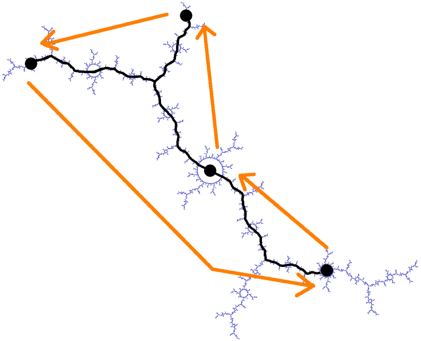

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where T_f is the Hubbard tree of f.

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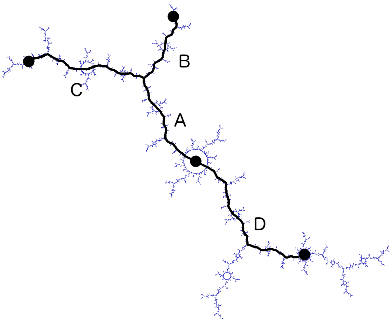

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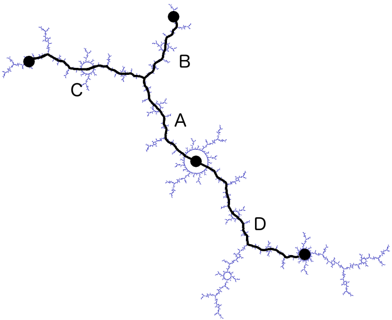

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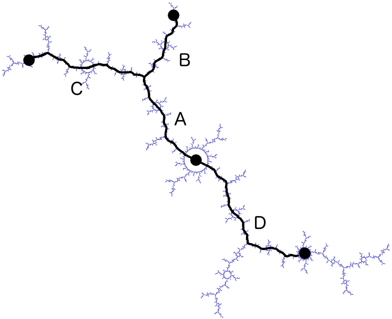

$A \to B$
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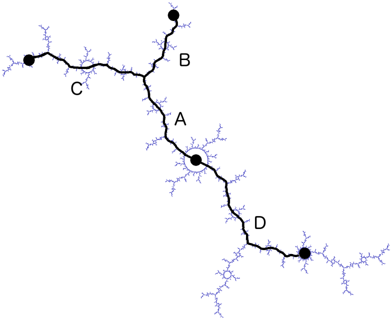

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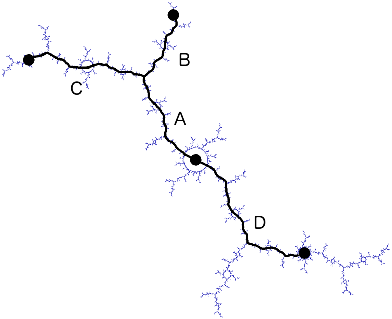

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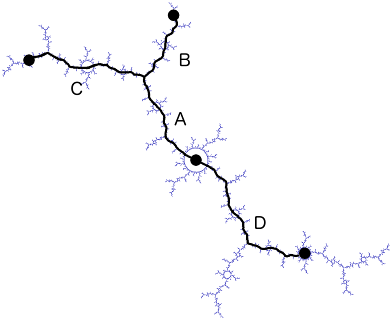

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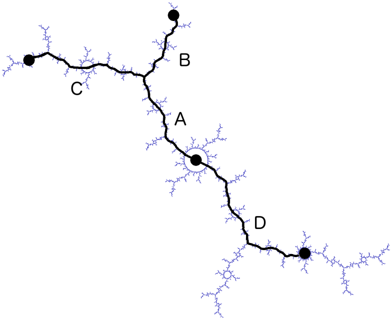

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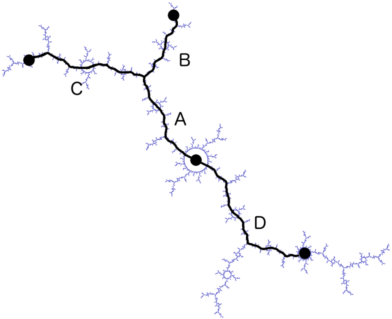

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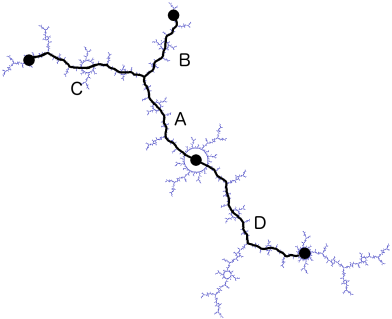

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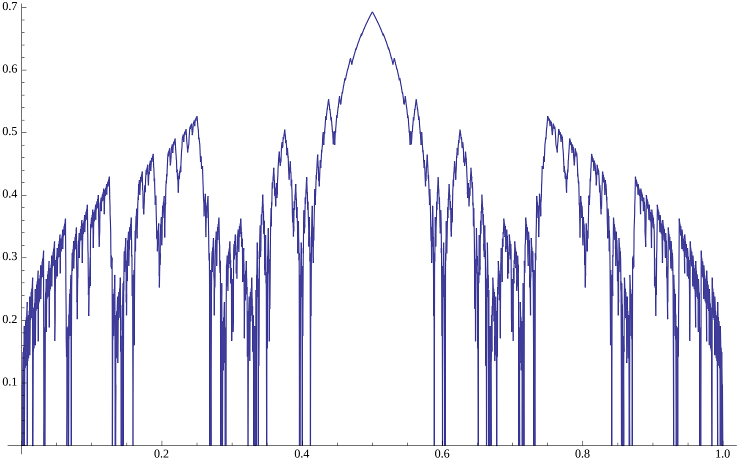
Question: How does $h(\theta)$ vary with the parameter θ?

