Gromov-Witten Invariants

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Lecture Guide

Chapter 1

The Moduli Spaces $\mathcal{M}_{g,n}$

Lecture 1 **[19.01.2017]**

Missed

Lecture 2 **[26.01.2017]**

We will discuss $\overline{\mathcal{M}}_{g,n}(X)$ more systematically, regarding $\overline{\mathcal{M}}_{g,n}(X)$ as a Deligne-Mumford Stack.

1.1 Crash Course on Stacks

1.1.1 Motivation

Let X be an algebraic variety over $\mathbb{C}$ and G an algebraic group (finite for example.) How can we define the quotient X/G in the general case of a non-free action? We want a nice definition of a quotient. In particular we want a map $X \to X/G$ with good properties; eg. X/G should be smooth if X is smooth etc. however this usually fails.

Example 1.1. $X = \mathbb{C}^2$, $G = \mathbb{Z}/2\mathbb{Z}$ with the action $(x, y) \to (-x, -y)$. The naive quotient is $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}) = \operatorname{Spec}\mathbb{C}[x, y]$ with the relations $uv = w^2$ where $u = x^2, v = y^2, w = xy$ so get a singular surface.

We will extend our notion of a space by constructing an inclusion $\mathrm{Sch}_{\mathbb{C}} \subset$ Bigger World. Let X be a scheme over $\mathbb{C}$. Define a functor $F_X : \mathrm{Sch}_{\mathbb{C}}^{op} \to \mathrm{Sets}$ by $F_X(S) = \mathrm{Hom}(S, X)$. Then X can be uniquely recoverd from F_X. Furthermore, given $X, Y \in \mathrm{Sch}_{\mathbb{C}}$, $\mathrm{Hom}(X, Y) = \mathrm{Hom}(F_X, F_Y)$. The most general notion of space is just a functor

$$F : \mathrm{Sch}_{\mathbb{C}}^{op} \to \mathrm{Sets}.$$

Ind-Schemes

Recall that an Ind-Scheme is a functor which is representable by an inductive limit of schemes.

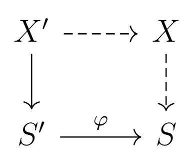
Example 1.2.

1. What is $\mathbb{A}^\infty$?
$$\operatorname{Spec}\mathbb{C}[x_1, x_2, x_3, \cdots] = \varprojlim_n \mathbb{A}^n$$
since
$$\mathbb{C}[x_1, x_2, \cdots] = \bigcup_{n\geq 1} \mathbb{C}[x_1, x_2, \cdots, x_n]$$

2. We have a sequence of embeddings $0 \subset \mathbb{A}^1 \subset \mathbb{A}^2 \subset \cdots$ given by the inclusions $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, 0)$. The inductive limit does not exist in the category of schemes but it does exist in the catgory of functors.

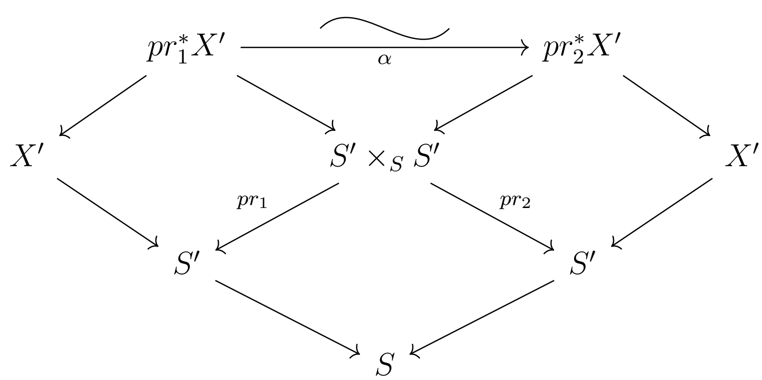
1.1.2 Descent Property

Given $X' \to S'$ and φ in the following diagram



we want conditions such that we can find X making this a Cartesian square, ie. want X such that $X' = X \times_S S'$.

