Moduli Space of Higgs Bundles

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Lecture Notes for MAT1305

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The basic structure of tangent bundles in Kähler geometry is

$$\begin{array}{ccc} T & \xrightarrow{\quad g \quad} & T^* \\ & {}_{I}\nwarrow \quad \nearrow_{\omega} & \\ & T & \end{array}$$

where I is a complex structure (so $I^2 = -1$), $g = g(-,-)$ is a Riemannian metric (so $g^* = g$ and g is positive definite), and ω is a symplectic form (so $\omega^* = -\omega$ and $\omega(X,Y) = -\omega(Y,X)$.)

Compatability of any two of these structures $\iff$ composite is as required.

Example 1. I, ω are compatible $\iff$ $I^*\omega = (-\omega I)^* = -\omega I$. Equivalently

$$\begin{aligned} & I^*\omega + \omega I = 1 \\ \iff & \omega(X, IY) + \omega(IX, Y) = 0 \\ \iff & I^*\omega I = \omega \\ \iff & \omega(IX, IY) = \omega(X, Y) \end{aligned}$$

Because of the complex structure I, $T \otimes \mathbb{C} = T_{1,0} \oplus T_{0,1}$ where $T_{1,0}$ is the $+i$ eigenspace and $T_{0,1}$ is the $-i$ eigenspace. So,

$$\bigwedge^2 T^*_{\mathbb{C}} = \bigwedge^2 T^*_{1,0} \oplus (T^*_{1,0} \otimes T^*_{0,1}) \otimes \bigwedge^2 T^*_{0,1}$$

and

$$\Omega^2_{\mathbb{C}} = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}.$$

The compatibility condition then becomes $\omega \in \Omega^{1,1}$.

3.1 Integrability of Kähler Structures

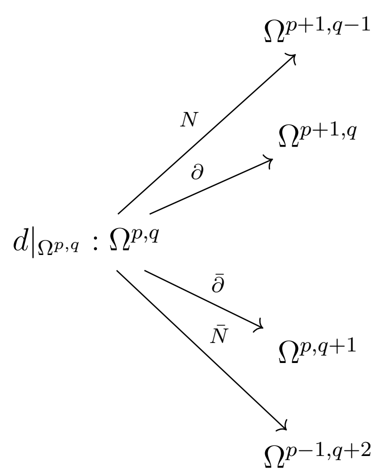
- I: $[T_{1,0}, T_{1,0}] \subset T_{1,0}$. Define the *Nijenhuis Tensor* $N \in \bigwedge^2 T^*_{1,0} \otimes T_{0,1}$ by $N(X,Y) = \pi_{0,1}[X,Y]$ where X and Y are of type $1,0$. I is integrable of this tensor vanishes. A result of Newlander-Nirenberg says that I is locally isomorphic to $\mathbb{C}^n$ when I is integrable.

- ω: ω is integrable if $d\omega = 0$. By a theorem of Darboux, if ω is integrable then ω is locally isomorphic to the standard symplectic structure $(\mathbb{R}^{2n}, \omega_{std})$.

If both integrability conditions are satisfied then this is a *Kähler structure.*

Remark 1. It is immediate from $\omega \in \Omega^{1,1}$ and $d\omega = 0$ that ω is locally $i\partial\bar{\partial}k$ for $k \in C^\infty(U, \mathbb{R})$ by the $\partial\bar{\partial}$ lemma. k is called the *Kähler potential.*

The de Rham differential splits as



and I is integrable $\iff d = \partial + \bar{\partial}$.

Lemma 1. *I, ω are each integrable if and only if $\nabla I = 0 \iff \nabla\omega = 0$. Here ∇ is the Levi-Civita connection, ie. the unique connection on T which preserves g ($Xg(Y,Z) = g(\nabla_X Y, Z + g(Y, \nabla_X Z) \iff \nabla g = 0$) and is torsion free ($\nabla_X Y - \nabla_Y X = [X,Y]$).*

Proof. If I is integrable, $IX = iX$, $IY = -iY$ and we want to show $I[X,Y] = i[X,Y]$. But

$$I(\nabla_X Y - \nabla_Y X) = (\nabla_X(IY) - \nabla_Y(IX)) = i[X,Y].$$

The rest is an exercise.