Core entropy as a function of external angle (W. Thurston)



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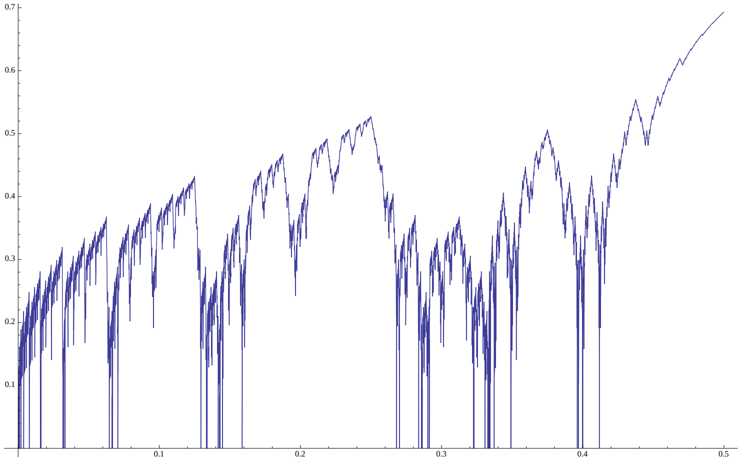

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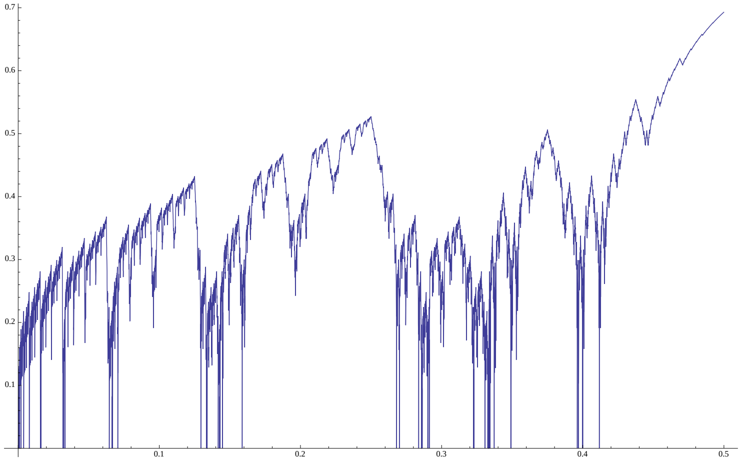

Question Can you see the Mandelbrot set in this picture?

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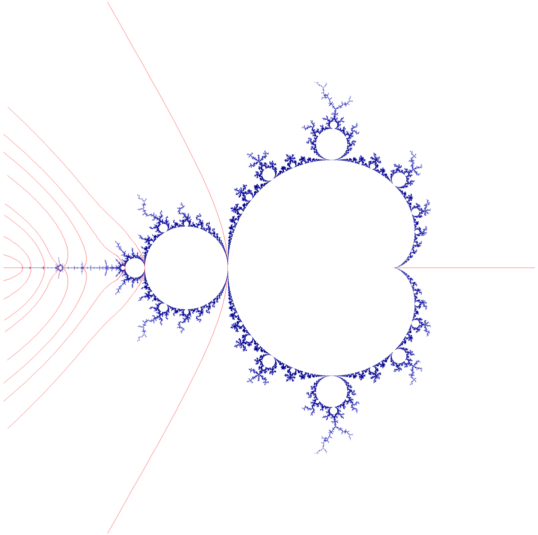
Rays landing on the real slice of the Mandelbrot set



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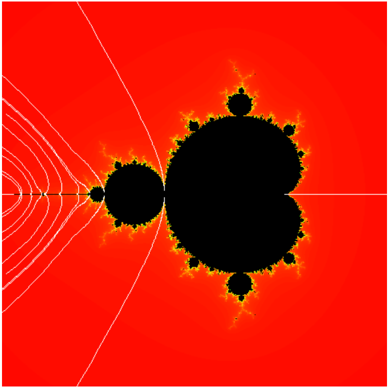
For instance, take $A = \mathcal{M} \cap \mathbb{R}$ the real section of the Mandelbrot set. How common is it for a ray to land on the real axis?



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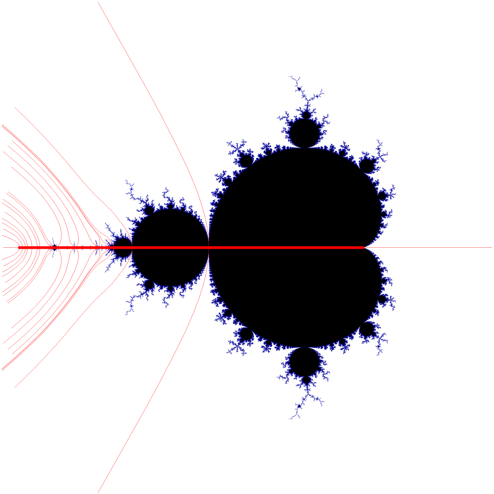

Sectioning $\mathcal{M}$

Given $c \in [-2, 1/4]$, we can consider the set of external rays which land on the real axis to the right of c:

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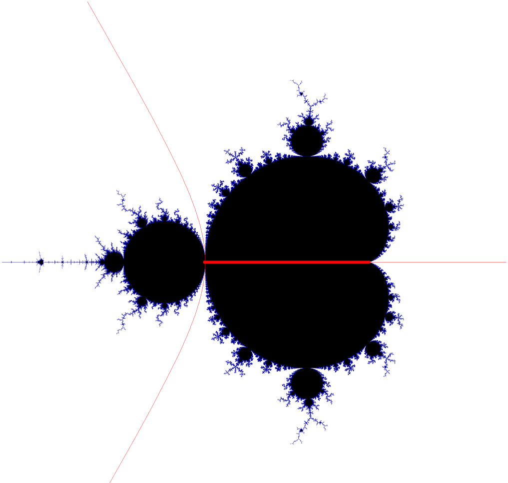

