Algebraic D-Modules

Instructor: Alexander Braverman

Lecture Notes for MAT1192F

Taught Fall of 2017

Typeset by Travis Ens

Last edited September 21, 2017

Contents

Lecture Guide

Chapter 1

Algebraic D-Modules

[14.09.2017]

1.1 Gamma Functions

Recall the gamma function

$$\Gamma(\lambda+1) = \int_0^\infty x^\lambda e^{-x} dx$$

for $\lambda \in \mathbb{C}$. This is convergent and defines a holomorphic function of λ for $\text{Re}\lambda > -1$.

Theorem 1.1. *The right hand side of the definition of the gamma function extends to a meromorphic function of λ with poles at the negative integers.*

Let $p \in \mathbb{R}[x_1, \cdots, x_n]$ be a polynomial and let $U \in \mathbb{R}^n$ be a connected component of $\mathbb{R}^n \backslash \text{Zeros}(p)$. Let φ be a rapidly decreasing function $\varphi : \mathbb{R}^n \to \mathbb{C}$. Recall the Schwartz space

$$S(\mathbb{R}^n) = \left\{ \varphi : \mathbb{R}^n \to \mathbb{C} : C^\infty, {}^{\text{any derivative } \psi \text{ of any order of } \varphi \text{ is such that}}_{\lim_{|\vec{x}| \to \infty} |\psi \cdot f(\vec{x})| = 0 \text{ for any } f \in \mathbb{R}[x_1, \cdots, x_n]} \right\}$$

For $p \in \mathbb{R}[x_1, \cdots, x_n]$ and $\varphi \in S(\mathbb{R}^n)$,

$$\int_U |p(x)|^\lambda \varphi(x) dx$$

is absolutely convergent for $Re\lambda \gg 0$ so that in this case we get a functional $S(\mathbb{R}^n) \to \mathbb{C}$, ie. we get an element $p_U^\lambda \in S^*(\mathbb{R}^n)$.

Example 1.1. $p(x) = x$, $U = \mathbb{R}_{>0}$. Then $p_U^\lambda(\varphi) = \int_0^\infty x^\lambda e^{-x} dx$ is the gamma function.

$$p_U^\lambda : \{\lambda : Re\lambda \gg 0\} \to S^*(\mathbb{R}^n)$$

Question (Gelfand, Sato, 1950s) Is there always a meromorphic continuation?

Theorem 1.2. *For all p and U, p_U^λ has meromorphic continuation to all of $\mathbb{C}$ with poles in finitely many arithmetic progressions with step 1.*

This was first preved by Atiyah, Bernstein-S. Gelfand but this is a "bad" proof (it uses reoslution of singularities.) An alternate "better" proof was given in Bernstein's 1972 thesis.
Goal: Produce a purely algebraic statement which implies the theorem.

The idea of the proof of Theorem 1.2 is to emulate the following argument for general polynomials. Consider the integral $\int_0^\infty x^\lambda \varphi(x)dx$ for φ rapidly decreasing. Using the key identity $\frac{d}{dx}(x^{\lambda+1}) = (\lambda+1)x^\lambda$,

$$\begin{aligned}\int_0^\infty x^\lambda \varphi(x)dx &= \int_0^\infty \frac{d}{dx}(x^{\lambda+1})\frac{1}{\lambda+1}\varphi(x)dx \\ &= -\frac{1}{\lambda+1}\int_0^\infty x^{\lambda+1}\varphi'(x)dx\end{aligned}$$

In the range $-2 < \mathrm{Re}(\lambda) \leq -1$ there is a problem only at $\lambda = -1$. Continue in the same way for the next interval and iterate.

