Derived Categories

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Lecture Guide

Chapter 1

Category Theory

Lecture 1 [02.02.2016]

1.1 Introduction

The first half of the course will be an introduction to homological algebra via derived categories. The second half will cover applications (eg. perverse sheaves or D-modules.)

1.2 Basic Definitions

Definition 1.1. A **category** $\mathcal{C}$ is a class of objects and a class of morphisms such that

1. For all $X, Y \in \operatorname{Ob} \mathcal{C}$ have a set $\operatorname{H\overline{om}}(X, Y)$ of morphisms from X to Y.
2. For all $X, Y, Z \in \operatorname{Ob} \mathcal{C}$ there is a composition morphism

$$\circ : \operatorname{H\overline{om}}(Y, Z) \times \operatorname{H\overline{om}}(X, Y) \to \operatorname{H\overline{om}}(X, Z)$$

which satisfy the properties

1. For all $X \in \operatorname{Ob} \mathcal{C}$ there exists $\mathrm{id}_X \in \operatorname{H\overline{om}}(X, X)$ such that for any $Y \in \operatorname{Ob}(\mathcal{C})$, and for all $f : X \to Y, g : Y \to X$, $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$.
2. Composition is associative.

Exercie 1.1. Prove id_X is unique and write down the associativity axiom.

Example 1.1. Sets, groups, abelian groups, topological spaces, sheaves of sets (or abelian groups) on a given topological space, and modules over a given ring all form categories.

For all $X, Y \in \operatorname{Ob}(\mathcal{C})$, $f : X \to Y$ and $g : Y \to X$ are said to bo **inverse** if $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. If f^{-1} exists we say that f is an **isomorphism**.

Definition 1.2. Let $\mathcal{C}$, $\mathcal{D}$ be two categories. A (covariant) **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of

1. For all $X \in \operatorname{Ob} \mathcal{C}$ an object $F(X) \in \operatorname{Ob} \mathcal{D}$

2. For all $X, Y \in \mathrm{Ob}\,\mathcal{C}$ a morphism $F_{X,Y} : \mathrm{H\overline{om}}(X,Y) \to \mathrm{H\overline{om}}(F(X), F(Y))$ which is compatible with compositions.

Given $\mathcal{C}$ we can define a new category $\mathcal{C}^{op}$ where $\mathrm{Ob}\,\mathcal{C}^{op} = \mathrm{Ob}\,\mathcal{C}$ and $\mathrm{H\overline{om}}_{\mathcal{C}^{op}}(X,Y) = \mathrm{H\overline{om}}_{\mathcal{C}}(Y,X)$. Then a **contraviariant functor** $F : \mathcal{C} \to \mathcal{D}$ is just a usual functor $F : \mathcal{C}^{op} \to \mathcal{D}$.

Example 1.2.

(i) Let $\mathcal{C} = Groups$ and $\mathcal{D} = Sets$. Then we have the forgetful functor $F : \mathcal{C} \to \mathcal{D}$.

(ii) Let k be a field and take $\mathcal{C} = Vect_k = \mathcal{D}$, the category of vector spaces over k. There is a contravariant functor $F : Vect_k \to Vect_k$ defined by $F(V) = V^*$

(iii) **Representable functors** Let $\mathcal{C}$ be a category, $X \in \mathrm{Ob}\,\mathcal{C}$. There are two functors $\ell_X : \mathcal{C} \to Sets$ and $r_X : \mathcal{C}^{op} \to Sets$ defined by $\ell_X(Y) = \mathrm{H\overline{om}}(X,Y)$ and $r_X(Y) = \mathrm{H\overline{om}}(Y,X)$. It turns out that X is completely determined by either ℓ_X or r_X.