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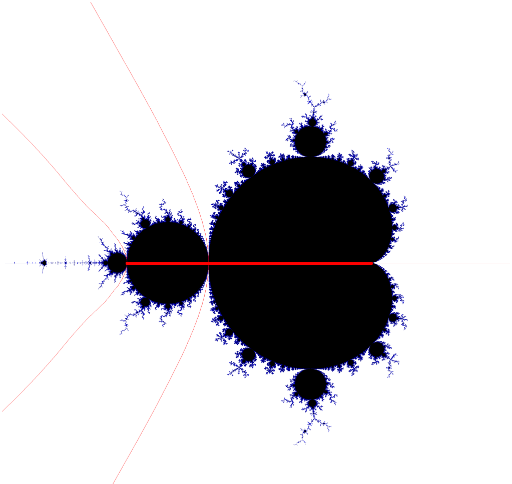

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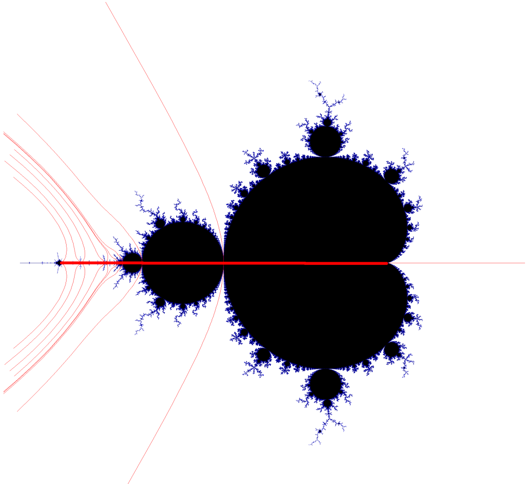

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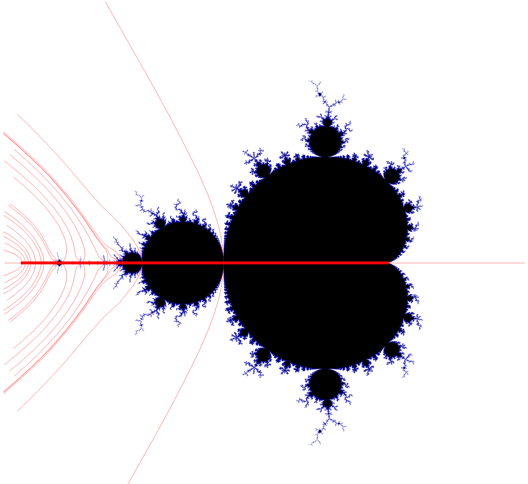

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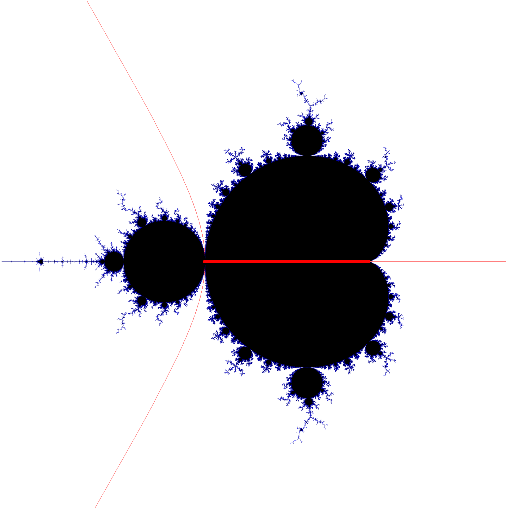

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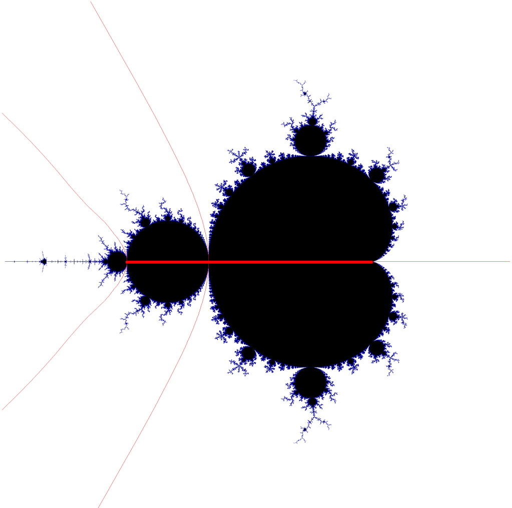

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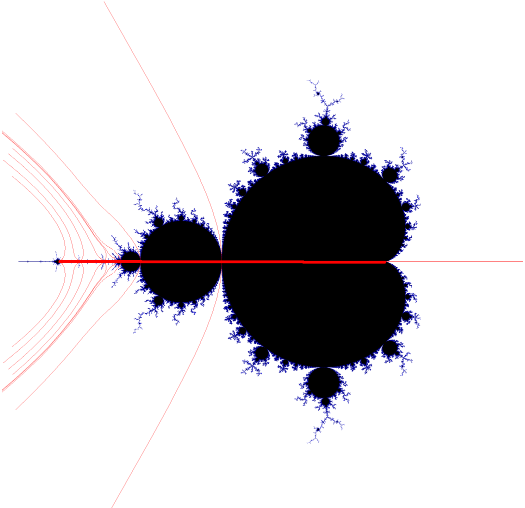