Let D_n be the algebra of linear differential operators in n variables with polynomial coefficients. $D_n \subset \mathrm{End}_{\mathbb{C}}\mathbb{C}[x_1, \cdots, x_n]$ is the algebra of linear combinations

$$\sum_{\alpha_1,\cdots,\alpha_n} f_{\alpha_1,\cdots,\alpha_n}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

Example 1.2.

$$D_1 = \mathbb{C}\left\langle x, \frac{d}{dx}\right\rangle \Big/ \left[\frac{d}{dx}, x\right] = 1$$

Theorem 1.3. *Given $p \in \mathbb{C}[x_1, \cdots, x_n]$ there exists $L \in D_n[\lambda]$ and $b(\lambda) \in \mathbb{C}[x_1, \cdots, x_n]$ such that $L(p^{\lambda+1}) = b(\lambda)p^\lambda$ where $\frac{\partial p^\lambda}{\partial x_i} = \lambda\left(\frac{\partial \varphi}{\partial x_i}\right)p^{\lambda-1}$. Such a $b(\lambda)$ is called the b function of p.*

Exercise 1.1. Show that Theorem 1.3 implies Theorem 1.2. Morevoer show that the poles will be at numbers of the form $\xi - n$ where $b(\xi) = 0$ and $n \in \mathbb{N}_{>0}$.

Let $D_n(\lambda) = D_n \otimes \mathbb{C}(\lambda)$ where $\mathbb{C}(\lambda)$ is the field of rational functions in λ. Define

$$M(p^\lambda) = \left\{fp^{\lambda+i} : i \in \mathbb{Z}, f \in \mathbb{C}(\lambda[x_1, \cdots, x_n]\right\} \Big/ (p \cdot p^{\lambda+i} = p^{\lambda+i+1}).$$

This is a $D_n(\lambda)$-module.

Claim 1.1. Theorem 1.3 is equivalent to $M(p^\lambda)$ being finitely generated over $D_n(\lambda)$.

Proof. $M(p^\lambda)$ is generated by all $p^{\lambda+i}$ for $i \in \mathbb{Z}$. Being finitely generated is equivalent to saying there exists i such that just the single element $p^{\lambda+i}$ generates M which is equivalent to saying $p^{\lambda+1}$ generates M. □

1.2 The Algebra D_n and Modules Over It

Let k be a field of characteristic 0 (usually $k = \mathbb{C}, \mathbb{C}(\lambda)$ for us.)

Lemma 1.1 (Definition)**.** *The following are equivalent.*

1. *The algebra $D_n \subset \mathrm{End}k[x_1, \cdots, x_n]$ spanned as a vector space by*

$$f_{\alpha_1,\cdots,\alpha_n}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

   for $\alpha_1, \cdots, \alpha_n \geq 0$, ie. the subalgebra of $\mathrm{End}k[x_1, \cdots, x_n]$ generated by multiplication by x_i for $i = 1, \cdots, n$ and $\frac{\partial}{\partial x_i}$ for $i = 1, \cdots, n$.

2. *The algebra D_n generated by symbols x_i and $\frac{\partial}{\partial x_j}$ for $i = 1, \cdots, n$ subject to the relations*

$$\left[\frac{\partial}{\partial x_i}, x_j\right] = \delta_{ij}$$
$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$$
$$[x_i, x_j] = 0$$

3. *The algebra of differential operators of order $\leq m$ is the subalgebra $D_{\leq m} \subset \mathrm{End}\, k[x_1, \cdots, x_n]$ defined inductively as follows:*

$$\begin{aligned} m = 0 :\ & \text{multiplication by } f \in k[x_1, \cdots, x_n] \\ m > 0 :\ & d : k[x_1, \cdots, x_n] \to k[x_1, \cdots, x_n] \text{ is an operator of order } \leq m \\ & \text{iff } [d, f] \in D_{\leq m-1} \text{ for all } f \in k[x_1, \cdots, x_n] \end{aligned}$$

Remark 1.1. There is an anti-automorphism $\sigma : D_n \to D_n$ defined by $x_i \mapsto x_i$ and $\frac{\partial}{\partial x_i} \mapsto -\frac{\partial}{\partial x_i}$. Hence left modules over D_n are equivalent to right modules.

Remark 1.2. $S(\mathbb{R}^n)$ is a left module over D_n and $S^*(\mathbb{R}^n)$ is a right module over D_n which we can convert to a left module using σ.

The main result we would like to formulate is the Bernstein inequality, which in vague terms says that D_n can't be too small.