Example 1.3. Let R be a commutative ring and let $\mathcal{C}$ be the category of R-modules. For any R-modules M and N, want $M \otimes_R N \in \mathrm{Ob}\,\mathcal{C}$. The condition

$$\mathrm{H\overline{om}}(M \otimes_R N, K) = \{R\text{-bilinear maps } f : M \times N \to K\}$$

uniquely determines the object $M \otimes_R N$.

Given $F, G : \mathcal{C} \to \mathcal{D}$ a morphism of functors (ie. a **natural transformation**) is given by the data of for every $X \in \mathcal{C}$ a map $F(X) \xrightarrow{\alpha_X} G(X)$ such that for any $f \in \mathrm{H\overline{om}}(X,Y)$,

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ \big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array}$$

Lemma 1.1. (**Yoneda Lemma**) *For all $X_1, X_2 \in \mathrm{Ob}\,\mathcal{C}$ the map $\mathrm{H\overline{om}}(X_2, X_1) \to \mathrm{H\overline{om}}(\ell_{X_1}, \ell_{X_2})$ is an isomorphism and the map $\mathrm{H\overline{om}}(X_1, X_2) \to \mathrm{H\overline{om}}(r_{X_1}, r_{X_2})$ is an isomorphism.*

Proof. Exercise. □

1.3 Adjoint Functors

Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ we say F is **left adjoint** to G (and G is **right adjoint** to F) if we are given a functorial isomorphism

$$\mathrm{H\overline{om}}(F(X), Y) \xrightarrow{\sim} \mathrm{H\overline{om}}(X, G(Y)).$$

Example 1.4. Let A, B be associative rings and let $\phi : A \to B$ be a ring homomorphism. Let $\mathcal{C}$ be the category of left B-modules and let $\mathcal{D}$ be the category of left A-modules. Let $G : \mathcal{C} \to \mathcal{D}$ be the obvious functor. Then G has a natural left adjoint given by $F(M) = B \otimes_A M$ for any A-module M.

The fact that these functors are adjoint just becomes the fact that

$$\mathrm{H\overline{om}}_A(M, N) = \mathrm{H\overline{om}}_B(B \otimes_A M, N)$$

for any A-module M and any B-module N.

Given any functor F there exists at most one (up to canonical isomorphism) adjoint G. For any category $\mathcal{C}$ we have the identity functor $Id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. Functors F and G are inverse if $G \circ F$ is isomorphic to $Id_{\mathcal{C}}$ and $F \circ G$ is isommorphic to $Id_{\mathcal{D}}$.

Exercie 1.2. Show that if F and G are inverse then G is both left and right adjoint to F.

If F has an inverse then we say that F is an **equivalence of categories**.

Example 1.5. For A a commutative $\mathbb{R}$-algebra, a norm on A is a map $\|\cdot\| : A \to \mathbb{R}_{\geq 0}$ such that

1. $\|ab\| \leq \|a\|\|b\|$
2. $\|a + b\| \leq \|a\| + \|b\|$
3. $\|ca\| = |c|\|a\|$ for $c \in \mathbb{R}$.
4. $\|a\| = 0 \iff a = 0$.

Note that $\|\cdot\|$ defines a topology on A in which A is complete.

For X a compact Hausdorff topological space, let A be the algebra of $\mathbb{R}$ valued continuous functions and define a norm by $\|f\| = \max_{x \in X} |f(x)|$.

Let $\mathcal{C}$ be the category of compact Hausdorff topological spaces and let $\mathcal{D}$ be the category of complete normed algebras (with morphism $\varphi : A \to B$ continuous.) Then the algebra of continuous functions on X gives us a functor $F : \mathcal{C}^{op} \to \mathcal{D}$ and a theorem due to Gelfand shows that this is an equivalence of categories.