□

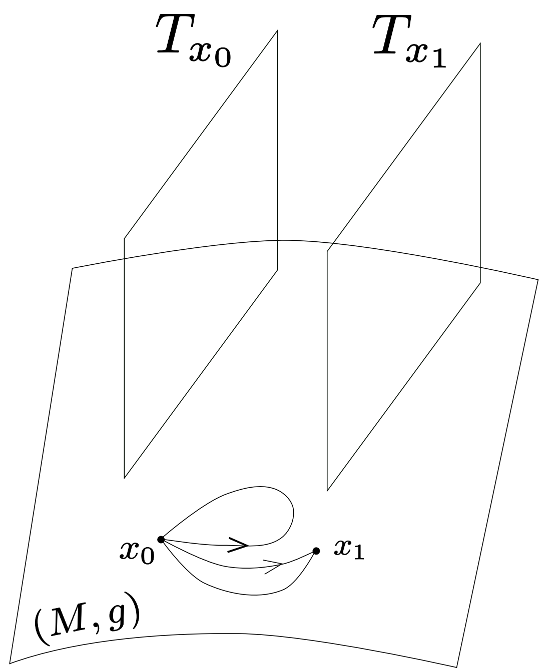
Remark 2. Given an n-dim. complex manifold with hol. tangent and ccotangent bundles $T_{1,0}$ (with coordinates $\frac{\partial}{\partial z_i}$) and $T^*_{1,0}$ (with coordinates dz_i).

Let $K = \bigwedge^n T^*_{1,0}$ be the canonical holomorphic line bundle.

$$\text{Kähler metric} \implies \text{Hermitian Metric on } K \implies \text{(unique) Chern connection.}$$

Curvature(Chern) $= F \in \Omega^2_{\mathbb{R}}$. By Poincaré-Lelong, $F = i\partial\bar{\partial}\log\|s\|_g$ where $\|s\|_g = \det g_{ij}$ (s is the hol. section $dz_1 \cdots dz_n$.) In the Kähler case, $F = \mathrm{Ric}_g(I-,-)$. Thus the Einstein equation $\mathrm{Ric} = 0 \iff K$ is flat in the Kähler case.

3.2 Holonomy Groups



We have a tangent bundle with structures I, ω, g which satisfy $\nabla I = 0$, $\nabla\omega = 0$ and $\nabla g = 0$. Hence parallel transport preserves the entire Kähler structure. In particular for a loop γ at x_0, we get an automorphism $O(n) \ni P_\gamma : T_{x_0} \to T_{x_0}$. Varying over all paths get $\{P_\gamma\} \subset_{\text{Lie Subgroup}} O(n)$, the *holonomy group* of (M, g).

In the Kähler case, get Special Holonomy: $\mathrm{Hol}(M,g) \subseteq U(n) \subset SO(2n,\mathbb{R})$.

$GL(2n,\mathbb{R})$

$\subset$ $\supset$

$\left(\text{aut of } (\mathbb{R}^{2n}, I)\right)$ $GL(n,\mathbb{C})$ $Sp(2n,\mathbb{R})$ $\left(\text{aut of } (\mathbb{R}^{2n}, \omega_{std})\right)$

$\supset$ $\subset$

max cpt subgp $U(n)$

Berger's List (1955) Of Possible Holonomy Groups

1. Locally symmetric spaces (loc. G/K with $\mathrm{Hol} = K$.) Lie theoretic classification by Cartan.

2. $M^n_{\mathbb{R}}$

Group $\subset O(n)$	Structure
$SO(n)$	orientable
$U\left(\frac{n}{2}\right)$	Kähler
$SU\left(\frac{n}{2}\right)$	Calabi-Yau (noncompact Calabi,compact Yau)
$Sp\left(\frac{n}{4}\right)$	Hyperkähler
$Sp\left(\frac{n}{4}\right) \cdot Sp(1)$	Quaternionic Kähler
$G_2 \subset O(7)$	G_2 structure
$\mathrm{Spin}(7) \subset O(8)$	Spin7 structure (noncompact Bryant, compact Joyce)

Hyperkähler

- Define
- Construct: KH reduction, examples
- Twistor space
- many moduli spaces (Higgs, monopoles,solns to Nahms eqns) coadjoint orbits for complex reductive groups, $\cdots$, have hyperkähler structures.

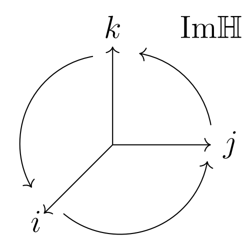
Lecture 2 **[13.01.2017]**

4 Hyperkähler Structures

Definition 1. A hyperkähler manifold is a Riemann manifold with holonomy $SP(m) \subset O(n = 4m)$.

Here $Sp(m)$ is the quaternionic unitary group which we now define. Let

$$\mathbb{H} = \{q = x_0 + ix_1 + kx_2 + kx_3 : x_i \in \mathbb{R}\}$$



where $i^2 = j^2 = k^2 = ijk = -1$.