Example 1.3. Take $n = 1$ and suppose that M is a nonzero D_1-module. Then $\dim_k M = \infty$ (this follows by considering traces in the relation $[d/dx, x] = 1$.)

If M is finitely generated we will introduce the functional dimension of M, a number $0 \leq d(M) \leq 2n$. The Bernstein inequality will be $d(M) \geq n$.

Definition 1.1. Let D be any k-algebra. A *filtration* on D is a decomposition $D = \cup_{i=0}^{\infty} F_i D$ where each $F_i D$ is a subspace such that $F_i D \cdot F_j D \subset F_{i+j} D$. We can then define the *associated graded*

$$\mathrm{gr} D = \bigoplus_{i=0}^{\infty} F_i D / F_{i-1} D = \bigoplus_{i=1}^{\infty} \mathrm{gr}_i D$$

which will satisfy $\mathrm{gr}_i D \cdot \mathrm{gr}_j D \subset \mathrm{gr}_{i+j} D$.

There are two common filtrations on D_n

1. Geometric Filtration: $F_i D_n = D_{\leq i}$ is the set of differential operators of order $\leq i$ and $F_0 D_n = k[x_1, \cdots, x_n]$.

2. Arithmetic Filtration:

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n} \in F_{\sum \alpha_i + \sum \beta_j} D$$

$F_0 D_n = k$, $F_1 D_n = k \oplus \mathrm{span}\left(x_1, \cdots, x_n, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$

For either of these filtrations, $\mathrm{gr} D_n \simeq k[x_1, \cdots, x_n, \xi_1, \cdots, \xi_n]$.
Note that in general, $\mathrm{gr}^F D$ is commutative $\iff$ $[F_i D, F_j D] \subset F_{i+j-1} D$.

Definition 1.2. Consider any D-algebra with filtration and let M be a D-module. A *filtration* on M is a union $M = \bigcup_{k=0}^{\infty} F_j M$ such that $F_i D \cdot F_j M \subset F_{i+j} M$. The *associated graded* of M is

$$\mathrm{gr}M = \bigoplus_{j=0}^{\infty} F_j M / F_{j-1} M = \bigoplus_{j=0}^{\infty} \mathrm{gr}_j M$$

$\mathrm{gr}M$ is naturally a graded module over $\mathrm{gr}D$ with $\mathrm{gr}_i D \cdot \mathrm{gr}_j M \subset \mathrm{gr}_{i+j} M$.

Lemma 1.2. *If $\mathrm{gr}^F M$ is finitely generated then so is M.*

Proof. Exercise. □

Definition 1.3.

1. A filtration F on M is called *good* if $\mathrm{gr}^F M$ is finitely generated over $\mathrm{gr}D$.
2. Two filtrations F and F' on M are called equivalent if there exist j_0 and j_1 such that

   $$F'_{j-j_0} M \subset F_j M \subset F'_{j+j_1} M$$

   for all j.

Proposition 1.1. *Let M be a finitely generated D-module. Then*

1. *M has a good filtration.*
2. *If F is a good filtration and F' is any filtration then there exists j_1 such that $F_j M \subset F'_{j+j_1} M$ for all j. In particular, any 2 good filtrations are equivalent.*

Proof.

1. F is a good filtration if there exists $N \geq 0$ such that for all i, $F_j D \cdot F_N M = F_{N+i} M$ where F_N is finitely generated over $F_0 D$. Pick $W \subset M$ which generates M and is finite-dimensional. Then $F_j M = F_j D \cdot W$.
2. Exercise.

□

Example 1.4. Let $D = \mathbb{C}[x]$ and F_i be polynomials of degree $\leq i$. Take $M = D$ and $F_j M = M$ for all j. This is a bad filtration.