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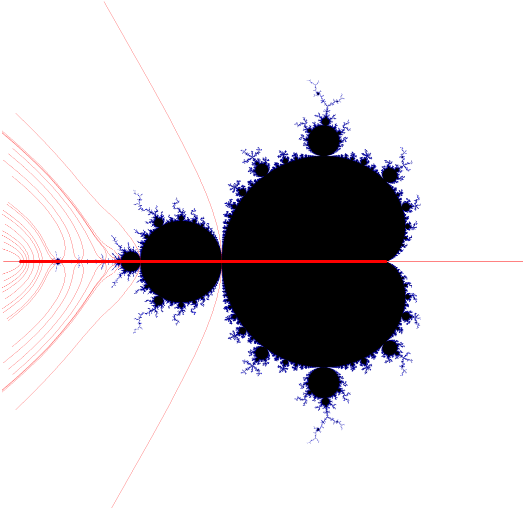

Entropy formula, real case

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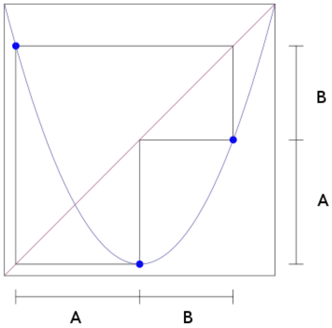

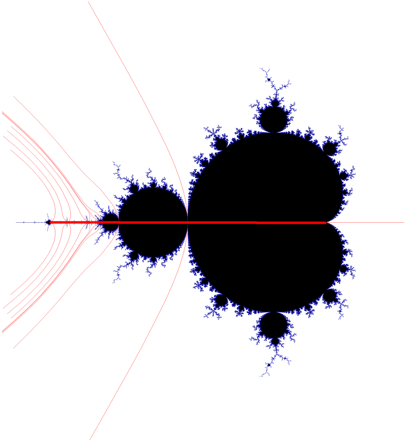

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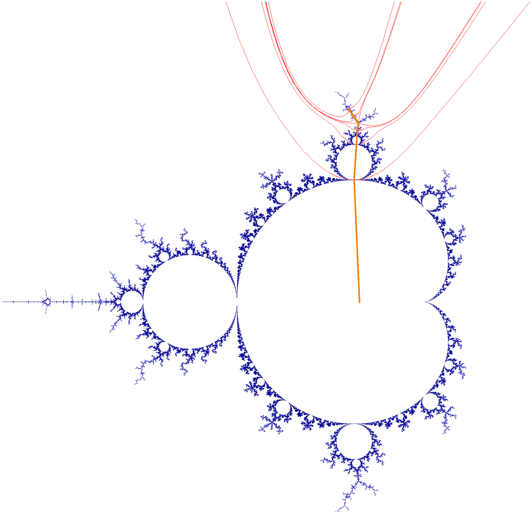
- It can be generalized to non-real veins.

Entropy formula along complex veins

A vein is an embedded arc in the Mandelbrot set.



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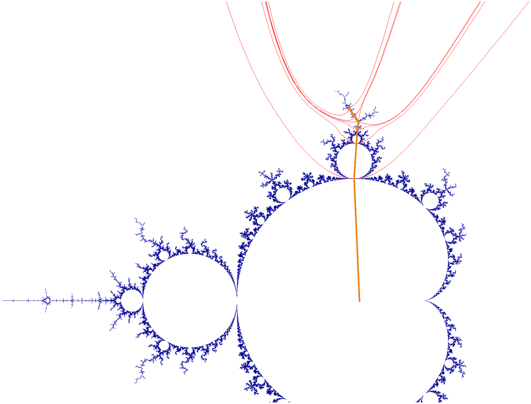

Given a parameter c along a vein, we can look at the set P_c of parameter rays which land on the vein between 0 and c.

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Let v be a vein in the Mandelbrot set, and let $c \in v$.

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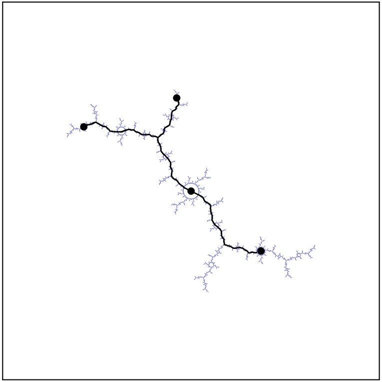
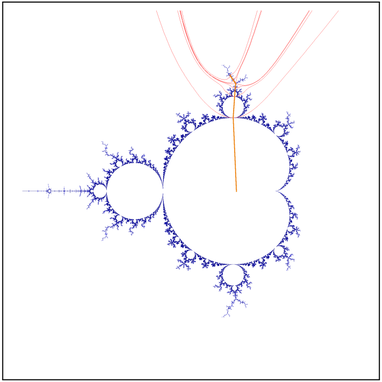
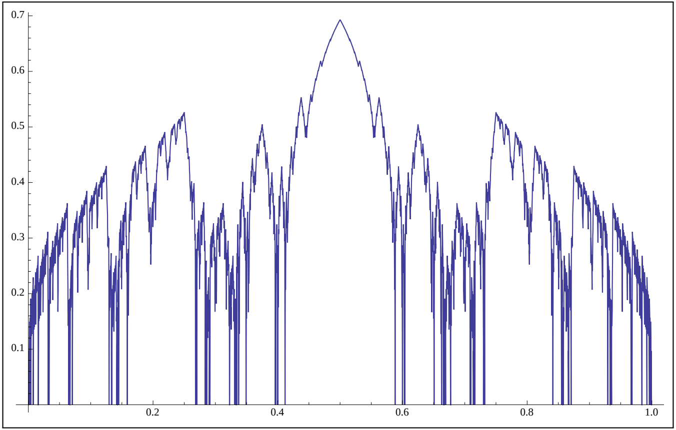




