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Lecture Guide

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Hyperkähler geometry and instantons
Hitchin (87-88)
Kapustin, Witten ('05)

2 Course Outline

I. Hyperkähler Geometry
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3 Kähler Geometry

The basic structure of tangent bundles in Kähler geometry is
\[ T \xrightarrow{\omega} T^* \xleftarrow{I} T \]

where \( I \) is a complex structure (so \( I^2 = -1 \)), \( g = g(-,-) \) is a Riemannian metric (so \( g^* = g \) and \( g \) is positive definite), and \( \omega \) is a symplectic form (so \( \omega^* = -\omega \) and \( \omega(X,Y) = -\omega(Y,X) \)).

Compatibility of any two of these structures \( \iff \) composite is as required.

Example 1. \( I, \omega \) are compatible \( \iff I^*\omega = (-\omega I)^* = -\omega I \). Equivalently

\[ I^*\omega + \omega I = 1 \]
\[ \iff \omega(X, IY) + \omega(IX, Y) = 0 \]
\[ \iff I^*\omega I = \omega \]
\[ \iff \omega(I^X, IY) = \omega(X, Y) \]
Because of the complex structure \( I, T \otimes \mathbb{C} = T_{1,0} \oplus T_{0,1} \) where \( T_{1,0} \) is the \( +i \) eigenspace and \( T_{0,1} \) is the \(-i\) eigenspace. So,

\[
\bigwedge^2 T^*_C = \bigwedge^2 T^*_{1,0} \oplus (T^*_{1,0} \otimes T^*_0) \otimes \bigwedge^2 T^*_0
\]

and

\[
\Omega^2_C = \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}.
\]

The compatibility condition then becomes \( \omega \in \Omega^{1,1} \).

### 3.1 Integrability of Kähler Structures

**I: \([T_{1,0}, T_{1,0}] \subset T_{1,0}\)**. Define the Nijenhuis Tensor \( N \in \bigwedge^2 T^*_{1,0} \otimes T^*_0 \) by \( N(X,Y) = \pi_{0,1}[X,Y] \) where \( X \) and \( Y \) are of type \( 1,0 \). \( I \) is integrable if this tensor vanishes. A result of Newlander-Nirenberg says that \( I \) is locally isomorphic to \( \mathbb{C}^n \) when \( I \) is integrable.

\( \omega: \omega \) is integrable if \( d\omega = 0 \). By a theorem of Darboux, if \( \omega \) is integrable then \( \omega \) is locally isomorphic to the standard symplectic structure \((\mathbb{R}^{2n}, \omega_{\text{std}})\).

If both integrability conditions are satisfied then this is a **Kähler structure**.

**Remark 1.** It is immediate from \( \omega \in \Omega^{1,1} \) and \( d\omega = 0 \) that \( \omega \) is locally \( i\partial \partial \bar{k} \) for \( k \in C^\infty(U, \mathbb{R}) \) by the \( \partial \bar{\partial} \) lemma. \( k \) is called the **Kähler potential**.

The de Rham differential splits as

\[
\begin{align*}
\Omega^{p+1,q-1} & \xrightarrow{\partial} \Omega^{p+1,q} & \xrightarrow{N} \Omega^{p,q} & \xrightarrow{\delta} \Omega^{p,q+1} & \xrightarrow{N} \Omega^{p-1,q+2} \\
\end{align*}
\]

and \( I \) is integrable \( \iff \) \( d = \partial + \bar{\partial} \).

**Lemma 1.** \( I, \omega \) are each integrable if and only if \( \nabla I = 0 \iff \nabla \omega = 0 \). Here \( \nabla \) is the Levi-Civita connection, ie. the unique connection on \( T \) which preserves \( g \) (\( Xg(Y,Z) = g(\nabla_X Y, Z + g(Y, \nabla_X Z) \iff \nabla g = 0 \)) and is torsion free (\( \nabla_X Y - \nabla_Y X = [X,Y] \)).