Lemma 1.3. *If $\mathrm{gr}D$ is left (or right) Noetherian then so is D. Recall that D is left Noetherian $\Longleftrightarrow$ any submodule of a finitely generated left module is finitely generated.*

Proof. Let M be a finitely generated D-module and $N \subset M$ a submodule. Let $F_j M$ be a good filtration on M. Define $F_j N = N \cap F_i M$. Then $\mathrm{gr}N \subset \mathrm{gr}M$ which implies that $\mathrm{gr}N$ is finitely generated over $\mathrm{gr}D$. Hence N is finitely generated over D by a previous lemma. □

Corollary 1.1. *D_n is left and right Noetherian and $\mathrm{gr}D_n \simeq k[x_1, \cdots, x_n, \xi_1, \cdots, \xi_n]$.*

1.2.1 Functional (Gelfand-Kirillov) Dimension

Let D be an algebra (over a field of any characteristic) with a filtration that satisfies the conditions

1. $\dim F_i D < \infty \ \ \forall i$
2. $\mathrm{gr} D \simeq k[y_1, \cdots, y_m]$

Theorem 1.4. *Let M be a finitely generated D-module.*

1. *Let $F_j M$ be a good filtration on M. Then there exists a polynomial $h_F(M)(t) \in \mathbb{Q}[t]$, called the* Hilbert polynomial *of M and F, such that $\dim_k F_j M = h_F(M)(j)$ for $j \gg 0$. Moreover*

$$h_F(M)(t) = \frac{c \cdot t^d}{d!} + \textit{lower order terms}$$

   for $0 \leq d \leq m$ and $c \in \mathbb{Z} > 0$.

2. *The numbers c and d do not depend on F. $d = d(M)$ is called the* functional *or* Gelfand-Kirillov dimension *of M.*

Proof.

1. It is enough to assume that $D = k[y_1, \cdots, y_n]$ at which point it reduces to a standard result from commutative algebra.
2. Let F, F' be two good filtrations on M. Then there exist numbers j_0, j_1 such that $F'_{j-j_0} \subset F_j M \subset F'_{j+j_1}$. But then $\dim F'_{j-j_0} \leq \dim F_j M \leq \dim F'_{j+j_1}$ and so $h_{F'}(j - j_0) \leq h_F(j) \leq h_{F'}(j + j + 1)$. This implies that h_F and h'_F have the same leading term.

□

Example 1.5. $d(M) = 0 \iff \dim M < \infty$.

Theorem 1.5 (Bernstein Inequality)**.** *For any finitely generated module M over D_n, $2n \geq d(M) \geq n$.*

Definition 1.4. Let M be a finitely generated D_n-module. M is called *holonomic* if $d(M) = n$.

Example 1.6.

1. $k[x_1, \cdots, x_n]$ is a D_n module. Let $F_i M$ be the filtration by degree of polynomials. $\mathrm{gr} M$ is isomorphic to $k[x_1, \cdots, x_n]$ as a vector space again and $\xi_i \in \mathrm{gr} D_n$ acts by 0.

$$\dim F_i k[x_1, \cdots, x_n] = \binom{n+i}{n} = \frac{(n+i) \cdots (i+1)}{n!} = \frac{i^n}{n!} + \text{lower order terms}$$

   so it is holonomic. Here $d = n$ and $-c = 1$.

2. For $M = D_n$, $d = 2n$ and $c = 1$.

[21.09.2017]

Missed material

Claim 1.2. Let M be any D-module and let F_jM be some filtration on M (maybe not good.) Assume that there exists a polynomial $h \in \mathbb{Q}[t]$ such that $h(j) = \dim F_jM$ for $j \gg 0$ and $h(t) = c \cdot \frac{t^n}{n!} + \cdots$. Then M is holonomic and length$(M) \leq c$. In particular, M is finitely generated.

Proof. Let $N \subset M$ be any finitely generated submodule. First we show N is holonomic and $c(N) \leq c$ (in particular, length$(N) \leq c$.)

There is a filtration on N given by $F_jN = F_jM \cap N$ and we can find a good filtration F'_jN such that $F'_jN \subset F_jN$ for all $j \gg 0$. Take any j_0 such that $F_{j_0}N$ generat4es N. For $j < j_0$ let $F'_jN = 0$ and for $j \geq j_0$ let $F'_jN = F_{j-j_0}D \cdot F_{j_0}N \subset F_jN$. This is a good filtration.