If we have such an X then we get an isomorphism $pr_1^* X' \xrightarrow{\alpha} pr_2^* X'$ between the pullbacks of X' to $S' \times_S S'$ in the following diagram.



Exercie 1.1. Produce some condition that α satisfies which "lives" on $S''' = S' \times_S S' \times_S S'$

Theorem 1.1. *If φ is faithfully flat (surjective and flat) then every α satisfying the cocycle condition uniquely determines X.*

In the language of Grothendieck topologies, this is equivalent to saying F_X is a sheaf in the Grothendieck topology where coverings are faithfully flat maps.

1.1.3 Stacks

Stacks are also functors, but in a more sophisticated sense.

Definition 1.1. A category $\mathcal{C}$ is called a **groupoid** if every morphism is an isomorphism.

Roughly speaking, stacks are "functors" $F : \mathrm{Sch}_{\mathbb{C}}^{op} \to \mathrm{Groupoids}$ satisfying nice properties. By a functor in this context, we mean to every $S \in \mathrm{Sch}_{\mathbb{C}}$ is associated a groupoid $F(S)$ and to every morphism $S_1 \xrightarrow{\varphi} S_2$ is associated a functor $F(\varphi) : F(S_2) \to F(S_1)$ such that given composable morphisms $S_1 \xrightarrow{\varphi} S_2 \xrightarrow{\phi} S_3$, there is a natural isomorphism $F(\varphi) \circ F(\phi) \simeq F(\psi \circ \varphi)$.

If X, Y are two stacks, $\mathrm{Hom}(X, Y)$ is a groupoid with

$$1 - \text{morphisms} = \text{objects of this groupoid}$$
$$2 - \text{morphisms} = \text{morphisms between 1-morphisms}$$

Example 1.3. X/G. What is a map $S \to X/G$? If the quotient exists and has all properties we want, than can form the pullback

$$\begin{array}{ccc} X \times_{X/G} S & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \longrightarrow & X/G \end{array} .$$

Given a scheme S, a principle G-bundle is a scheme W with an action of G such that locally in the étale topology, W is isomorphic to $X \times G$, ie. there exists a surjective étale map $f : X' \to X$ such that $X' \times_X W \simeq X' \times G$. If all schemes are of finite type, then you can replace étale by flat.

So can define the functor $F_{X/G} : \mathrm{Sch}_{\mathbb{C}}^{op} \to \mathrm{Groupoids}$ by

$$F_{X/G}(S) = \left\{ \begin{matrix} \text{a principle } G\text{-bundle } W \text{ over} \\ S \text{ and a } G \text{ equiv. map } W \to X \end{matrix} \right\}$$

and this makes sense for any group action.

A map $f : X \to Y$ between two stacks is called representable if for all $\varphi : S \to Y$ where S is a scheme in

$$\begin{array}{ccc} S \times_Y X & \longrightarrow & S \\ \downarrow & & \downarrow \varphi \\ X & \xrightarrow{f} & Y \end{array} ,$$

$S \times_Y X$ is also a scheme.

For example if $X \to Y$ are both schemes, G acts on both X and Y and the map is G-equivariant then $X/G \to Y/G$ is representable.

Definition 1.2. A stack X is an **algebraic stack** if there exists a surjective map $X' \xrightarrow{f} X$ where X' is a scheme and f is a smooth representable map.

Definition 1.3. A stack X is a **Deligne-Mumford stack** if there exists a surjective map $X' \xrightarrow{f} X$ where X' is a scheme and f is a finite étale representable map.

Definition 1.4. An **algebraic space** is a Deligne-Mumford stack with trivial automorphism groups at every point.

Every point of a Deligne-Mumford stack has a finite group of automorphisms.

Example 1.4. $X \to X/G$ is always an algebraic stack, and it is a Deligne-Mumford stack if and only if the stabilizer of every point is finite.