**Proof.** If \( I \) is integrable, \( IX = iX, IY = -iY \) and we want to show \( I[X,Y] = i[X,Y] \). But

\[
I(\nabla_X Y - \nabla_Y X) = (\nabla_X (iY) - \nabla_Y (iX)) = i[X,Y].
\]

The rest is an exercise.
Remark 2. Given an \( n \)-dim. complex manifold with hol. tangent and ccotangent bundles \( T_{1,0} \) (with coordinates \( \frac{\partial}{\partial z_i} \)) and \( T_{1,0}^* \) (with coordinates \( dz_i \)).

Let \( K = \bigwedge^n T_{1,0}^* \) be the canonical holomorphic line bundle.

Kähler metric \( \implies \) Hermitian Metric on \( K \) \( \implies \) (unique) Chern connection.

Curvature(Chehn) = \( F \in \Omega^2_\mathbb{R} \). By Poincaré-Lelong, \( F = i\partial\bar{\partial}\log \| s \|_g \) where \( \| s \|_g = \det g_{ij} \) (s is the hol. section \( dz_1 \cdots dz_n \)). In the Kähler case, \( F = \text{Ric}_g(I-,-) \). Thus the Einstein equation \( \text{Ric} = 0 \iff K \) is flat in the Kähler case.

3.2 Holonomy Groups

We have a tangent bundle with structures \( I, \omega, g \) which satisfy \( \nabla I = 0, \nabla \omega = 0 \) and \( \nabla g = 0 \). Hence parallel transport preserves the entire Kähler structure. In particular for a loop \( \gamma \) at \( x_0 \), we get an automorphism \( O(n) \ni P_\gamma : T_{x_0} \to T_{x_0} \). Varying over all paths get \( \{ P_\gamma \} \subset \text{Lie Subgroup} \ O(n) \), the holonomy group of \( (M, g) \).

In the Kähler case, get Special Holonomy: \( \text{Hol}(M, g) \subset U(n) \subset SO(2n, \mathbb{R}) \).
Berger’s List (1955) Of Possible Holonomy Groups

1. Locally symmetric spaces (loc. $G/K$ with $\text{Hol} = K$.) Lie theoretic classification by Cartan.

2. $M_n^\mathbb{R}$

<table>
<thead>
<tr>
<th>Group $\subset O(n)$</th>
<th>Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SO(n)$</td>
<td>orientable</td>
</tr>
<tr>
<td>$U\left(\frac{n}{2}\right)$</td>
<td>Kähler</td>
</tr>
<tr>
<td>$SU\left(\frac{n}{2}\right)$</td>
<td>Calabi-Yau (noncompact Calabi, compact Yau)</td>
</tr>
<tr>
<td>$Sp\left(\frac{n}{4}\right)$</td>
<td>Hyperkähler</td>
</tr>
<tr>
<td>$Sp\left(\frac{n}{4}\right) \cdot Sp(1)$</td>
<td>Quaternionic Kähler</td>
</tr>
<tr>
<td>$G_2 \subset O(7)$</td>
<td>$G_2$ structure</td>
</tr>
<tr>
<td>$\text{Spin}(7) \subset O(8)$</td>
<td>Spin7 structure (noncompact Bryant, compact Joyce)</td>
</tr>
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Hyperkähler

- Define
- Construct: KH reduction, examples
- Twistor space
- many moduli spaces (Higgs, monopoles, solutions to Nahms eqns) coadjoint orbits for complex reductive groups, ..., have hyperkähler structures.

4 Hyperkähler Structures

Definition 1. A hyperkähler manifold is a Riemann manifold with holonomy $SP(m) \subset O(n = 4m)$.

Here $Sp(m)$ is the quaternionic unitary group which we now define. Let

$$\mathbb{H} = \{ q = x_0 + ix_1 + kx_2 + kx_3 : x_i \in \mathbb{R} \}$$

where $i^2 = j^2 = k^2 = ijk = -1$.

Every quaternion $q$ has a conjugate $\bar{q} = x_0 - ix_1 - jx_2 - kx_3$ and $qq = x_0^2 + x_1^2 + x_2^2 + x_3^2$.