Since $\dim F'_jN \leq \dim F_jN \leq \dim F_jM$ it follows that $h_{F'}(N)(j) \leq h(j)$ for $j \gg 0$. Because $\deg h_{F'}(N) \geq n$ and $\deg h = n$ this implies $\deg h_{F'}(N) = n$ and so $c(N) \leq c$.

Next, assume that $) \subset M_1 \subset M)2 \subset M_3 \subset M_4 \subset \cdots \subset M$ are finitely generated submodules with $M_i / M_{i-1} \neq 0$ for all i. Then $m \leq c$ so M is finitely generated and so by the previous argument that M is holonomic of length $\leq c$. □

Take the base field $\mathbb{C}(\lambda)$, pick $p \in \mathbb{C}[x_1, \cdots, x_n]$ and define

$$M(p^\lambda) = \{q \cdot p^{\lambda+i}, q \in \mathbb{C}(\lambda)[x_1, \cdots, x_n]\} \Big/ (pq)p^{\lambda+i} = qp^{\lambda+i+1}$$

Note that D_n acts on $M(p^\lambda)$.

Theorem 1.6. *$M(P^\lambda)$ is holonomic (in particular finitely generated.)*

Proof. Define

$$F_jM = \left\{qp^{\lambda-j} : \deg q \leq j(m+1)\right\}$$

where $m = \deg p$.

To see that this is a filtration observe that $x_iqp^{\lambda-j} = x_ipqp^{\lambda-j-1}$. and

$$\frac{\partial}{\partial x_i}(qp^{\lambda-j}) = \frac{\partial q}{\partial x_i}p^{\lambda-j} + q\frac{\partial p}{\partial x_i}p^{\lambda-j-1} = p^{\lambda-j-1}\left(q\frac{\partial p}{\partial x_i} + p\frac{\partial q}{\partial x_i}\right)$$

$\dim F_kM = \binom{j(m+1)+n}{n}$ = polynomials in j of degree n, which proves $M(p^\lambda)$ is holonomic and this implies $M(p^\lambda)$ is finitely generated. □

1.2.2 Proof of Bernstein's Inequality (Theorem 1.5)

Let F_jM be a good filtration on a module M. Assume $F_0M \neq 0$.

Lemma 1.4. *The map $F_i\mathcal{D}$ to $\operatorname{Hom}(F_iM, F_{2i}M)$ is injective.*

Proof. There is always a map $F_iD \to \operatorname{Hom}(F_jM, F_{j+1}M)$ so the map in the lemma is the $i = j$ case.

For all $a \in F_iD$ there exists $\alpha \in F_i$ such that $a(\alpha) \neq 0$. Observe that $[F_iD, F_jD] \subset F_{i+j-2}D$.

Assuming a is not constant, there exists $m = 1, \cdots, n$ such that either $[a, x_m] \neq 0$ or $[a, \partial_m] \neq 0$, and this is equivalent to D having center the constants. Lets assume WLOG that $[a, x_m] \neq 0$. If $a \in F_iD$ then $[a, x_m] \in F_{i-1}D$. i

We will show that $a(F_iM) \neq 0$. By induction we assssume that this is true for all smaller i's. Then there exists $\alpha \in F_{i-1}M$ such that $[a, x_m](\alpha) = ax_m(\alpha) - x_ma(\alpha) \neq 0$.

$a(\alpha) = 0$ since $\alpha \in F_{i-1}M \subset F_iM$ and if $a(F_iM) = 0$, then $ax_m(\alpha) = 0$ which would contradict $[a, x_m](\alpha) \neq 0$. Hence $a(F_iM) \neq 0$.

□

To see that the lemma implies the theorem, notice that

$$\dim F_iD = \frac{i^2n}{(2n!)} + \text{lower order terms}$$

but also

$$\dim F_i\mathcal{D} \leq \dim F_iM \cdot \dim F_{2i}(M) = h(i) \cdot h(2i) = c(M)^2 \frac{i^d}{d!} \frac{(2i)^d}{d!} + \text{lower order terms.}$$

Comparing the degrees of these two polynomials in i, we get $n \leq d$ proving the theorem.