Every quaternion q has a conjugate $\bar{q} = x_0 - ix_1 - jx_2 - kx_3$ and $q\bar{q} = x_0^2 + x_1^2 + x_2^2 + x_3^2$.

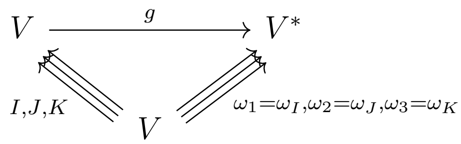
May view $\mathbb{H}$ as $\mathbb{C}^2$ via $q = (x_0 + ix_1) + (x_2 + ix_3)j \in \mathbb{C} \oplus \mathbb{C}j$ so the extra structure on $\mathbb{C}^2$ is multiplication by j. More precisely, we have $J : L_j : \mathbb{C}^2 \to \mathbb{C}^2$ such that for $\lambda \in \mathbb{C}$, $J(\lambda v) = \bar{\lambda}(Jv)$ ie. J is a complex linear automrophism. So we have $J : \mathbb{C}^2 \to \overline{\mathbb{C}^2}$ such that

$$\mathbb{C}^2 \xrightarrow{K} \overline{\mathbb{C}^2} \xrightarrow{\bar{J}} \mathbb{C}^2, \qquad \bar{J} \circ K = -1$$

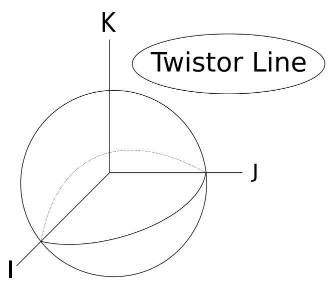
Any such J is called a quternionic structure on a complex vector space.

Focus on a certain structure enjoyed by $\mathbb{R}^4 = \mathbb{H}$. There are three complex structures $L_i = I, L_j = J, L_k = K$ with the euclidean metric g_{euc}, $g(q,q) = q\bar{q}$.

These define 3 Kähler structures



$$\omega_1 = dx_0 \wedge dx_1 + dx_2 \wedge dx_3$$
$$\omega_2 = dx_0 \wedge dx_2 - dx_1 \wedge dx_3$$
$$\omega_3 = dx_0 \wedge dx_3 + dx_1 \wedge dx_2$$



In fact, for all $(\alpha, \beta\gamma) \in S^2$,

$$\alpha I + \beta J + \gamma K = I_{(\alpha,\beta,\gamma)}$$

is also a complex structure. The corresponding Kähler form is

$$\omega_{(\alpha,\beta,\gamma)} = gI_{\alpha,\beta,\gamma} = \alpha\omega_1 + \beta\omega_2 + \gamma\omega_3$$

$\mathbb{R}^{4m} = \mathbb{H}^m \cong (\mathbb{C}^{2m}, J = L_J)$ w.r.t complex structure defined by $L_i = I$ is a left quaternionic module. Have

$$q_\ell = x_0^\ell + ix_1^\ell + jx_2^\ell + kx_3^\ell, \quad \ell = 1, \cdots, m$$

Definition 2. $Sp(m)$ is the stabilizer in $O(4m, \mathbb{R})$ of the structure (g, I, J, K) (and hence $\omega_1, \omega_2, \omega_3$).

$Sp(m)$ is a compact (simply) connected Lie group with $\dim_{\mathbb{R}} = 2m^2 + m$.

m=1: coincides with $SU(2) = S^3$

m=2: coincides with Spin(5)

This is sometimes defined as a subgroup of $GL(m, \mathbb{H})$ preserving $(,)$. Here

$$GL(m, \mathbb{H}) = \left\{ [a_{ij}]_{i,j=1}^m : A_{ij} \in \mathbb{H} \right\}$$

with quaternionic action $A \cdot (q^1, \cdots, q^m) = (q^1, \cdots, q^m)\bar{A}^T$. c.f. $U(n) \subset GL(n, \mathbb{C})$ preserving $(,)$.