May view $\mathbb{H}$ as $\mathbb{C}^2$ via $q = (x_0 + ix_1) + (x_2 + ix_3)j \in \mathbb{C} \oplus \mathbb{C}j$ so the extra structure on $\mathbb{C}^2$ is multiplication by $j$. More precisely, we have $J : L_j : \mathbb{C}^2 \to \mathbb{C}^2$ such that for $\lambda \in \mathbb{C}$, $J(\lambda v) = \bar{\lambda}(Jv)$ ie. $J$ is a complex linear automorphism. So we have $J : \mathbb{C}^2 \to \overline{\mathbb{C}^2}$ such that

$$\mathbb{C}^2 \xrightarrow{K} \overline{\mathbb{C}^2} \xrightarrow{J} \mathbb{C}^2$$

Any such $J$ is called a quaternionic structure on a complex vector space.
Focus on a certain structure enjoyed by $\mathbb{R}^4 = \mathbb{H}$. There are three complex structures $L_i = I, L_j = J, L_k = K$ with the euclidean metric $g_{\text{euc}}, g(q,q) = q\bar{q}$.

These define 3 Kähler structures

$$
\begin{align*}
V &\xrightarrow{\varphi} V^* \\
I,J,K &\text{Twistor Line}
\end{align*}
$$

$$
\omega_1 = dx_0 \wedge dx_1 + dx_2 \wedge dx_3 \\
\omega_2 = dx_0 \wedge dx_2 - dx_1 \wedge dx_3 \\
\omega_3 = dx_0 \wedge dx_3 + dx_1 \wedge dx_2
$$

In fact, for all $(\alpha, \beta, \gamma) \in S^2$,

$$
\alpha I + \beta J + \gamma K = I_{(\alpha, \beta, \gamma)}
$$

is also a complex structure. The corresponding Kähler form is

$$
\omega_{I,J,K} = gI_{\alpha, \beta, \gamma} = \alpha \omega_1 + \beta \omega_2 + \gamma \omega_3
$$

$\mathbb{R}^{4m} = \mathbb{H}^m \cong (\mathbb{C}^{2m}, J = L_J)$ w.r.t complex structure defined by $L_i = I$ is a left quaternionic module. Have

$$
q_\ell = x_0^\ell + ix_1^\ell + jx_2^\ell + kx_3^\ell, \quad \ell = 1, \ldots, m
$$

**Definition 2.** $Sp(m)$ is the stabilizer in $O(4m, \mathbb{R})$ of the structure $(g, I, J, K)$ (and hence $\omega_1, \omega_2, \omega_3$).

$m=1$: coincides with $SU(2) = S^3$

$m=2$: coincides with $\text{Spin}(5)$

This is sometimes defined as a subgroup of $GL(m, \mathbb{H})$ preserving $(,)$. Here

$$
GL(m, \mathbb{H}) = \{[a_{ij}]_{i,j=1}^m : A_{ij} \in \mathbb{H}\}
$$

with quaternionic action $A \cdot (q^1, \ldots, q^n) = (q^1, \ldots, q^n)A^T$. c.f. $U(n) \subset GL(n, \mathbb{C})$ preserving $(,)$. Similarly, by privileging $I$, we view $((\mathbb{H}^m, I), J) = (\mathbb{C}^{2m}, J)$ as a complex vector space with quaternionic structure. Have complex coordinates

$$
z_{2p-1} = x_p^0 + ix_p^1, \quad z_{2p} = x_p^2 + ix_p^3
$$

and then

$$
g = \sum_{\ell} |dz_\ell|^2
$$

$$
\omega_I = \frac{i}{2} \sum_{\ell} dz_\ell \wedge d\bar{z}_\ell
$$

$$
\omega_J + i \omega_K = dz_1 \wedge dz_2 + dz_3 \wedge dz_4 + \cdots
$$

Note that $\omega_J + i \omega_K$ is a holomorphic (complex) symplectic form on $\mathbb{C}^{2m}$. Hence $Sp(m) = U(2m) \cap Sp(2m, \mathbb{C})$ ($Sp(2m, \mathbb{C})$ is the complex symplectic group.)

