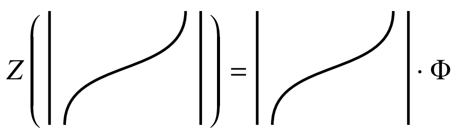
Braidors and Associators

Abstract. Studying braids in an annulus rather than braids in a disk yields a theory of braidors, the analogue of Drinfeld associators. A priori, braidors appear weaker than associators and furthermore the equations make sense in more spaces than the associator equations do. However, computational evidence suggests the braidor equations are in fact equivalent to the associator equations. This suggests it may be useful to review the array of ideas related to associators, e.g. Grothendieck-Teichmüller groups, multiple zeta values etc., in the simpler context of braidors in the hopes of gaining new information.

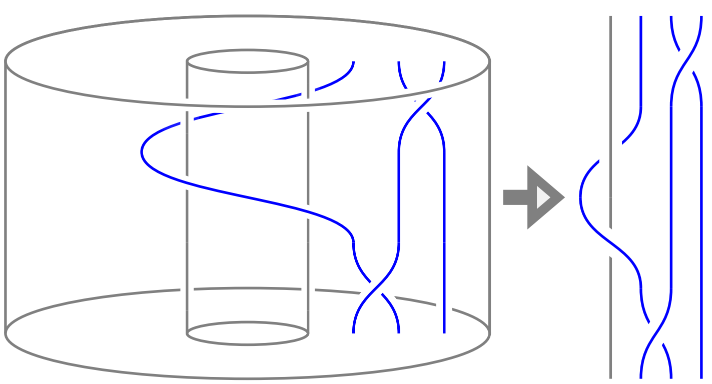
Associators.

PaB = category/operad of parenthesized braids
PaCD = category/operad of parenthesized chord diagrams
Associators $\iff$ operad morphisms $Z : \mathbf{PaB} \to \mathbf{PaCD}$



$\Phi \in \widehat{\mathfrak{t}_3}$ is a Drinfeld associator.

Annular Braids.



0 1 2 3

Permutation: $(0, 3, 1, 2) \in S_{1,3}$

$B_{1,n}$ = n-component braids in the annulus

$$= \left\langle \tau, \sigma_1, \cdots, \sigma_{n-1} \,\middle|\, \begin{matrix} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ [\sigma_i, \sigma_j] = 1 \quad |i-j|>0, \\ [\tau, \sigma_1\tau\sigma_1] = 1, \\ [\tau, \sigma_i] = 1 \quad i>1, \end{matrix} \right\rangle$$

$= \{B \in B_{n+1} : \text{first strand ends in first position}\}$ [B][KP]

For pure braid groups,

$$PB_{1,n} = PB_{n+1} = \mathfrak{t}_{n+1}$$

Definition 1.
For A a commutative, associative $\mathbb{Q}$–algebra, define $\boldsymbol{B_a}$ = category of braids in the annulus:

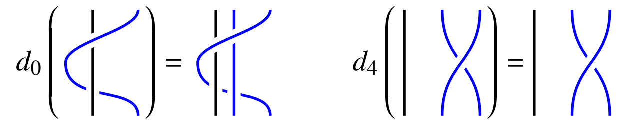
$$\mathrm{Obj}_{\boldsymbol{B_a}} = \text{ finite ordinals } \boldsymbol{n} = \{0, 1, \cdots, n\}$$

$$\mathrm{Mor}_{\boldsymbol{B_a}}(\boldsymbol{m}, \boldsymbol{n}) = \begin{cases} \varnothing & \boldsymbol{m} \neq \boldsymbol{n} \\ \bigsqcup_{P \in S_{1,n}} A\left[B_{1,n}^P\right] & \boldsymbol{m} = \boldsymbol{n} \end{cases}$$

$P \in S_{1,n}$ is a permutation fixing 0
$B_{1,n}^P$ = group of annular braids with underlying permutation P

Category With Operations.
$\{PaB_n\}_{n\in\mathbb{N}}$ (almost) forms a cosimplicial set
Coface Operators $\{d_i\}$: strand doubling
Codegeneracies $\{s_i\}$: strand deletion