The core entropy as a function of external angle

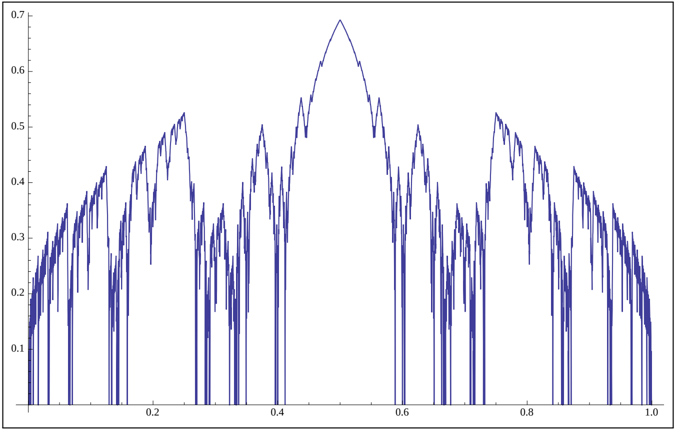
Question (Thurston, Hubbard):
Is $h(\theta)$ a continuous function of θ?



The Main Theorem: Continuity

Theorem (T.)

The core entropy function $h(\theta)$ extends to a continuous function from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.



Regularity properties of the core entropy

In fact:

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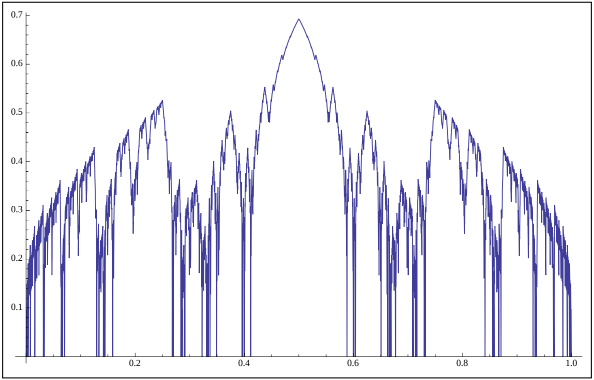

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(Conjectured by Isola-Politi, 1990)

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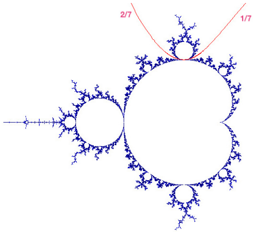

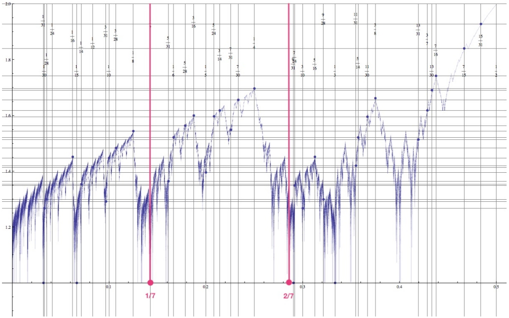

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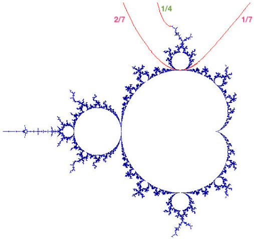

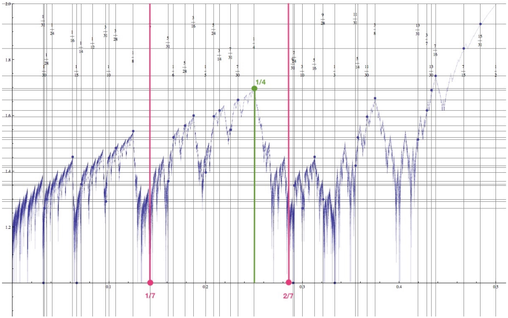

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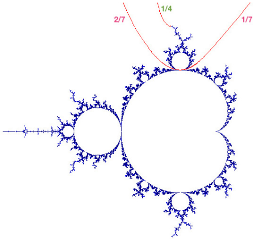
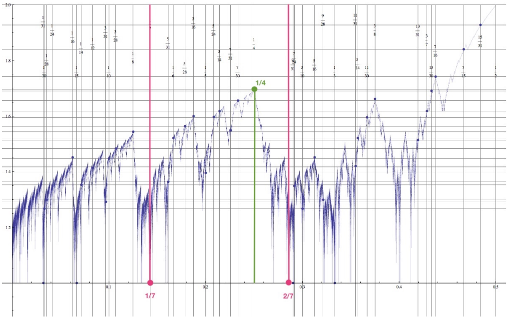
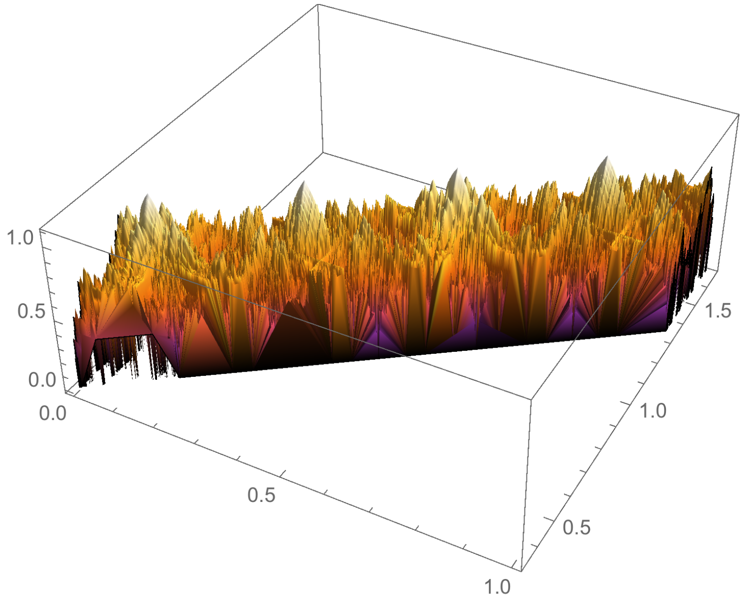