In fact, the hyperkahler structure is completely determined by $(\omega_I, \omega_J, \omega_K)$:
Lemma 2. \[\text{Stab}(\omega_I, \omega_J, \omega_K) = Sp(m)\]

Proof.

\[\begin{array}{ccc}
V & \xrightarrow{\omega} & V^* \\
\downarrow{g} & & \uparrow{\omega} \\
I & & J
\end{array}\]

\[\begin{aligned}
\omega_1 &= gI \\
\omega_2 &= gJ \\
\omega^{-1}\omega_2 &= I^{-1}g^{-1}gJ = -K \\
\omega_K \circ (-K) &= g
\end{aligned}\]

Warning 1. Not so for \(I, J, K\).

Example 2. Consider dimension 4 on \(\mathbb{R}^4 = V\).

Lemma 3. \((\omega_1, \omega_2, \omega_3)\) has stabilizer \(Sp(1) \iff \)

\[\begin{aligned}
\omega_1 \land \omega_1 &= \omega_2 \land \omega_2 = \omega_3 \land \omega_3 \neq 0 \\
\omega_1 \land \omega_2 &= \omega_1 \land \omega_3 = \omega_2 \land \omega_3 = 0
\end{aligned}\]

Proof. Consider \(\Omega_1 = \omega_2 + i\omega_3\) (\(I\) holomorphic symplectic form \(dz_1 \land dz_2\)). Then \(\Omega_1 \land \Omega_1 = 0\) so \(\Omega_1\) decomposes as \(\Omega_1 = \theta_1 \land \theta_2\)

\[\begin{array}{ccc}
\omega_3 & \Omega^2 \\
\downarrow{\omega_1} & & \uparrow{\omega_2}
\end{array}\]

\(\Omega_1 \land \overline{\Omega_2} = 2\omega_1^2 \neq 0 \iff \theta_1 \land \theta_2 \land \overline{\theta_1} \land \overline{\theta_2} \neq 0\) so we have a basis \((\theta_1, \theta_2, \overline{\theta_1}, \overline{\theta_2})\) for \(V^* \otimes \mathbb{C}\). But this is equivalent to a decomposition \(V^* \otimes \mathbb{C} = \langle \theta_1, \theta_2 \rangle \oplus \langle \overline{\theta_1}, \overline{\theta_2} \rangle\). This yields a complex structure by regarding \(\langle \theta_1, \theta_2 \rangle\) as the 1,0 eigenspace and \(\langle \overline{\theta_1}, \overline{\theta_2} \rangle\) as the 0,1 eigenspace.

So far, we have a complex structure \(I\) on \(\text{Stab} \subset GL(2, \mathbb{C})\) but \(\Omega_1\) is also preserved which implies \(\text{Stab} \subset SL(2, \mathbb{C})\) and since \(\omega_1\) is also preserved, \(\omega_1 \in \Omega_{i1}^{1,1}\).

Expand \(\omega_1 = \sum_{a, \beta = 1, 2} h_{a, \beta} \theta_a \land \overline{\theta}_\beta\) where \(h_{a, \beta}\) is a \(2 \times 2\) Hermitian matrix and so \(h_{a, \beta}\) is conjugate to

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix}
\]

We have

\[
\lambda \mu \theta_1 \theta_2 \overline{\theta}_1 \overline{\theta}_2 = (\det h_{a, \beta}) \theta_1 \theta_2 \overline{\theta}_1 \overline{\theta}_2 = \omega_1^2 = \frac{1}{2} \Omega \land \overline{\Omega} = \frac{1}{2} \theta_1 \land \theta_2 \land \overline{\theta}_1 \land \overline{\theta}_2
\]

which implies both \(\lambda, \mu\) have the same sign.

Note 1. \(M^4\) a compact oriented manifold, \(H^2 \times H^2 \rightarrow H^4 = \mathbb{R}\) a symmetric, nondegenerate inner product on \(H^2\). \(b_+(M^4) = \#\) of positive directions signature \((b_+, b_2-b_+)\). In order to have a Kähler structure need what?

\(k3\) has signature \((3, 19)\), \(\dim H^2 = 22\) supports a hyperkähler structure.
The Manifold Case

If \((M^{4m}, g)\) has \(\text{hol}(\nabla^{LC}) \subset Sp(m)\) then all the \(Sp(m)\)-invariant structure on \(\mathbb{H}^m\) passes to corresponding structure on \(TM\), which is flat. So we get a Riemannian manifold \((M, g, I, J, K)\) with three complex structures, 3 Kähler structures and is flat: \(\nabla I = \nabla J = \nabla K \implies 3\) Kähler structures.