1.4 Abelian Categories

Let $\mathcal{C}$ be a category and suppose we are given a pair of morphisms

$$\begin{array}{ccc} X & & \\ & \searrow^{f} & \\ & & Z \\ & \nearrow_{g} & \\ Y & & \end{array} .$$

Then the diagram

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

where $\overline{\text{Hom}}(W, X \times_Z Y) = \{\alpha : W \to X, \beta : W \to Y : f \circ \alpha = g \circ \beta\}$ uniquely (up to canonical isomorphism) defines the object $X \times_Z Y$, called the **fibered product** of X and Y over Z. We can also talk about $X \times Y$ with no Z where $X \times Y = X \times_Z Y$ with Z the final object of $\mathcal{C}$ ie. for all X, $\overline{\text{Hom}}(X, Z)$ consists of one element.

Dually, given morphisms

$$\begin{array}{ccc} & & X \\ & \nearrow^{f} & \\ Z & & \\ & \searrow_{g} & \\ & & Y \end{array}$$

the diagram

$$\begin{array}{ccc} X \sqcup_Z Y & \longleftarrow & X \\ \uparrow & & \uparrow \\ Y & \longleftarrow & Z \end{array}$$

uniquely determines the object $X \sqcup_Z Y$, called the **fibered sum** or **pushout.**

Definition 1.3. An additive category over a commutative ring k is given by

1. A category $\mathcal{C}$
2. All $\mathrm{H}\overline{\mathrm{om}}(X, Y)$ are k-modules and compositions are compatible with this
3. There exists an object 0 such that for all X,

$$\mathrm{H}\overline{\mathrm{om}}(X, 0) = \mathrm{H}\overline{\mathrm{om}}(0, X) = \{0\}.$$

   This means that 0 is both final and initial.

4. Binary products exist and they satisfy the conditions that for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$, $X \times Y = Y \times X$ coincide. Similarly for $X \sqcup Y$.

Example 1.6. For R a ring the categories of left R-modules, free R-modules and projective R-modules are all abelian.

Let $\mathcal{C}$ be a category. A **subcategory** $\mathcal{D}$ of $\mathcal{C}$ is given by a subclass $\mathrm{Ob}\,\mathcal{D} \subset \mathcal{C}$ and for all $X, Y \in \mathrm{Ob}\,\mathcal{D}$ a map $\mathrm{H}\overline{\mathrm{om}}_{\mathcal{D}}(X, Y) \hookrightarrow \mathrm{H}\overline{\mathrm{om}}_{\mathcal{C}}(X, Y)$ which takes the identity to the identity. If $\mathrm{H}\overline{\mathrm{om}}_{\mathcal{D}}(X, Y) = \mathrm{H}\overline{\mathrm{om}}_{\mathcal{C}}(X, Y)$ for all X, Y then $\mathcal{D}$ is called a **full subcategory.**

Let $\mathcal{C}$ be an additive category and $f : X \to Y$. Then the **kernel** of f, $\mathrm{Ker}\, f \in \mathrm{Ob}\,\mathcal{C}$, is the object determined by the condition $\mathrm{H}\overline{\mathrm{om}}_{\mathcal{C}}(Z, \mathrm{Ker}\, f) = \{\phi : Z \to X : f \circ \varphi = 0\}$

Dually the **cokernel** is defined by $\mathrm{H}\overline{\mathrm{om}}_{\mathcal{C}}(\mathrm{Coker}\, f, Z) = \{\phi : Y \to Z : \varphi \circ f = 0\}$.

Definition 1.4. The additive category $\mathcal{C}$ is called **abelian** if

- All kernels and cokernels exist.
- The sequence $\mathrm{Ker}\, f \xrightarrow{\alpha} X \to Y \xrightarrow{\beta} \mathrm{Coker}\, f$ yields a map $\mathrm{Coker}\,\alpha \to \mathrm{Ker}\,\beta$ which we require to always be an isomorphism.

Example 1.7. The category of R-modules for R an associative ring and the category of sheaves of abelian groups on a topological space X are abelian categories.