Annular Case. Throw away everything except d_0, d_{n+1}



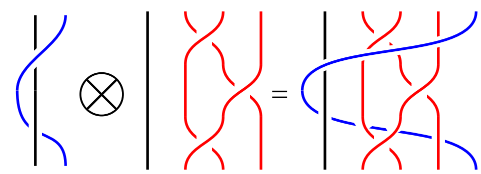
Monoidal Structure. Can view $\{\mathbf{PaB}_n\}$ as an operad in groupoids with partial composition

$$P_1 \circ_i P_2 = \text{glue } P_2 \text{ into the } i\text{th strand of } P_1$$

Annular Case. $\boldsymbol{B_a}$ is a (strict) monoidal category with tensor product:

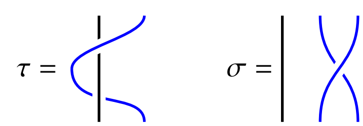
On Objects: $\boldsymbol{m} \otimes \boldsymbol{n} = \boldsymbol{m} + \boldsymbol{n}$

On Morphisms:



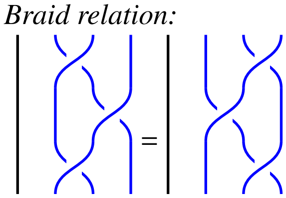
Coproduct. Define the coproduct $\square$ by making any annular braid $B \in B_{1,n}$ grouplike.

Claim 1. *$\boldsymbol{B_a}$ is generated as a strict monoidal category (by repeated applications of d_0 and d_{n+1}) by $\tau^{\pm 1}$ and $\sigma^{\pm 1}$ where*

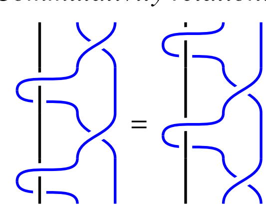


subject to the relations generated by

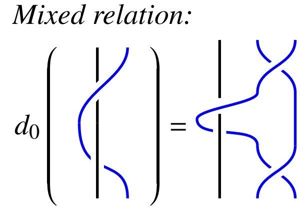
- *Braid relation:*



- *Commutativity relation:*



- *Mixed relation:*



- *(Locality in Space and Scale)*

Filtrations and Completions. Define the unipotent filtration/completion of the category $\boldsymbol{B_a}$ by applying the corresponding operation to each $A[B_{1,n}^P]$.
Let $I_{1,n}$ be the augmentation ideal in $\sqcup_{P\in S_{1,n}} A[B_{1,n}^P]$.

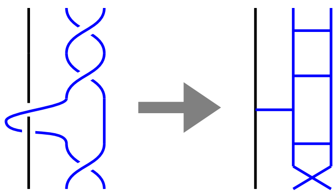
Definition 2. Let ${B_a}^{(m)} = B_a/\mathcal{F}_m B_a$ be the m**th unipotent quotient** of B_a:

$$\mathrm{Mor}_{{B_a}^{(m)}}(\mathbf{n},\mathbf{n}) = \bigsqcup_{P\in S_{1,n}} A[B^P_{1,n}]/(I^P_{1,n})^m$$

Let $\widehat{B_a} = \varprojlim_{m\to\infty} {B_a}^{(m)}$ be the **unipotent completion** of B_a:

$$\mathrm{Mor}_{{B_a}^{(m)}}(\mathbf{n},\mathbf{n}) = \bigsqcup_{P\in S_{1,n}} \varprojlim_{m\to\infty} A[B^P_{1,n}]/(I^P_{1,n})^m$$

Chord Diagrams for Annular Braids. Chord diagrams remember only crossing information:



$= (1,2)\cdot t_{12}t_{02}t_{12}t_{12}$

Algebraically, the space of chord diagrams is the associated graded of the space of braids.