If we privilege \(I\) we get \((M, I)\) of complex dimension \(2m\)

\[ \Omega_1 = \omega_2 + i\omega_3 \text{ holomorphic symplectic (2, 0) form} \]

\[ \Omega_1^m \in \bigwedge^{2m} T^*_{1,0} = K \text{ the canonical bundle} \]

\[ \implies K = \mathcal{O} \text{ trivial CY manifold. Recall } \nabla^{LC}, K \text{ flat so } \text{curv}(\text{Chen}) = \text{Ric} I = 0 \]

4.1 Examples of Hyperkähler Structures in dimension 4

Recall that we showed a hyperkähler structure in dimension 4 is a triple \((\omega_1, \omega_2, \omega_3)\) such that \(\omega_1^2 = 0, \omega_1\omega_2 = \omega_1\omega_3 = \omega_2\omega_3 = 0\) and \(d\omega_i = 0\) (integrable.)

**Example 3 (Local).**

Take an open in \(\mathbb{R}^4 = (\mathbb{C}^2, (z_1, z_2))\) with \(I = i\),

\[ \Omega_1 = dz_1 \wedge dz_2 = \omega_2 + i\omega_3 \]

\[ \omega_1 = i\partial \bar{\partial} \varphi \quad \varphi \in C^\infty(U, \mathbb{R}) \]

\[ = i \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j \]

For example can take \(\varphi = z_1 \bar{z}_1 + z_2 \bar{z}_2\) so \(i\partial \bar{\partial} \varphi = i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)\). This simply gives \(\mathbb{H}^4\).

The only remaining condition is

\[ \omega_1^2 = \text{vol}_{\text{Euc}} \iff \det \left( \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right) = 1 \]

The equation \(\det \left( \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right) = 1\) is the (complex) Monge-Ampère equation. Calabi: Suppose \(z_i = x_i + y_i\) and \(\varphi(z_1, z_2, \bar{z}_1, \bar{z}_2) = \varphi(x_1, x_2)\). Then we get the real Monge-Ampère equation:

\[ \det \left( \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right) = 1. \]

This is a nonlinear PDE so assume \(\varphi\) is \(SO(2)\) invariant (ie. radial) to get an ODE "cohomogeneity one". Then can solve the equation to get

\[ \varphi(x) = \int_0^{\|x\|} (c + r^{2c})^{1/2} dr. \]

For example, for \(c = 1\) get \(\frac{1}{2} (c + r)^{1/2} + \frac{1}{2} \sinh^{-1}(r) + K\). Diagrams missing
Gibbons Hawking Examples

1.

\[ U \subset \mathbb{R}^3, \quad M = S^1 \times U \text{ with coordinates } (\theta, x_1, x_2, x_3), \quad V(x_1, x_2, x_3) > 0. \]

\[ g_m = V^{-1}(d\theta + A)^2 + V(dx_1^2 + dx_2^2 + dx_3^2) \]

where

\[ A = A_1(x)dx_1 + A_2(x)dx_2 + A_3(x)dx_3 \quad \text{with } A_i \in C^\infty(U, \mathbb{R}) \]

We have a trivial \( S^1 \) bundle with principal connection \( d\theta A \) on the principal \( S^1 \) bundle \( M \rightarrow U \). \( A \) is called the connection one form.

\( g_M \) diagonal \( \implies \) on basis for \( T^* \)

\[ e_0 = V^{-\frac{1}{2}}(d\theta + A) \]
\[ e_1 = V^{-\frac{1}{2}}dx_1 \]
\[ e_2 = V^{-\frac{1}{2}}dx_2 \]
\[ e_3 = V^{-\frac{1}{2}}dx_3, \]

\[ g_M = e_0^2 + e_1^2 + e_2^2 + e_3^2. \]

\[ \omega_1 = (d\theta + A) \wedge dx_1 + V(dx_2 \wedge dx_3) \]
\[ \omega_2 = (d\theta + A) \wedge dx_2 - V(dx_1 \wedge dx_3) \]
\[ \omega_3 = (d\theta + A) \wedge dx_3 + V(dx_1 \wedge dx_2) \]

which has stabiliser \( Sp(1) \). To get a hyperkähler structure we need \( d\omega_i = 0 \):

\[ d\omega_1 = dA \wedge dx_1 + dV \wedge dx_2 \wedge dx_3 = 0 \]
\[ \left( \frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2} + \frac{\partial V}{\partial x_1} \right) dx_1 \wedge dx_2 \wedge dx_3 = 0 \]