Similarly, by privileging I, we view $((\mathbb{H}^m, I), J) = (\mathbb{C}^{2m}, J)$ as a complex vector space with quaternionic structure. Have complex coordinates

$$z_{2p-1} = x_0^p + ix_1^p, \quad z_{2p} = x_2^p + ix_3^p$$

and then

$$g = \sum_\ell |dz_\ell|^2$$
$$\omega_I = \frac{i}{2} \sum_\ell dz_\ell \wedge d\bar{z}_\ell$$
$$\omega_J + i\omega_K = dz_1 \wedge dz_2 + dz_3 \wedge dz_4 + \cdots$$

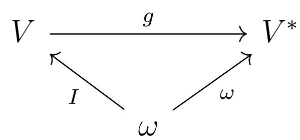
Note that $\omega_j + i\omega_K$ is a holomorphic (complex) symplectic form on $\mathbb{C}^{2m}$. Hence $Sp(m) = U(2m) \cap Sp(2m, \mathbb{C})$ ($Sp(2m, \mathbb{C})$ is the complex symplectic group.)

In fact, the hyperkähler structure is completely determined by $(\omega_I, \omega_J, \omega_K)$:

Lemma 2.

$$\mathrm{Stab}(\omega_I, \omega_J, \omega_K) = Sp(m)$$

Proof.



$$\begin{aligned}
&\omega_1 = gI \\
&\omega_2 = gJ \\
&\omega^{-1}\omega_2 = I^{-1}g{-}1gJ = -K \\
&\omega_K \circ (-K) = g
\end{aligned}$$

Warning 1. Not so for I,J,K.

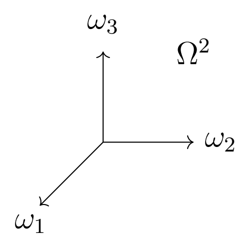
□

Example 2. Consider dimension 4 on $\mathbb{R}^4 = V$.

Lemma 3. $(\omega_1, \omega_2, \omega_3)$ *has stabilizer* $Sp(1) \iff$

$$\begin{cases} \omega_1 \wedge \omega_1 = \omega_2 \wedge \omega_2 = \omega_3 \wedge \omega_3 \neq 0 \\ \omega_1 \wedge \omega_2 = \omega_1 \wedge \omega_3 = \omega_2 \wedge \omega_3 = 0 \end{cases}$$

Proof. Consider $\Omega_1 = \omega_2 + i\omega_3$ (I holomorphic symplectic form $dz_1 \wedge dz_2$.) Then $\Omega_1 \wedge \Omega_1 = 0$ so Ω_1 decomposes as $\Omega_1 = \theta_1 \wedge \theta_2$



$\Omega_1 \wedge \overline{\Omega_2} = 2\omega_1^2 \neq 0 \iff \theta_1 \wedge \theta_2 \wedge \bar\theta_1 \wedge \bar\theta_2 \neq 0$ so we have a basis $(\theta_1, \theta_2, \bar\theta_1, \bar\theta_2)$ for $V^* \otimes \mathbb{C}$. But this is equivalent to a decomposition $V^* \otimes \mathbb{C} = \langle \theta_1, \theta_2 \rangle \oplus \langle \bar\theta_1, \bar\theta_2 \rangle$. This yields a complex structure by regarding $\langle \theta_1, \theta_2 \rangle$ as the $1,0$ eigenspace and $\langle \bar\theta_1, \bar\theta_2 \rangle$ as the $0,1$ eigenspace.

So far, we have a complex structure I on $\mathrm{Stab} \subset GL(2, \mathbb{C})$ but Ω_1 is also preserved which imples $\mathrm{Stab} \subset SL(2, \mathbb{C})$ and since ω_1 is also preserved, $\omega_1 \in \Omega_I^{1,1}$.

Expand $\omega_1 = \sum_{\alpha,\beta=1,2} h_{\alpha\bar\beta} \theta_\alpha \wedge \bar\theta_\beta$ where $h_{\alpha,\bar\beta}$ is a 2×2 Hermitian matrix and so $h_{\alpha,\bar\beta}$ is conjugate to

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

We have

$$\lambda\mu\theta_1\theta_2\bar\theta_1\bar\theta_2 = (\det h_{\alpha,\bar\beta})\theta_1\theta_2\bar\theta_1\bar\theta_2 = \omega_1^2 = \frac{1}{2}\Omega \wedge \bar\Omega = \frac{1}{2}\theta_1 \wedge \theta_2 \wedge \bar\theta_1 \wedge \bar\theta_2$$

which implies both λ, μ have the same sign. □

Note 1. M^4 a compact oriented manifold, $H^2 \times H^2 \to H^4 = \mathbb{R}$ a symmetric, nondegenerate inner product on H^2. $b_+(M^4) = \#$ of positive directions signature $(b_+, b_2 - b_+)$. In order to have a Kähler structure need what?

$k3$ has signature $(3, 19)$, $\dim H^2 = 22$ supports a hyperkähler structure.