A **filtration** on a ring R is a collection of $R_0 \subset R_1 \subset R_2 \subset \cdots$ such that $\bigsqcup_i R_i = R$, $R_i \cdot R_j \subset R_{i+j}$ and $1 \in R_0$. For M an R-module, a filtration on M is a collection $M_0 \subset M_1 \subset \cdots$ such that $\bigsqcup_i M_i = M$ and $R_i \cdot M_j \subset M_{i+j}$.

A **grading** on R is a direct sum decomposition $R = \bigoplus R_i$ such that $R_i \cdot R_j \subset R_{i+j}$ and similarly for modules.

Exercie 1.3. Let $\mathcal{C}$ be the category of filtered modules. Show that $\mathcal{C}$ is additive but not abelian. Show that graded modules do form an abelian category.

Let $\mathcal{A}$ be an abelian category. A **short exact sequence** in $\mathcal{A}$ is a diagram $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ in which $(X, \alpha) = \mathrm{Ker}\,\beta$, $(Z, \beta) = \mathrm{Coker}\,\alpha$ and $\mathrm{Im}\,\alpha = \mathrm{Ker}\,\beta$.

Example 1.8. $Y = X \oplus Z$ defines a short exact sequence. $\mathcal{A}$ is **semisimple** if any short exact sequence is split, ie. of the form $Y = X \oplus Z$.

Lecture 2 [09.02.2016] (Notes taken and typed by Anne Dranovski)

Last time

- categories and functors
- additive (morphisms form an abelian group, $\exists$ 0 object, $\exists$ direct sums) and abelian ($\exists$ kernels, cokernels, $\operatorname{Ker} f \longrightarrow X \longrightarrow Y \longrightarrow \operatorname{Coker} f$)

More examples

Example 1.9. Let R be an associative ring. Take $\mathcal{A}$ = left R-modules.

Example 1.10. Let X be a topological space. Take $\mathcal{A}$ = category of sheaves of abelian groups on X.

1.5 Additive functors

Let $\mathcal{A}, \mathcal{B}$ be additive categories.

Definition 1.5. $F : \mathcal{A} \to \mathcal{B}$ is called additive if

1. $F(\mathcal{O}_\mathcal{A}) = \mathcal{O}_\mathcal{B}$
2. $\forall X, Y \in \operatorname{Ob} \mathcal{A}, F_{X,Y} : \operatorname{H\overline{om}}(X, Y) \to \operatorname{H\overline{om}}(F(X), F(Y))$ is a hom of abelian groups
3. $\forall X, Y \in \operatorname{Ob} \mathcal{A},\ F(X \oplus Y) \xrightarrow{\ \sim\ } F(X) \oplus F(Y)$ is an isomorphism.

Exercie 1.4. Show that a priori there exists a map in one direction (i.e. a quasi-isomorphism? that we require to be an isomorphism) in (3).

Example 1.11. For all $X \in \operatorname{Ob} \mathcal{A}$, the associated (left-exact) functor

$$l_X : \mathcal{A} \longrightarrow \mathsf{Ab} : Y \mapsto \operatorname{H\overline{om}}(X, Y)$$

is additive.

Example 1.12. Let $\mathcal{A}$ be the category of left R-modules. Let M be a right R-module. Then

$$F : \mathcal{A} \longrightarrow \mathsf{Ab} : X \mapsto M \otimes_R X$$

is additive. Variant: if R is commutative, regard F as $\mathcal{A} \to \mathcal{A}$.

Example 1.13. Let X be a topological space. Let $\mathcal{A}$ be the category of sheaves of abelian groups on X. Then

$$F : \mathcal{A} \longrightarrow \mathsf{Ab} : \mathcal{F} \mapsto \Gamma(X, \mathcal{F})$$

is additive.

Exercie 1.5. Show that example (1.11) is a special case of example (3).

Remark 1.1. In the category of modules over a commutative ring R the functor $F(M) = M \otimes_R M$ is **not** additive.