This is a scalar equation in \( \mathbb{R}^3 \). Including the other equations \( d\omega_i = 0 \), get \( dA = -\star_{\mathbb{R}^3} dV \), or equivalently, \( \nabla V = 0 \), so \( V \) is harmonic on \( \mathbb{R}^3 \implies dV \) closed \( \implies \) exact if \( U \) is simply connected \( \implies \exists A \) with \( dA = -\star V \). On all of \( \mathbb{R}^3 \), \( V = 1 \) and so just get \( \mathbb{R}^3_{Euc} \times S^1 \) (\( A \) is unique up to gauge.)

Observe that if we use \( V = \frac{1}{r} \) in \( \mathbb{R}^3 \setminus \{0\} \) (warning: \( H^2 \neq 0 \)),

\[ \star dV = \star(-r^{-2}dr) = -r^{-2}(r^2d\Omega) = -d\Omega \]

\[ \int_{S^2} \star dV \neq 0 \implies \star dV \neq dA \]
Generalization: $M^4 \xrightarrow{\phi} \mathbb{R}^3 \backslash \{p_1, \ldots, p_k\} = B$ a principal $S^1$ bundle which is Hopf about each point $p_i$, ie. $c_1(M) = 1$ on each $S^2_{p_i}$ or $M|_{S^2_{p_i}}$ is the Hopf fibre $S^3 \to S^2$. \textbf{Diagrams Missing}

Now, choose a principal connection $\eta \in \Omega^1(M)$ which is $S$-invariant ($L_{\partial \theta} \eta = 0$) and $\iota_{\partial \theta} = 1$ where $\partial \theta$ is the vector field generated by the $S^1$ action.

- ker $\eta$ is the horizontal distribution.
- $d\eta = \pi^* F$, $F$ is the curvature of $\eta \in \Omega^2(B)$ (since $\iota_{\partial \theta} d\eta = L_{\partial \theta} \eta = 0$)

Given $M \to B \subset \mathbb{R}^3$, $\eta \in \Omega^1(M)$ and $V \in C^\infty(B, \mathbb{R})$ can build the metric

$$ g = V^{-1} \eta^2 + V \pi^* (dx_1^2 + dx_2^2 + dx_3^2) $$

As before, define $e_0 = V^{-\frac{1}{2}} \eta, e_i = V^{\frac{1}{2}} dx_i, w_0 = \cdots$. Then we get a hyperkähler structure $\iff F = - \ast dV$ (so $V$ is harmonic) and by assumption $\int_{S^2_{p_i}} F = 1 \implies \int_{S^2_{p_i}} \ast dV = 1$ so $V \sim \frac{1}{r}$ at each $p_i$.

$$ V = c + \sum_{i=0}^k \frac{1}{|x - p_i|_{\mathbb{R}^3}} $$

For $c \neq 0$ these are called (multi) Taub-NUT Hyperkähler metrics (ALF (asymptotically locally flat): $\mathbb{R}^3 \times S^1$)

For $c = 0$ $A_k$ ALE (asymptotically locally euclidean: $\mathbb{R}^4$) hyperkähler structures. These are also called gravitationally instantons in the physics literature.

These two cases are very different in terms of asymptotic geometry but both have two \textbf{amazing} properties:

1. Extend smoothly to $M \sqcup \{p_1, \ldots, p_k\}$ to give a smooth 4-dimensional manifold similar to the way $\mathbb{R}^4 \xrightarrow{\text{Hopf}} \mathbb{R}^3.$

2. Resulting $(M, g)$ is complete.

\textit{Example 4.} $c = 0, k = 0$ (one point), $V = \frac{1}{r}$ gives $(\mathbb{R}^4, g_{\text{Euc}})$.