1.2.3 Outlook

We want a systematic theory of modules over D_n. Or, more generally, of $D(X)$-modules where X is a smooth affine algebraic variety $X = \operatorname{Spec} R$. Recall that we defined differential operators of order $\leq k$ to be maps $L : R \to R$ such that $[L, f]$ is a differential operator of order $\leq m - 1$ for all $f \in R$. Alternatively, $D(X)$ is the algebra generated by $R = \mathcal{O}(X)$ and by Vect_X = vector fields on X.

D_n-modules $\sim$ to a system of linear PDE's with polynomial coefficients, ie. we have functions $f_1, \cdots, f_r$ which we want to satisfy the equation $\sum_{i,j} L_{ij}(f_i) = 0$ for $L_{ij} \in D_n$.

A finitely generated D-module M has generators $\xi_1, \cdots, \xi_r$ with relation $\sum_i L_{ij}(\xi_i)$. So, a system of differential equations is equivalent to a D-module plus a system of generators $D^r \twoheadrightarrow M$. A solution in some functional space $\mathcal{F}$ is then an element of $\operatorname{Hom}_D(M, \mathcal{F})$. Using this dictionary can say a system of differential equations is holonomic if it's associated D-module is.

If we have a cyclic system (ie. generated by one element)

$$(*) \qquad L_1(f) = 0, \cdots, L_k(f) = 0$$

for $L_1, \cdots, L_k \in D_n$, this is equivalent to $M = D\big/I$ where I is the left ideal generated by $L_1, \cdots, L_k$. So, a solution of $(*)$ in some function space $\mathcal{F} \iff$ an element of $\operatorname{Hom}_D(M, \mathcal{F})$. Conversely, given $f \in \mathcal{F}$ get a module $M(f) = D \cdot f$.

Theorem 1.7. *Any holonomic D_n module is cyclic (ie. generated by one element.)*

Example 1.7. In the $n = 1$ case, $\mathbb{C}[x, x^{-1}]$ is a D-module generated by x^{-1}, ie. $\mathbb{C}[x, x^{-1}] = D\Big/\left(x\frac{d}{dx} + 1\right)$.

Claim 1.3. D_n is a simple algebra.

Proof. Assume that I is a proper two-sided ideal in D_n. Pick some $0 \neq L \in I$. Then there exists m such that either $[L, m] \neq 0$ or $[L, \partial_m] \neq 0$. Both of these commutators lie in the ideal I still.

$L \in F_iD$ for some i and $[L, x_m], [L, \partial_m] \in F_{i-1}D$. Iterate this procedure and after i steps, observe that there exists $L' \in F_0D \cap I$. L' is a nonzero constant and hence $I = D_n$. □

As a result notice that $D \to \operatorname{End} M$ is injective.

Claim 1.4. Let A be any simple algebra which as a left module over itself has infinite length. Then any A-module of finite length is cyclic.

Proof. Let M be a module of finite length. Use induction on length(M). Given an exact sequence $0 \to K \to M \xrightarrow{\pi} N \to 0$, if $K, N \neq 0$ then $\ell(N), \ell(K) < \ell(M)$ so K, N is cyclic: $N = A \cdot n$, $n \in N$. Let $I = \mathrm{Ann}_A(n)$

$I \neq 0$ since $I = 0$ would imply N is free. Let $m \in \pi^{-1}(n)$. $M' = A \cdot m$, $M' \to N$ is surjective. $\ker(M' \to N) \subset K$ implies either 0 or k.

If kernel=K, then $M' = M$ and M is cyclic. If kernel=0, then $M' - N$. $\mathrm{Ann}_A(m) = \mathrm{Ann}_A(n) = I$. $\mathrm{Ann}_A(m + v) = I$ for all $v \in K$ which implies I kills K.

$A \xrightarrow{\alpha} \mathrm{End}K$, $K \neq 0$ is not injective. But then $\mathrm{Ker}\,\alpha \subset A$ is a proper two-sided ideal, a contradiction.

□

Exercise 1.2. D_n has infinite length as a module over itself.