The Manifold Case

If (M^{4m}, g) has $\mathrm{hol}(\nabla^{LC}) \subset Sp(m)$ then all the $Sp(m)$-invariant structure on $\mathbb{H}^m$ passes to corresponding structure on TM, which is flat. So we get a Riemannian manifold (M, g, I, J, K) with three complex structures, 3 Kähler structures and is flat: $\nabla I = \nabla J = \nabla K \implies$ 3 Kähler structures.

If we privilege I we get

$\implies$ complex manifold (M, I) of complex dimension $2m$

$\implies$ $\Omega_1 = \omega_2 + i\omega_3$ holomorphic symplectic $(2, 0)$ form

$\implies$ $\Omega_1^m \in \bigwedge^{2m} T^*_{1,0} = K$ the canonical bundle

$\implies$ $K \cong \mathcal{O}$ trivial CY manifold. Recall ∇^{LC}, K flat so $\mathrm{curv(Chen)} = \mathrm{Ric}I = 0$

Lecture 3 **[18.01.2017]**

4.1 Examples of Hyperkähler Structures in dimension 4

Recall that we showed a hyperkähler structure in dimension 4 is a triple $(\omega_1, \omega_2, \omega_3)$ such that $\omega_i^2 = 0$, $\omega_1\omega_2 = \omega_1\omega_3 = \omega_2\omega_3 = 0$ and $d\omega_i = 0$ (integrable.)

Example 3 (Local).

Take an open in $\mathbb{R}^4 = (\mathbb{C}^2, (z_1, z_2))$ with $I = i$,

$$\begin{aligned}\Omega_1 &= dz_1 \wedge dz_2 = \omega_2 + i\omega_3 \\ \omega_1 &= i\partial\bar{\partial}\varphi \qquad\qquad \varphi \in C^\infty(U, \mathbb{R}) \\ &= i\frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j\end{aligned}$$

For example can take $\varphi = z_1\bar{z}_1 + z_2\bar{z}_2$ so $i\partial\bar{\partial}\varphi = i(dz_1 \wedge d\bar{z}_1 + dz_2 d\bar{z}_2)$. This simply gives $\mathbb{H}^4$.

The only remaining condition is

$$\omega_1^2 = \mathrm{vol}_{Euc} \iff \det\left(\frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}\right) = 1$$

The equation $\det\left(\frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}\right) = 1$ is the (complex) Monge-Ampère equation. Calabi: Suppose $z_i = x_i + y_i$ and $\varphi(z_1, z_2, \bar{z}_1, \bar{z}_2) = \varphi(x_1, x_2)$. Then we get the real Monge-Ampère equation:

$$\det\left(\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right) = 1.$$

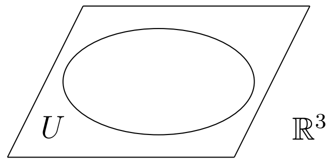
This is a nonlinear PDE so assume φ is $SO(2)$ invariant (ie. radial) to get an ODE "cohomgeneity one". Then can solve the equation to get

$$\varphi(x) = \int_0^{\|x\|} (c + r^{2c})^{\frac{1}{2}} dr.$$

For example, for $c = 1$ get $\frac{1}{2}(c + r)^{\frac{1}{2}} + \frac{1}{2}\sinh^{-1}(r) + K$. Diagrams missing

Gibbons Hawking Examples

1.



$U \subset \mathbb{R}^3$, $M = S^1 \times U$ with coordinates (θ, x_1, x_2, x_3), $V(x_1, x_2, x_3) > 0$.

$$g_m = V^{-1}(d\theta + A)^2 + V(dx_1^2 + dx_2^2 + dx_3^2)$$

where

$$A = A_1(x)dx_1 + A_2(x)dx_2 + A_3(x)dx_3 \quad \text{with } A_i \in C^\infty(U, \mathbb{R})$$

We have a trivial S^1 bundle with principal connection $d\theta A$ on the principal S^1 bundle $M \xrightarrow{\pi} U$. A is called the connection one form.

g_M diagonal $\implies$ on basis for T^*

$$\begin{aligned} e_0 &= V^{-\frac{1}{2}}(d\theta + A) \\ e_1 &= V^{-\frac{1}{2}}dx_1 \\ e_1 &= V^{-\frac{1}{2}}dx_2 \\ e_1 &= V^{-\frac{1}{2}}dx_3, \end{aligned}$$

$g_M = e_0^2 + e_1^2 + e_2^2 + e_3^2$.