Have $\mathbb{R}^4 = \mathbb{C}^2$ with $S^1$ action

$$ e^{i \theta} \cdot (z_1, z_2) = (e^{i \theta} z_1, e^{-i \theta} z_2) $$

$$ \partial \theta = i(z_1 - z_2) $$

$$ g(\partial \theta, \partial \theta) = V^{-1} (\eta(\partial \theta))^2 = V^{-1} $$

$$ ||\partial \theta|| = V^{-1} = |z_1|^2 + |z_2|^2 $$

Map

$$ \pi : \mathbb{R}^4 \to \mathbb{R}^3 $$

$$ (z_1, z_2) \mapsto \left\{ \begin{array}{l}
  x_1 = |z_1|^2 - |z_2|^2 \\
  x_2 = i x_3 = 2 z_1 z_2
\end{array} \right. $$

(map is quadratic and restricted to $S^3$ is just the Hopf map.) Get a principal $S^1$ bundle on $\mathbb{R}^3 \backslash 0$ with critical value 0.

$$ r^2 = x_1^2 + x_2^2 + x_3^2 = (|z_1|^2 - |z_2|^2)^2 + 4 |z_1|^2 |z_2|^2 = \left(|z_1| + |z_2|\right)^2 $$

$$ V^{-1} = \sqrt{r^2} = r \quad V = \frac{1}{r} $$
Recall that we constructed the $A_k$ ALE hyperkähler manifolds: Given
\[ \mathbb{R}^3 \setminus \{ p_1, \ldots, p_k \} \leftarrow M, \text{principle } S^1\text{bundle with } c_1 = 1 \text{ about each } p_i \]
\[ \eta \in \Omega^1(\tilde{M}) \text{ } S^1\text{-invariant connection} \]
\[ d\eta = -\pi^* \ast dV \]
\[ V \text{ a harmonic function on } \mathbb{R}^3 \text{ with } \frac{1}{r} \text{ singularities at } p_i \]
get a metric
\[ g = V\pi^* g_{\mathbb{R}^3} + V^{-1}\eta^2. \]
Taking \( V = c + \sum_{i=0}^k \frac{1}{|x-p_i|} \) with \( c = 0 \) gave $A_k$ ALE and \( c \neq 0 \) gives the multi Taub-NUT metrics.
In either case, $M = \tilde{M} \cup \{ p_1, \ldots, p_k \}$ is a smooth 4-manifold with complete hyperkähler metric.

**Description of Hyperkähler Structure**

**Finish Diagram**

**Example 5.**

1. \( V = \frac{1}{r} \Leftrightarrow (\mathbb{R}^4, g_{\mathbb{R}^4}) \text{ flat hyperkähler} \)

2. $A_1$ diagram missing

   associated to straightline $-p \to p$ get minimal surface $S^2 \subset M \implies H^2 = \langle S^2 \rangle$.

   If $x_2|_\ell = x_3|_\ell = 0$, then

   \[ \omega_1 = (d\theta + A) \wedge dx_1 + Vdx_2 \wedge dx_3 \]

   \[ \omega_2 = (d\theta + A) \wedge dx_2 - Vdx_1 \wedge dx_3 \]

   \[ \omega_3 = (d\theta + A) \wedge dx_3 + Vdx_1 \wedge dx_2 \]

   \( \omega_1 \) is $I$ (Kähler) and $\Omega_1 = \omega_2 + i\omega_3$. So $\Omega_1|_{S^2} = 0$ implies $S^2$ is Lagrangian for $\omega_2, \omega_3$ and a complex curve with respect to $I$. In the physics terminology, this is a $(B, A, A)$ brane (the $B$ refers to complex geometry, the $A$ to symplectic.)

**Fact:** For a generic $x \in S^2$, no line $\overline{p_ip_j}$ is parallel to $x$. Hence $(M, I_x)$ has no rational curves and in fact is an affine algebraic variety but for $x$ parallel to $\overline{p_ip_j}$ get rational curves in corresponding complex structure.
5 Penrose Twistor Space

Start with a hyperkähler manifold \((M^{4n}, g, I, J, K)\) and let \(S^2\) be the sphere of complex structures with coordinates \((x_1, x_2, x_3) \leftrightarrow x_1 I + x_2 J + x_3 K = I_x\). Then the **twistor space** is \(Z = M \times S^2\).