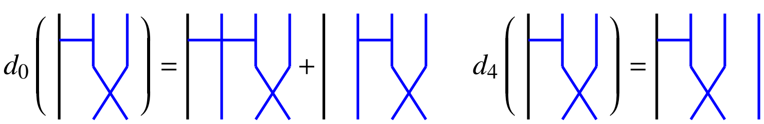
Definition 3. Let $CD_a = CD_a(A)$ be the category of chord diagrams for annular braids:

Obj_{CD_a} = finite ordinals $\boldsymbol{n} = \{0, 1, \cdots, n\}$
Mor_{CD_a} = A-linear combinations of formal products $P\cdot D$

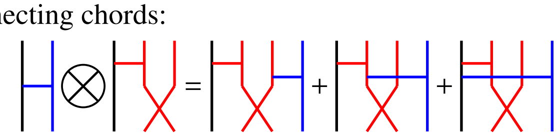
with $P \in S_{1,n}$ and $D \in \mathfrak{t}_{n+1}(A)$.

Structure. CD_a can be given all the same structure that was given to B_a:

Doubling:

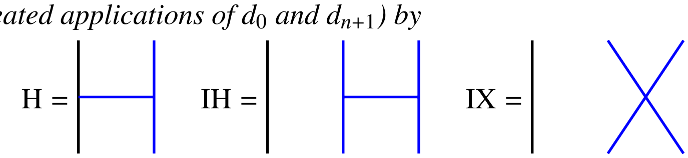


Monoidal Structure: Glue into the core and sum all ways of connecting chords:



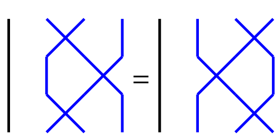
Coproduct: Define $\square$ by making individual chords primitive.

Claim 2. *CD_a is generated as a strict monoidal category (by repeated applications of d_0 and d_{n+1}) by*

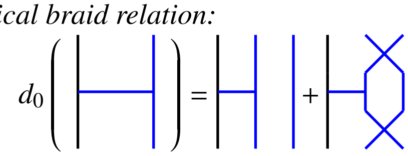


subject to the relations generated by

- *Braid relation:*

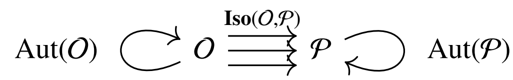


- *Semi-classical braid relation:*



Filtrations and Completions. Chord diagrams are graded by the number chords. $\widehat{CD_a}$ = degree completion of CD_a (this is also the unipotent completion of CD_a.)

General Setup.



Aut($\mathcal{O}$) and Aut($\mathcal{P}$) act simply and transitively on **Iso**($\mathcal{O},\mathcal{P}$)
Example. $\mathcal{O} = \widehat{\mathbf{PaB}}$, $\mathcal{P} = \widehat{\mathbf{PaCD}}$. Then **Iso**($\widehat{\mathbf{PaB}}, \widehat{\mathbf{PaCD}}$) is the set of Drinfeld associators, and Aut($\widehat{\mathbf{PaB}}$) = $\widehat{\mathbf{GT}}$ and Aut($\widehat{\mathbf{PaCD}}$) = $\widehat{\mathbf{GRT}}$ are the (pro-unipotent versions of the) Grothendieck-Teichmüller groups as defined by Drinfeld [BN, D].

Braidors. Braidor are monoidal functors $Z \in \mathrm{Iso}(\widehat{B_a}, \widehat{CD_a})$ where Z is required to preserve the underlying permutation of a braid and gr$Z$ must be the identity.
Determined by

$$Z(\sigma) = \mathrm{IX}\cdot B \qquad Z(\tau) = R$$

where $B \in \hat{\mathfrak{t}}_3$ and $R = \exp(t_{01})$. Ensuring the relations in B_a are satisfied yields:

Definition 4. A **braidor** is a grouplike (nondegenerate) element $B \in \hat{\mathfrak{t}}_3$ which satisfies the equations

$$B^{0,1,2}B^{02,1,3}B^{0,2,3} = B^{01,2,3}B^{0,1,3}B^{03,1,2} \qquad \text{(Braidor Eqn.)}$$
$$d_0(R) = R^{01,2} = BR^{0,2}B^{0,2,1} \qquad \text{(Mixed Eqn.)}$$
$$R^{0,1}BR^{0,2}B^{0,2,1} = BR^{0,2}B^{0,2,1}R^{0,1} \qquad \text{(Commutativity Eqn.)}$$

Annular Grothendieck-Teichmüller Groups.