1.2 Moduli Spaces

1.2.1 Coarse Moduli Space

If X is a Deligne-Mumford stack of finite type over $\mathbb{C}$, can construct an algebraic space $\overline{X}$, called the **coarse moduli space** of X and a map $X \to \overline{X}$ such that $\overline{X}(\mathbb{C})$ are the isomorphism classes of $X(\mathbb{C})$.

Theorem 1.2.

1. *Let X be a smooth projective variety. Then there exists a Deligne-Mumford stack of finite type $\mathcal{M}^{\beta}_{g,n}(X)$ (stable maps), where $\beta \in H_2(X, \mathbb{Z})$, with quasi-projective coarse moduli space.*

2. *Similary, have $\overline{M}^{\beta}_{g,n}$ with a projective coarse moduli space.)*

3. *If $X = pt$ then all these stacks are smooth and irreducible.*

Note 1.1. X/G with G finite is a Deligne-Mumford stack and furthermore, every Deligne-Mumford stack with quasi-projective coarse moduli space can be obtained by gluing these together. For X affine, say $X = \operatorname{Spec} A$, the coarse moduli space is $\operatorname{Spec} A^G$.

1.2.2 The Stack $\mathcal{M}_{g,n}$

What is $\mathcal{M}^{\beta}_{g,n}(X)$ as a stack?

Example 1.5. Let $X = pt$ and $n = 0$. Define the functor

$$S \mapsto \left\{ \mathcal{C} \xrightarrow{\varphi} S : \begin{array}{c} \mathcal{C} \text{ a scheme} \\ \varphi \text{ a flat projective morphism} \\ \text{The fiber of } \varphi \text{ over every geometric point of } S \\ \text{is a smooth projective irreducible curve of genus } g \end{array} \right\}$$

Claim 1.1.

1. This is a smooth algebraic stack

2. If $g \geq 2$ then it is a Deligne-Mumford stack

Proof. (Sketch) 2 essentially follow from 1. Suppose C is a smooth projective irreducible curve of genus g. For any $d \in \mathbb{Z}$, can define $\operatorname{Pic}^d(C)$ which classifies line bundles $\mathcal{L}$ of degree d over C as a stack via

$$S \mapsto \left\{ \begin{array}{c} \text{line bundles } \mathcal{L} \text{ over } S \times C \text{ such that} \\ \text{for all geometric points } s \in S,\ \mathcal{L}|_{\{s\}} \times C \text{ has degree } d \end{array} \right\}$$

$\operatorname{Pic}^d_{st}(C) = \operatorname{Pic}^d(C)/\mathbb{C}^*$ with trivial action. Choose $c \in C$. A map $S \to \operatorname{Pic}^d(C)$ is equivalent to a line bundle $\mathcal{L}$ of degree d over $S \times C$ and a trivialization on $S \times \{c\}$. Choose $\mathcal{L}_{S \times \{c\}} \simeq \mathcal{O}_{S \times \{c\}}$.

Theorem 1.3. *This is representable by a smooth projective variety.*

Exercie 1.2. $\operatorname{Pic}^d_{st}(C) = \operatorname{Pic}^d(C)/\mathbb{C}^*$ (with the trivial action.)

Choose $d \gg 0$ (need to satisfy the condition that any line bundle $\mathcal{L}$ over C of degree d is very ample. It is enough to require that $d > 2g$.)