$$\begin{aligned} \omega_1 &= (d\theta + A) \wedge dx_1 + V(dx_2 \wedge dx_3) \\ \omega_2 &= (d\theta + A) \wedge dx_2 - V(dx_1 \wedge dx_3) \\ \omega_3 &= (d\theta + A) \wedge dx_3 + V(dx_1 \wedge dx_2) \end{aligned}$$

which has stabiliser $Sp(1)$. To get a hyperkähler structure we need $d\omega_i = 0$:

$$d\omega_1 = dA \wedge dx_1 + dV \wedge dx_2 \wedge dx_3 = 0$$
$$\left(\frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2} + \frac{\partial V}{\partial x_1}\right) dx_1 \wedge dx_2 \wedge dx_3 = 0$$

This is a scalar equation in $\mathbb{R}^3$. Including the other equations $d\omega_i = 0$, get $dA = - \star_{\mathbb{R}^3} dV$, or equivalently, $\nabla V = 0$, so V is harmonic on $\mathbb{R}^3 \implies dV$ closed $\implies$ exact if U is simply connected $\implies \exists A$ with $dA = -\star V$. On all of $\mathbb{R}^3$, $V = 1$ and so just get $\mathbb{R}^3_{Euc} \times S^1$ (A is unique up to gauge.)

<u>Observe</u> that if we use $V = \frac{1}{r}$ in $\mathbb{R}^3 \backslash \{0\}$ (warning: $H^2 \neq 0$),

$$\star dV = \star(-r^-2dr) = -r^{-2}(r^2 d\Omega) = -d\Omega$$

$$\int_{S^2} \star dV \neq 0 \implies \star dV \neq dA$$

Generalization: $M^4 \xrightarrow{\pi} \mathbb{R}^3 \backslash \{p_1, \cdots, p_k\} = B$ a principal S^1 bundle which is Hopf about each point p_i, ie. $c_1(M) = 1$ on each $S^2_{p_i}$ or $M|_{S^2_{p_i}}$ is the Hopf fibre $S^3 \to S^2$. Diagrams Missing

Now, choose a principal connection $\eta \in \Omega^1(M)$ which is S-invariant ($L_{\partial_\theta}\eta = 0$) and $\iota_{\partial_\theta} = 1$ where ∂_θ is the vector field generated by the S^1 action.

- $\ker \eta$ is the horizontal distribution.
- $d\eta = \pi^* F$, F is the curvature of $\eta \in \Omega^2(B)$ (since $\iota_{\partial_\theta} d\eta = L_{\partial_\theta}\eta = 0$)

Given $M \to B \subset \mathbb{R}^3$, $\eta \in \Omega^1(M)$ and $V \in C^\infty(B, \mathbb{R})$ can build the metric

$$g = V^{-1}\eta^2 + V\pi^*(dx_1^2 + dx_2^2 + dx_3^2)$$

As before, define $e_0 = V^{-\frac{1}{2}}\eta, e_i = V^{\frac{1}{2}}dx_i, w_0 = \cdots$. Then we get a hyperkähler structure $\iff$ $F = -\star dV$ (so V is harmonic) and by assumption $\int_{S^2_{p_i}} F = 1 \implies \int_{S^2_{p_i}} \star dV = 1$ so $V \sim \frac{1}{r}$ at each p_i.

$$V = c + \sum_{i=0}^{k} \frac{1}{|x - p_i|_{\mathbb{R}^3}}$$

For $c \neq 0$ these are called (multi) Taub-NUT Hyperkähler metrics (ALF (asymptotically locally flat): $\mathbb{R}^3 \times S^1$)

For $c = 0$ A_k ALE (asymptotically locally euclidean: $\mathbb{R}^4$) hyperkähler structures. These are also called graviational instantons in the physics literature.

These two cases are very different in terms of asymptotic geometry but both have two **amazing** properties:

1. Extend smoothly to $M \sqcup \{p_1, \cdots, p_k\}$ to give a smooth 4-dimensional manifold similar to the way $\mathbb{R}^4 \xrightarrow{\text{Hopf}} \mathbb{R}^3$.
2. Resulting $(M, g_)$ is complete.

Example 4. $c = 0, k = 0$ (one point), $V = \frac{1}{r}$ givese $(\mathbb{R}^4, g_{Euc})$.