$\widehat{\mathbf{GT}}_a$: $\psi \in \mathrm{Aut}(\widehat{B_a})$ is determined by

$$\psi(\sigma) = \Sigma_1, \quad \psi(\tau) = \Sigma_2,$$

where $\Sigma_1 \in \widehat{PB_{1,2}}, \Sigma_2 \in \widehat{PB_{1,2}}$ and the the relations in Claim 2 must hold.

Definition 5. As a set, the group $\widehat{\mathbf{GT}}_a$ is the collection of all grouplike nondegenerate pairs $(\Gamma_1, \Gamma_2) \in \widehat{PB_{1,1}} \times \widehat{PB_{1,2}}$ which satisfy the equations

$$\Sigma_1^{0,1,2}\Sigma_1^{02,1,3}\Sigma_1^{0,2,3} = \Sigma_1^{01,2,3}\Sigma_1^{0,1,3}\Sigma_1^{03,1,2} \qquad \text{(Braidor Eqn.)}$$
$$d_0(\Sigma_2) = \Sigma_2^{01,2} = \Sigma_1\Sigma_2^{0,2}\Sigma_1^{0,2,1} \qquad \text{(Mixed Eqn.)}$$
$$\Sigma_2^{0,1}\Sigma_1\Sigma_2^{0,2}\Sigma_1^{0,2,1} = \Sigma_1\Sigma_2^{0,2}\Sigma_1^{0,2,1}\Sigma_2^{0,1} \qquad \text{(Commutativity Eqn.)}$$

$\widehat{\mathbf{GRT}}_a$: $\phi \in \mathrm{Aut}(\widehat{CD_a})$ is determined by

$$\phi(\mathrm{H}) = \Gamma_1, \quad \phi(\mathrm{IH}) = \Gamma_2, \quad \phi(\mathrm{IX}) = \mathrm{IX}\cdot\Gamma_3,$$

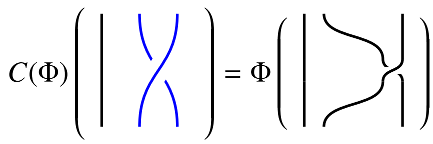
$\Gamma_1 \in \hat{\mathfrak{t}}_2$ and $\Gamma_2, \Gamma_3 \in \hat{\mathfrak{t}}_3$
Requiring ϕ to preserve the relations in Claim 2,

Definition 6. As a set, the group $\widehat{\mathbf{GRT}}_a$ is the collection of all grouplike nondegenerate triples $(\Gamma_1, \Gamma_2, \Gamma_3) \in \hat{\mathfrak{t}}_2\times\hat{\mathfrak{t}}_3\times\hat{\mathfrak{t}}_3$ which satisfy the equations

$$\Gamma_3^{0,1,2}\Gamma_3^{02,1,3}\Gamma_3^{0,3,2} = \Gamma_3^{01,2,3}\Gamma_3^{0,1,3}\Gamma_3^{03,1,2}$$
(Classical Braidor Eqn.)

$$d_0(\Gamma_1) = \Gamma_2 + \Gamma_3\Gamma_1^{0,2}\Gamma_3^{0,2,1}$$
(Semiclassical Braidor Eqn.)

From Associators to Braidors. Every associators yields a braidor; there is a map $C : \mathrm{Iso}(\widehat{\mathbf{PaB}}, \widehat{\mathbf{PaCD}}) \hookrightarrow \mathbf{BRAID}$



ie. $C(\Phi) = \Phi e^{\frac{1}{2}t_{12}}\Phi^{-0,2,1}$.

Results.