The core entropy for cubic polynomials



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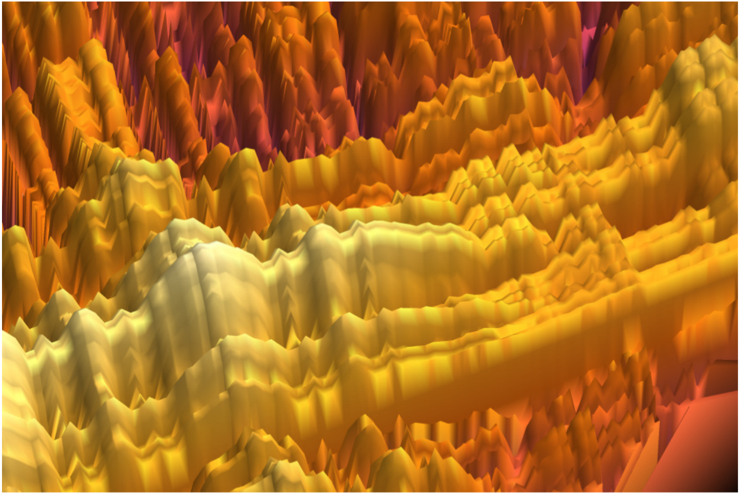

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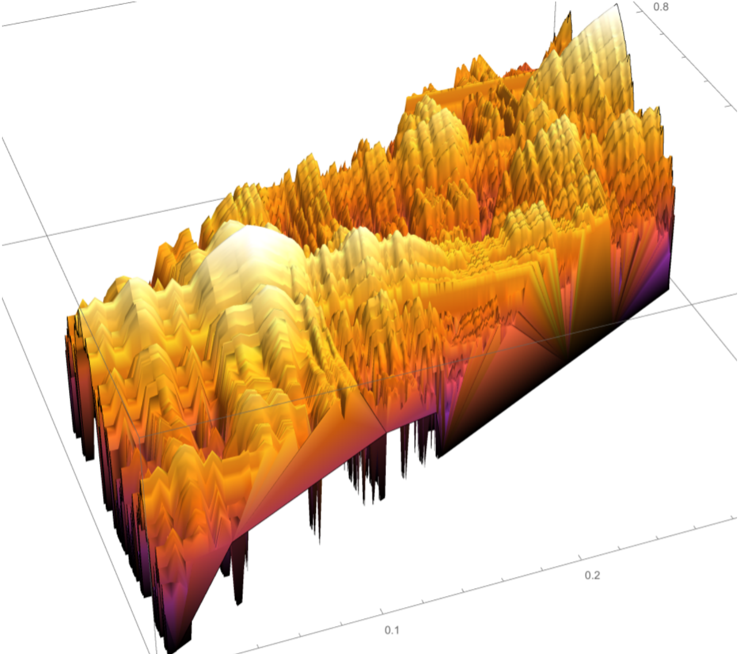
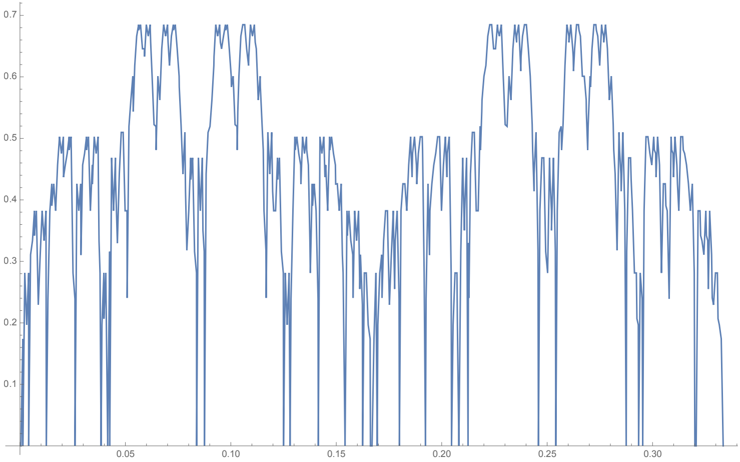


The unicritical slice



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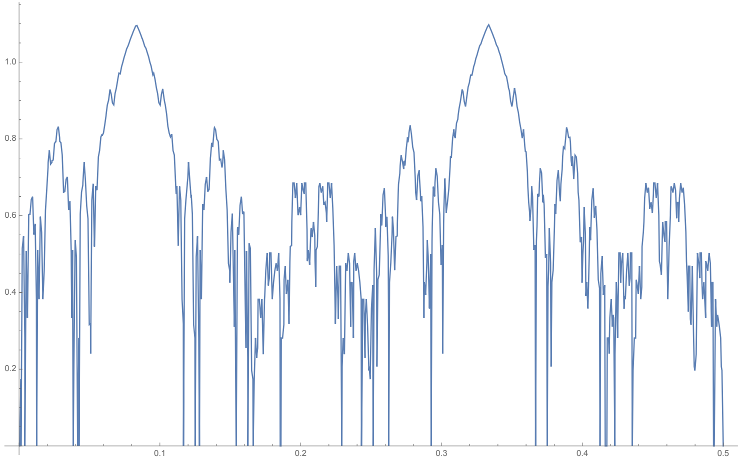

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Continuity in higher degree, combinatorial version

For polynomials of degree d, the analog of the circle at infinity for the Mandelbrot set is the set $PM(d)$ of primitive majors.