Given \((m, x) \in Z, T_{(m,x)} = T_m M \oplus T_x S^2, \mathbb{I} = I_x \oplus I_{\text{std}} \circ T_{(m,x)} Z\).

**Theorem 1.** \((Z, \mathbb{I})\) is a complex \((2n + 1)\) dimensional manifold.

*Proof.* Strategy: Identify generators for \(\Omega^2_{Z^{2n+1,0}}\) and show these are closed.

The holomorphic symplectic structure for \(I_x\) is what?

\[
I_x = x_1 I + x_2 J + x_3 K \quad (x_1, x_2, x_3) \in S^2
\]

\[
\Omega_x = a \omega_1 + b \omega_2 + c \omega_3 \quad [a : b : c] \in \mathbb{C}P^2 \quad a^2 + b^2 + c^2 = 0
\]

smooth conic in \(\mathbb{P}^2\)

\[
\frac{\vec{\eta} \times \vec{\eta}}{\|\eta \times \vec{\eta}\} = \frac{1}{1 + \zeta \zeta} (1 - \zeta \zeta, i (\zeta - \zeta), -(\zeta + \zeta))
\]

So the holomorphic symplectic form for the complex structure \(I_\zeta\) is

\[
\Omega_\zeta = 2i \zeta \omega_1 + (1 + \zeta^2) \omega_2 + i(1 - \zeta^2) \omega_3 = \Omega_1 + 2i \zeta \omega_1 + \zeta \omega_1 + \zeta^2 \Omega_1.
\]

Finally for the holomorphic volume form for \(Z^{2n+1}\).

\[
\Theta = \Lambda \Omega_\zeta \wedge \cdots \wedge \Omega_\zeta \wedge d\zeta
\]

defines a complex structure and

\[
d\Theta = (d_M + d_{\mathbb{S}^2})\Theta = 0
\]

*Example 6.* If \(\mathbb{H} = \mathbb{C}^2\) then \(Z^3\) is a complex 3-fold with complex structure determined by

\[
\omega_1 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)
\]

\[
\omega_2 + i\omega_3 = dz_1 \wedge d\bar{z}_2
\]

\[
\Omega_\zeta = \Omega_1 + 2i \zeta \omega_1 + \zeta^2 \bar{\Omega}_1
\]

\[
= dz_1 \wedge d\bar{z}_2 + 2i \zeta \frac{i}{2}(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2) + \zeta^2 (d\bar{z}_1 \wedge d\bar{z}_2) - \zeta (dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2)
\]

\[
= (dz_1 + \zeta d\bar{z}_2) \wedge (d\bar{z}_2 - \zeta d\bar{z}_1).
\]
Hence there are complex coordinates

\[
\begin{cases}
(\zeta, z_1 + \zeta \bar{z}_2, z_2 - \zeta \bar{z}_1) & \zeta \neq \infty \\
(\zeta^{-1}, \zeta^{-1} z_1 + \bar{z}_2, \zeta^{-1} z_2 - \bar{z}_1) & \zeta \neq 0
\end{cases}
\]

on \( \mathbb{C} \times \mathbb{C}^2 \). If we glue these, get

\[
Z = \text{tot}\left(\mathcal{O}(1) \oplus \mathcal{O}(1) \rightarrow \mathbb{P}^1\right)
\]

since the transition functions are \((\zeta, \zeta)\) for the 2 factors.

Diagram missing

Warning: Each \( p \in M \) defines \( p \times S^2 \subset Z \) and this is obviously a complex curve, hence defines \( S_p \in H^0(\mathbb{P}^1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \). There exists a “real structure” on \( Z, \sigma : Z \overset{C^\infty}{\rightarrow} Z \) preserving all the structure:

- antiholomorphic involution
- compatible with \( \Omega_\zeta \)
- sends sections to sections

The sections fices by \( \sigma \) are \( \mathbb{R}^4 \subset \mathbb{C}^4 \), \( \mathbb{R}^4 \) being the “real” twistor lines and \( \mathbb{C}^4 \) being all twistor lines.