Theorem 1. *Braidors can be constructed degree by degree (and rational braidors exist.)*

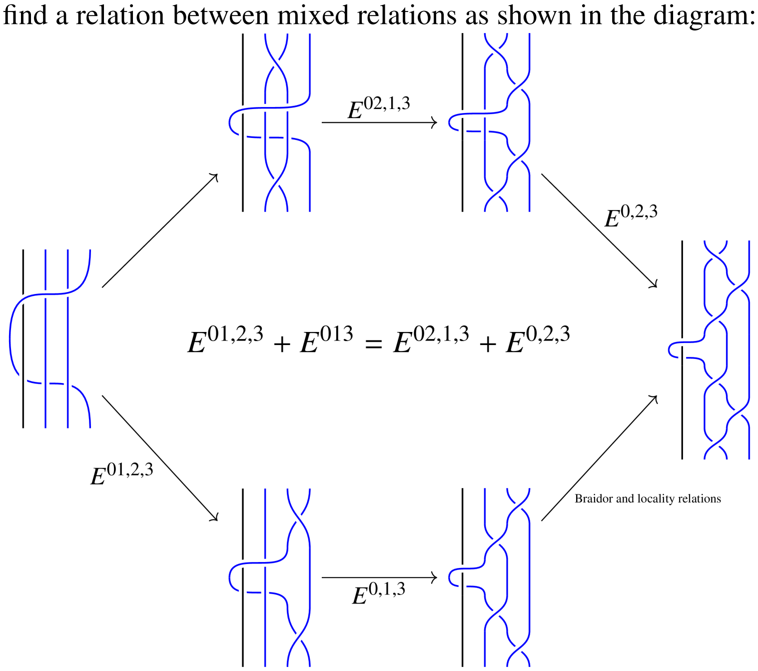
The proof of this theorem is inspired by and very similar to Drinfeld's proof of the same result for associators, especially as reformulated in [BN] although each step is simpler in the annular case.

Lemma 1. *The braid equation implies the semiclassical braid equation.*

Proof. Assume $\Gamma_3 \in \mathfrak{t}_3$ satisfies the braidor and locality relations and let E be the error in the semiclassical braidor equation, ie.

$$1 + E = d_0(\Gamma_1) - \Gamma_2 - \Gamma_3\Gamma_1^{0,2}\Gamma_3^{0,2,1}.$$

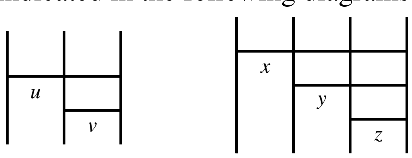
The semiclassical braidor can be derived from the mixed relation in $\boldsymbol{B_a}$, so to find an equation satisfied by E, we need to find a relation between mixed relations as shown in the diagram:



Each move in the diagram in which the mixed relation is applied will pick up an error E installed on the relevant strands. Comparing the errors along the two paths in the diagram, we get

$$E^{01,2,3} - E^{02,1,3} + E^{0,1,3} - E^{0,2,3} = 0. \tag{1}$$

Consider the free Lie algebras $FL[u, v]$ and $FL[x, y, z]$ contained in $\mathfrak{t}_2$ and $\mathfrak{t}_3$ as indicated in the following diagrams:



Every term in Equation 1 lies in $FL[x, y, z]$ so upon restriction, E becomes a Lie polynomial $F \in \mathrm{Lie}[u, v]$ and Equation 1 becomes

$$F(x + y, z) - F(x + z, y) + F(x, y) - F(x, z) = 0 \tag{2}$$

Applying the Lie morphism $FL_3 \mapsto FL_2$, $\{x \mapsto x, y \mapsto y, z \mapsto 0\}$ to equation 2 implies $\langle F, u\rangle = 0$ while applying the Lie morphism $\{x \mapsto 0, y \mapsto y, z \mapsto z\}$ implies $F(u, v) - F(v, u) = \langle F, v\rangle(v - u)$, ie. that F is symmetric up to an error in degree 1.

We have the following diagram of maps between free lie algebras and free associative algebras:

$$\begin{tikzcd}
FL[u,v] \arrow[r, "\Psi"] \arrow[d, hook] & FL[x,y,z] \arrow[d, hook] & & \\
FA[u,v] \arrow[r, "\overline{\Psi}"] \arrow[rrr, bend right, "\Theta"'] & FA[x,y,z] \arrow[r, "\pi"] & FA[x,y]\cdot z \arrow[r] & FA[x,y].
\end{tikzcd}$$

Ψ, and $\overline{\Psi}$ are obtained by sending a series f to $f(x+y, z) - f(x + z, y) + f(x, y) - f(x, z)$ and π is the projection onto words ending in z. Notice that $\ker\Psi \subseteq \ker\Theta$.