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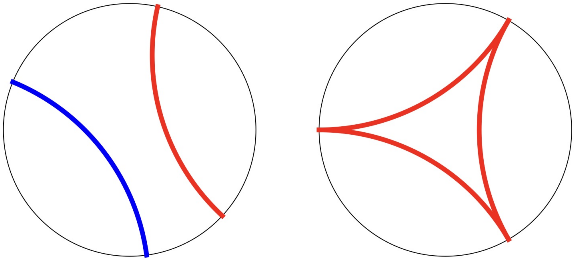

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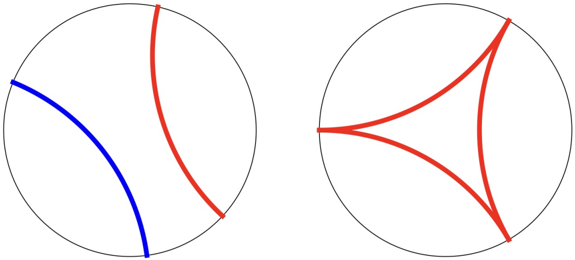

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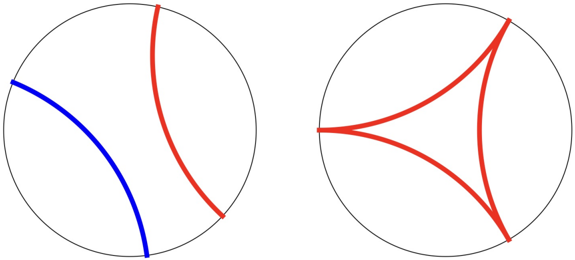

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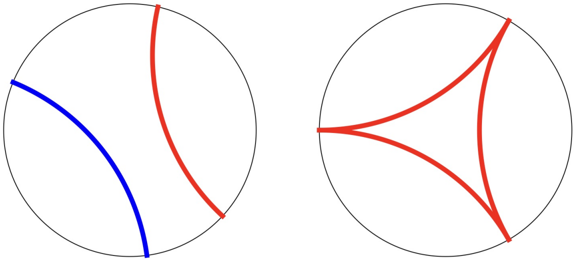

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Example. $\pi_1(PM(3)) = \langle x, y \ : x^2 = y^3 \rangle$

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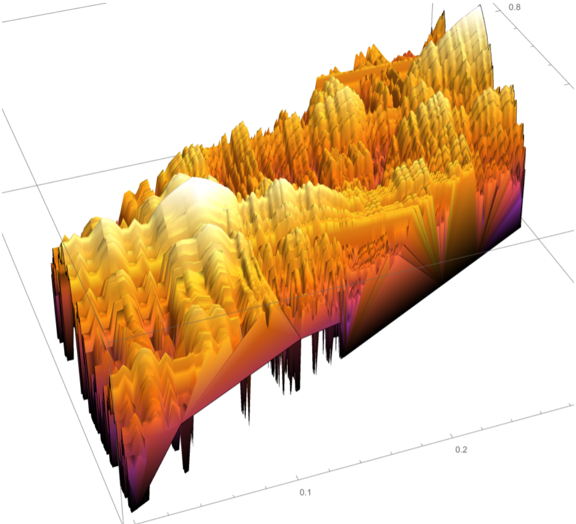

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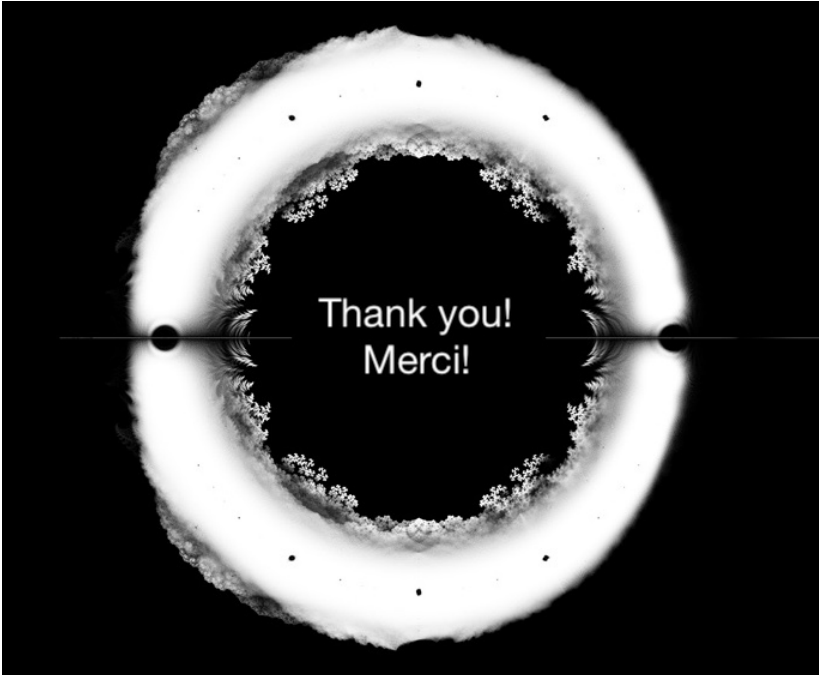
Thank you!
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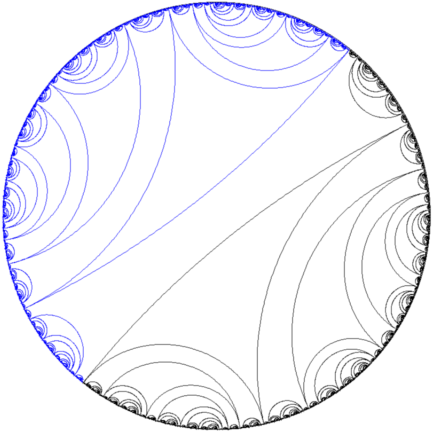