Have $\mathbb{R}^4 = \mathbb{C}^2$ with S^1 action

$$\begin{aligned} &e^{i\theta} \cdot (z_1, z_2) = \left(e^{i\theta} z_1, e^{-i\theta} z_2\right) \\ &\partial_\theta = i(z_1 - z_2) \\ &g(\partial_\theta, \partial_\theta) = V^{-1}(\eta(\partial_\theta))^2 = V^{-1} \\ &\|\partial_\theta\| = V^{-1} = |z_1|^2 + |x_2|^2 \end{aligned}$$

Map

$$\begin{aligned} \pi :& \mathbb{R}^4 \to \mathbb{R}^3 \\ &(z_1, z_2) \mapsto \begin{cases} x_1 = |z_1|^2 - |z_2|^2 \\ x_2 + ix_3 = 2z_1 z_2 \end{cases} \end{aligned}$$

(map is quadratic and restricted to S^3 is just the Hopf map.) Get a principal S^1 bundle on $\mathbb{R}^3 \backslash 0$ with critical value 0.

$$r^2 = x_1^2 + x_2^2 + x_3^2 = (|z_1|^2 - |z_2|^2)^2 + 4|z_1|^2|z_2|^2 = \Big(|z_1| + |z_2|^2\Big)^2$$

$$V^{-1} = \sqrt{r^2} = r \qquad V = \frac{1}{r}$$

Lecture 4 [20.01.2017]

Recall that we constructed the A_k ALE hyperkähler manifolds: Given

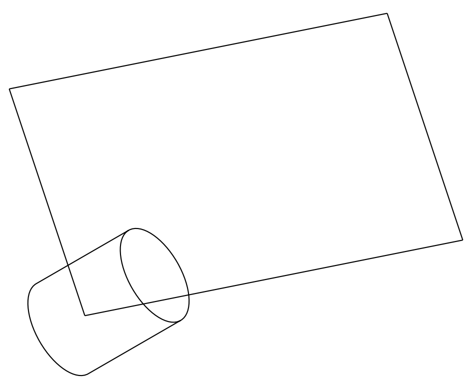
$\mathbb{R}^3 \backslash \{p_1, \cdots, p_k\} \leftarrow \tilde{M}$principle S^1bundle with $c_1 = 1$ about each p_i
$\eta \in \Omega^1(\tilde{M})$ S^1-invariant connection
$d\eta = -\pi^* \star dV$
V a harmonic function on $\mathbb{R}^3$ with $\dfrac{1}{r}$ singularities at p_i

get a metric

$$g = V\pi^* g_{\mathbb{R}^3 3} + V^{-1}\eta^2.$$

Taking $V = c + \sum_{i=0}^{k} \frac{1}{|x-p_i|}$ with $c = 0$ gave A_k ALE and $c \neq 0$ gives the multi Taub-NUT metrics. In either case, $M = \tilde{M} \cup \{p_1, \cdots, p_k\}$ is a smooth 4-manifold with complete hyperkähler metric.

Description of Hyperkähler Structure



Finish Diagram

Example 5.

1. $V = \frac{1}{r} \iff (\mathbb{R}^4, g_{\mathbb{R}^4})$ flat hyperkähler

2. A_1 diagram missing
   associated to straightline $-p \to p$ get minimal surface $S^2 \subset M \implies H^2 = \langle S^2_\ell \rangle$.
   If $x_2|_\ell = x_3|_\ell = 0$, then

$$\begin{aligned}
\omega_1 &= (d\theta + A) \wedge dx_1 + V dx_2 \wedge dx_3 \\
\omega_2 &= (d\theta + A) \wedge dx_2 - V dx_1 \wedge dx_3 \\
\omega_3 &= (d\theta + A) \wedge dx_3 + V dx_1 \wedge dx_2
\end{aligned}$$

   ω_1 is I (Kähler) and $\Omega_1 = \omega_2 + i\omega_3$. So $\Omega_1|_{S_{\ell^2}} = 0$ implies S_{ℓ^2} is Lagrangian for ω_2, ω_3 and a complex curve with respect to I. In the physics terminology, this is a (B, A, A) brane (the B refers to complex geometry, the A to symplectic.)

   Fact: For a generic $x \in S_{cx^2}$, no line $\overline{p_i p_j}$ is parallel to x. Hence (M, I_x) has no rational curves and in fact is an affine algebraic variety but for x parallel to $\overline{p_i p_j}$ get rational curves in corresponding complex structure.

5 Penrose Twistor Space

Start with a hyperkähler manifold (M^{4n}, g, I, J, K) and let S^2 be the sphere of complex structures with coordinates $(x_1, x_2, x_3) \leftrightarrow x_1 I + x_2 J + x_3 K = I_x$. Then the **twistor space** is $Z = M \times S^2$.

Given $(m, x) \in Z$, $T_{(m,x)} = T_m M \oplus T_x S^2$. Since $I_x \circlearrowleft T_m M$ and $I_{std} \circlearrowleft T_x S^2$, $\mathbb{I} = I_x \oplus I_{std} \circlearrowleft T_{(m,x)} Z$.