Regarding $F \in FL[u, v] \subseteq FA[u, v]$ as an element in the algebra of noncommutative series in u and v, write $F(u, v) = F_1(u, v)u + F_2(u, v)v$. Symmetry of F implies $F_1(u, v) = F_2(v, u) - \langle F, v\rangle$.

Tracing through the maps, $\Theta F = 0$ is equivalent to

$$F_2(x, y) = \sum_{n=0}^{\infty}\langle F_2, u^n\rangle\Big((x + y)^n - y^n\Big) - \langle F, v\rangle$$

and hence

$$F(u, v) = \sum_{n=0}^{\infty}\langle F_2, u^n\rangle\Big(\left[(u + v)^n - u^n\right]u + \left[(u + v)^n - v^n\right]v\Big) - \langle F, v\rangle v \tag{3}$$

In order for $F(u, v)$ to be a Lie polynomial, F must be primitive but it is clear from equation 3 that primitivity implies $F_2 = 0$. Thus $F(u, v) = \langle F, v\rangle v$ and since braidors are fixed in degree 1, $F = 0$. □

Lemma 2. *The natural homomorphism* $\widehat{\mathbf{GRT}}_a^{(m)} \to \widehat{\mathbf{GRT}}_a^{(m-1)}$ *is surjective.*

Proof. Since $\widehat{\mathbf{GRT}}_a^{(m)}$ are connected reduced algebraic group schemes, it is enough to prove the statement on the level of Lie algebras, defined by the linearization

$$\gamma^{0,1,2} + \gamma^{02,1,3} + \gamma^{0,3,2} = \gamma^{01,2,3} + \gamma^{0,1,3} + \gamma^{03,1,2}.$$

But clearly if γ is of degree m and satisfies this, it can be extended by taking the $m + 1$st degree component to be 0 for example. □

Lemma 3. *The natural map* $\mathbf{BRAID}^{(m)} \to \mathbf{BRAID}^{(m-1)}$ *is surjective.*

Proof. Since $\mathrm{Iso}(\widehat{\mathbf{PaB}}, \widehat{\mathbf{PaCD}}) \hookrightarrow \mathbf{BRAID}$, there exists at least one braidor, so there is one degree $m - 1$ braidor which extends to degree m. Since $\mathbf{GRT}_a^{(m-1)}$ acts transitively all degree $m - 1$ braidors must extend by the previous lemma. □

Twists for Associators.

$$\begin{array}{ccc} \mathbf{Twists}_{\text{Tangles}} & \circlearrowright & \mathbf{Assoc}_{\text{Tangles}} \\ \uparrow & & \uparrow \\ \{\text{id}\} = \mathbf{Twists}_{\mathfrak{t}_2} & \circlearrowright & \mathbf{Assoc}_{\mathfrak{t}_3} \end{array}$$

Twists, invertible, nondegenerate elements in $\widehat{\mathfrak{t}_2}$, act on associators via

$$\Phi \mapsto F_{2,3}^{-1} F_{1,23}^{-1} \Phi F_{12,3} F_{1,2}$$

Problem: No nontrivial twists in $\hat{\mathfrak{t}}_2$.
Solution [LM]: Embed $\widehat{\mathfrak{t}_2}$ into chord diagrams for tangles. Then twists act transitively.

Twists for Braidors.

Instead of embedding the Drinfeld-Kohno algebra into chord diagrams for tangles, we can instead embed it into spaces of derivations of free Lie algebras as considered in [AT].
Let $A = \mathbf{TDer}, \mathbf{SDer}, \mathbf{KV}$ or $\widehat{\mathbf{KV}}$.