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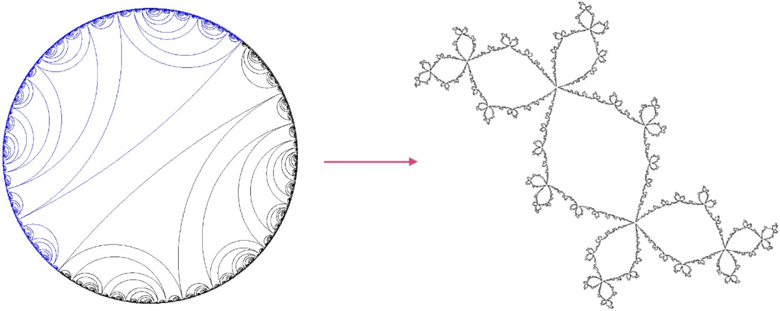

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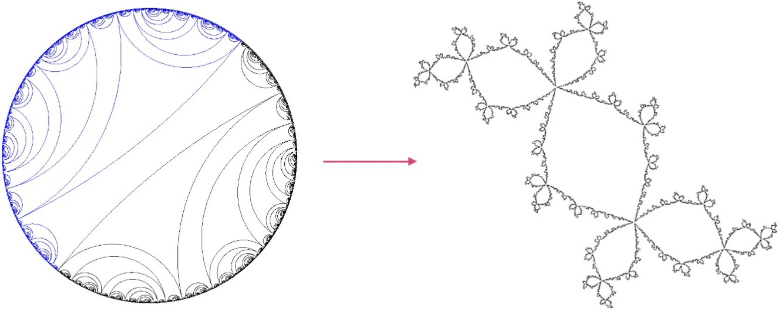

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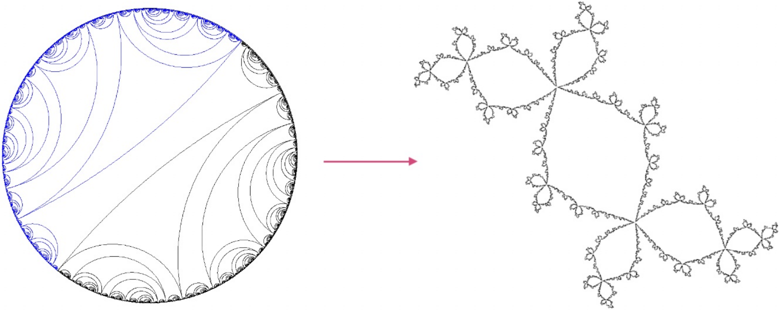

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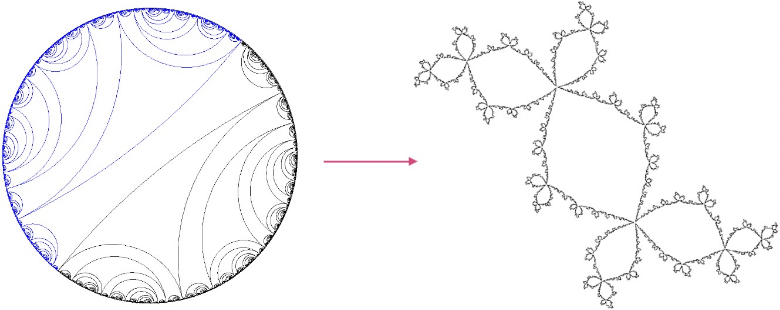

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(Compare: Milnor-Thurston, Baillif-deCarvalho, Sousa-Ramos, ...)

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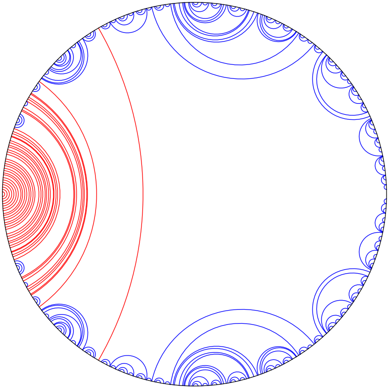

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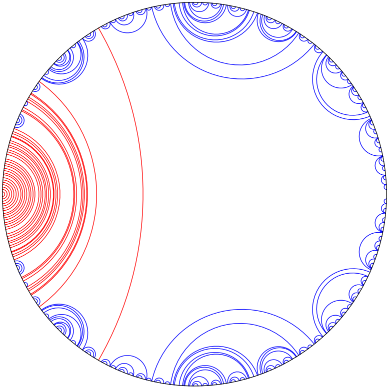

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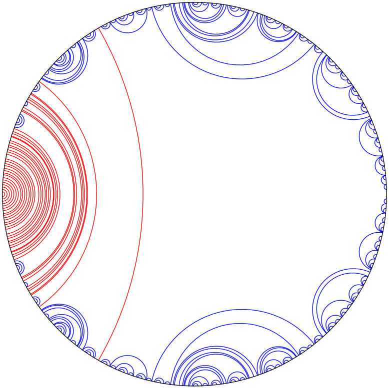

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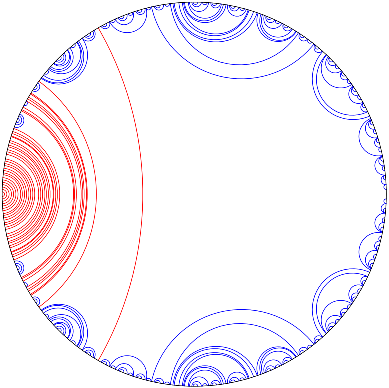

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