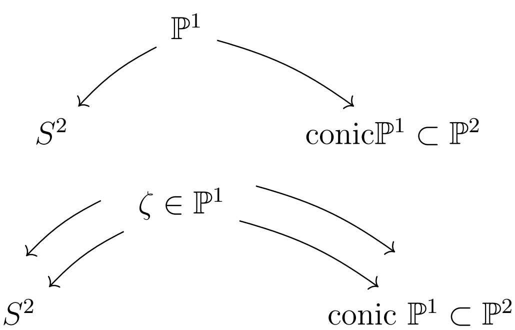
Theorem 1. *$(Z, \mathbb{I})$ is a complex $(2n+1)_{\mathbb{C}}$ dimensional manifold.*

Proof. Strategy: Identify generators for $\Omega_Z^{(2n+1,0)}$ and show these are closed.

The holomorphic symplectic structure for I_x is what?

$$I_x = x_1 I + x_2 J + x_3 K \quad (x_1, x_2, x_3) \in S^2$$

$$\Omega_x = a\omega_1 + b\omega_2 + c\omega_3 \quad [a : b : c] \in \mathbb{C}P^2 \quad \frac{a^2 + b^2 + c^2 = 0}{\text{smooth conic in}\mathbb{P}^2}$$



$$\frac{\vec{\eta} \times \overline{\vec{\eta}}}{\|\eta \times \bar{\eta}} = \frac{1}{1 + \zeta\bar{\zeta}} (1 - \zeta\bar{\zeta}, i\left(\bar{\zeta} - \zeta), -(\zeta + \bar{\zeta})\right)$$

So the holomorphic symplectic form for the complex structure I_ζ is

$$\Omega_\zeta = 2i\zeta\omega_1 + (1 + \zeta^2)\omega_2 + i(1 - \zeta^2)\omega_3 = \Omega_1 + 2i\zeta\omega_1 + \zeta\omega_1 + \zeta^2\bar{\Omega}_1.$$

Finally for the holomorphic volume form for Z^{2n+1}.

$$\Theta = \Omega_\zeta \underbrace{\wedge \cdots \wedge}_{\eta} \Omega_\zeta \wedge d\zeta$$

defines a complex structure and

$$d\Theta = (d_M + d_{S^2})\Theta = 0$$

□

Example 6. If $\mathbb{H} = \mathbb{C}^2$ then Z^3 is a complex 3-fold with complex structure determined by

$$\begin{aligned}
\omega_1 &= \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) \\
\omega_2 + i\omega_3 &= dz_1 \wedge dz_2 \\
\Omega_\zeta &= \Omega_1 + 2i\zeta\omega_1 + \zeta^2\bar{\Omega}_1 \\
&= dz_1 \wedge dz_2 + 2i\zeta\frac{i}{2}(dz_1 \bar{d}z_1 + dz_2 d\bar{z}_2) + \zeta^2(d\bar{z}_1 \wedge d\bar{z}_2) - \zeta(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2) \\
&= (dz_1 + \zeta d\bar{z}_2) \wedge (dz_2 - \zeta d\bar{z}_1).
\end{aligned}$$

Hence there are complex coordinates

$$\begin{cases} (\zeta, z_1 + \zeta \bar{z}_2, z_2 - \zeta \bar{z}_1) & \zeta \neq \infty \\ (\zeta^{-1}, \zeta^{-1} z_1 + \bar{z}_2, \zeta^{-1} z_2 - \bar{z}_1) & \zeta \neq 0 \end{cases}$$

on $\mathbb{C} \times \mathbb{C}^2$. If we glue these , get

$$Z = \operatorname{tot}\Big(\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{P}^1 \Big)$$

since the transition functions are (ζ, ζ) for the 2 factors.

Diagram missing

Warning:Each $p \in M$ defines $p \times S^2 \subset Z$ and this is obviously a complex curve, hence defines $S_p \in H^0(\underbrace{\mathbb{P}^1}_{=H^0(\mathcal{O}(1)) \oplus H^0(\mathcal{O}(1))}, \mathcal{O}(1) \oplus \mathcal{O}(1))$. There exists a "real structure" on Z, $\sigma : Z \xrightarrow{C^\infty} Z$ preserving all the structure:

- antiholomorphic involution
- compatible with Ω_ζ
- sends sections to sections

The sections fices by σ are $\mathbb{R}^4 \subset \mathbb{C}^4$, $\mathbb{R}^4$ being the "real" twistor lines and $\mathbb{C}^4$ being all twistor lines.