Definition 7. A **braidor in A** is an $B \in A$ which satisfies the all the conditions and equations a braidor in $\widehat{\mathfrak{t}_3}$ did.

Definition 8. A **twist** in A is an invertible nondegenerate element $F \in A_2$. Twists act on braidors in A via

$$B \mapsto B^F = F^{-1,2} F^{-12,3} B F^{1,3} F^{13,2}$$

Then

$$C\left(\Phi^F\right) = C(\Phi)^F.$$

Lemma 4. *If twists act transitively on braidors in KV, then every usual braidor B is of the form $C(\Phi)$ for some associator Φ.*

Proof. Pick an associator Φ_0 and let $B_0 = C(\Phi_0)$. Given any other braidor B, can write $B = B_0^F$. But then $B = C(\Phi_0^F)$. □

Question: If $C(\Phi_0^F)$ and Φ_0 are in $\widehat{\mathfrak{t}_3}$, is Φ_0^F is in $\widehat{\mathfrak{t}_3}$ also?

Conjectures/Computational Evidence.

Conjecture 1. *The braid equation implies the commutativity and mixed equations.*

- *Verified to degree 10 in the Drinfeld-Kohno algebra.*

Conjecture 2. *$C : \mathrm{Iso}\left(\widehat{\mathbf{PaB}}, \widehat{\mathbf{PaCD}}\right) \to \mathbf{BRAID}$ is a bijection.*

- *Up to degree 10, the dimension of the space of braidors is equal to that of associators.*

Conjecture 3. *$\widehat{\mathbf{GT}}_a \cong \widehat{\mathbf{GT}}$ and $\widehat{\mathbf{GRT}}_a \cong \widehat{\mathbf{GRT}}$.*

Conjecture 4. *Braidors in* TAut *and* SAut *generally fail to extend but braidors in* KV *do extend. Furthermore all braidors in* KV *come from braidors in the Drinfeld-Kohno algebra.*

- *Verified up to degree* 8.

Conjecture 5. *Twists act transitively in $\widehat{KV}$ but not in* KV, SDer *or* TDer.

- *Define differential graded complexes*

$$\cdots \to \mathfrak{a}_n \xrightarrow{d^n} \mathfrak{a}_{n+1} \to \cdots$$

where $\mathfrak{a}_n = \mathfrak{tder}_n, \mathfrak{sder}_n, \mathfrak{kv}_n$ or $\widehat{\mathfrak{kv}_n}$ and the differential is given by

$$d^n = \sum_{j=1}^{n} (-1)^j \left(\Phi^{(0j),1,\cdots,\hat{j},\cdots n+1} - \Phi^{1,\cdots,\hat{j},\cdots,n+1}\right).$$

Transitivity of the action by twists $\iff H^2(A) = 0.$

- *Computations show*

Degree	$\mathfrak{tder}$	$\mathfrak{sder}$	$\mathfrak{kv}$	$\widehat{\mathfrak{kv}}$
1	5	4	2	0
2	3	0	0	0
3	5	2	1	0
4	7	0	0	0
5	13	6	1	0
6	19	0	0	0

where the entries of the table are the dimension of $H_2(A)$ in the given degree

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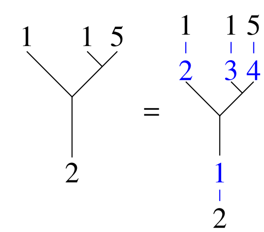
Future Work.

- Prove the conjectures!
- In order to compute cohomology, it is useful to know $\mathfrak{tder}, \mathfrak{sder}, \mathfrak{tr}_n$ etc. admit decompositions of the form

$$\mathfrak{sder}_n = \bigoplus_m Lie_m^{(1,\cdots,1)} \otimes_{S_{m+1}} (\mathbb{Q}^n)^{\otimes(m+1)}$$



Question: How does div : $\mathfrak{tder} \to \mathfrak{tr}_n$ interact with this decomposition?

- Does $\mathbf{GRT}_a$ inject into $\widehat{\mathfrak{kv}}$? What about the relationship with double shuffle?
- An associator gives a solution to the Kashiwara-Vergne problem. Does a braidor also?

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