For $\mathcal{L} \in \operatorname{Pic}^2(C)$, $H^1(C, \mathcal{L}) = 0$ and by Riemann-Roch $\dim H^0(C, \mathcal{L}) = d + 1 - g$ so assume we have chosen an isomorphism $H^0(C, \mathcal{L}) \simeq \mathbb{C}^{d+1-g}$. Prome this to a functor $\operatorname{Sch}^{op}_{\mathbb{C}} \to \operatorname{Sets}$. Explicitly,

$$S \mapsto \left\{ \begin{array}{c} \mathcal{L}\text{−line bundle of degree } d \text{ over } S \times C \\ \text{given } p_* \mathcal{L} \simeq \mathcal{O}_S^{d+1-g} \end{array} \right\}$$

Given $\{C, \mathcal{L}, \alpha\}$ get a closed embedding $C \hookrightarrow \mathbb{P}^{d-g}$ (deg $C(= i_*[c]) = d$) and furthermore this embedding determines the triple.
Hilbert Schemes: $Y \subset \mathbb{P}^n$ a closed subscheme. $\mathbb{C}[Y] \leftarrow \mathbb{C}[x_1, \cdots, x_n]$ homogeneous graded coordinate ring. Can define the Hilbert polynomial $h_Y \in \mathbb{Q}[t]$ such that $h_Y(k) = \dim \mathbb{C}[y]_k$ for $\mathbb{Z} \ni k \gg 0$. If $\dim Y = m$ then $h_Y = d\frac{t^m}{m!} + \cdots$ and $d = \deg Y$.

Theorem 1.4. *Given $h \in \mathbb{Q}[t], h(k) \in \mathbb{Z}_{>0}$ for $k \gg 0$ there exists a projective scheme which classifies subschemes $Y \subset \mathbb{P}^n$ with $h_Y = h$.*

Exercie 1.3. The condition that Y is non-singular is open and the condition that Y is connected is closed.

Exercie 1.4. h_C depends only on g and d.

$\mathbb{C}[C]_k = H^0(C, \mathcal{L}^{\otimes K}) = kd + 1 - g$ and $h(t) = dt + 1 - g$. So, smooth and irreducible C plus $\mathcal{L}$ of degree d plus α gives a locally closed embedding $\hookrightarrow \text{Hilb}(\mathbb{P}^{d-g})$. $X_{d,g}$-scheme.

We have a surjective map $X_{d,g} \xrightarrow{\kappa} \mathcal{M}_g$.

Exercie 1.5. This is a smooth representable map.

Every fiber of κ is isomorphic to

$$\text{Pic}^d(C) \times \text{GL}(d + 1 - g) \Big/ \mathbb{C}^*$$

□

How to prove that $\mathcal{M}_g$ is smooth (for $g \geq 2$?) A point x in a Deligne Mumford stack X is smooth if $\dim T_x X = \dim X^x$ (X^x is any irreducible component of X containing x.) So to guarantee that X is smooth, we need to guarantee that $\dim T_x X$ is constant.

$T_x X$ classifies maps $\text{Spec}\, \mathbb{C}[\varepsilon]/\varepsilon^2 \xrightarrow{\varphi} X$ together with an isomorphism $\varphi|_{\text{Spec}\,\mathbb{C}} \xrightarrow{\sim} x : \text{Spec}\, \mathbb{C} \to X$.

Lemma 1.1. *Given $C \in \mathcal{M}_g$, $T_C\mathcal{M}_g = H^1(C, T_C)$ where T_C is the tangent bundle.*

Proof. $T_C\mathcal{M}_g$ classifies schemes $\mathbb{C}$ with $\mathbb{C} \to \text{Spec}\, \mathbb{C}[\varepsilon]/\varepsilon^2$ flat plus an isomorphism $\mathbb{C}|_{pt} \simeq C$.

Lemma 1.2. *If C is any smooth variety such infinitesimal deformations are in one to one correspondence with $H^1(C, T_C)$.*

□

If the genus of C is g, then $\deg T_c = 2 - 2g$ and $\deg \Omega_c = 2g - 2$. Hence by Riemann-Roch,

$$\dim H^0(T_c) - \dim H^1(T_C) = 2 - 2g + 1 - g = 3g - 3$$

If $g \geq 2$ then $H^0(T_C) = 0$ (follows since $\deg T_C < 0$). Hence $\dim H^1(C, T_c) = 3g - 3$ which implies $\mathcal{M}_g$ is smooth of dimension $3g - 3$.