Let $\mathcal{A}, \mathcal{D}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be additive. (Question: do we need additive? is $F(0) = 0$ enough?)

Definition 1.6 ("The most important definition of homological algebra")**.** F is called exact if for every $0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$ short exact sequence in $\mathcal{A}$,

- $(X, \alpha) = \operatorname{Ker} \beta$
- $(Z, \beta) = \operatorname{Coker} \alpha$
- $\operatorname{Im} \alpha = \operatorname{Ker} \beta$

Example 1.14. Let $\mathcal{A}$ be an abelian category. Let Ab be the category of abelian groups. Let $W \in \operatorname{Ob} \mathcal{A}$. Is $l_W : \mathcal{A} \to \mathcal{A} : X \mapsto \underline{\operatorname{Hom}}(W, X)$ an exact functor? No. But, it's left-exact.

Definition 1.7. An additive (though, again, may make sense for others?) functor F is called left exact if for every short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

the sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z)$$

is exact.

Exercie 1.6. Show that r_W is also left exact. (Recall $r_W : \mathcal{A}^{\mathsf{op}} \to \mathsf{Ab}$.)

Definition 1.8. $\mathcal{A}$ is called semisimple if any short exact sequence splits. That is, whenever

$$0 \longrightarrow X \xrightarrow{\ \alpha\ } Y \xrightarrow{\ \beta\ } Z \longrightarrow 0$$

is short exact, $Y \cong X \oplus Z$. In this case, every additive $F : \mathcal{A} \to \mathcal{B}$ is exact.

Example 1.15. Let $\mathcal{A}$ be the category of k-vector spaces. Every additive $F : \mathcal{A} \to \mathcal{B}$ will be exact.

Remark 1.2. Interesting cohomology theory comes from functors that are only left or only right exact.

Definition 1.9. Right exact.

Exercie 1.7. Let M be a right R-module. Show that $F_M : \mathcal{A} \to \mathsf{Ab} : X \mapsto M \otimes_R X$ is always right exact.

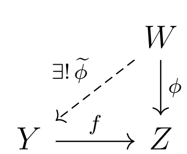
Remark 1.3. The main goal of homological algebra: make all functors exact.

Given a functor that is only left exact or right exact we would like to define some notion of its derived functor. Whatever that means..

"As an aside" there exist additive functors that are neither left nor right exact. Take the localization functor for example. These functors come from composing left and right exact functors.

Example 1.16. Let CA be the category of R-modules. Take $W = R^n$ a free module. Then $\underline{\operatorname{Hom}}(R^n, X) = X \oplus \cdots \oplus X$ (n times) and l_W is exact.

Definition 1.10. An object W of an abelian category is called projective if l_W is exact. Equivalently, for every surjective map $Y \to Z$, $\underline{\operatorname{Hom}}(W, Y) \to \underline{\operatorname{Hom}}(W, Z)$ surjects. That is, for every hom ϕ of $W \to Z$, there is a (uniqe lift) $\widetilde{\phi}$ such that the following diagram commutes.

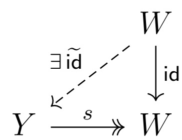


Free modules are projective

Lemma 1.2. *An R-module W is projective if and only if W is a direct summand of a free module.*

Exercie 1.8. Show that a direct summand of a projective object is projective.

Towards a converse: any R-module W is a quotient of a free module. That is, there exists a free Y and a surjection such that the following diagram commutes.



We can cook up $Y = \operatorname{Im}(\widetilde{\mathsf{id}}) \oplus \operatorname{Ker}(s)$ and if W is projective then it's a direct summand, $W = \operatorname{Im}(\widetilde{\mathsf{id}})$.

Fact: If $R = \mathbb{Z}$ (more generally, if it's commutative?) then all finitely generated projective modules are free.

Example 1.17. Let $R = k[X_1, \ldots, X_n]$ for k a field. Serre's conjecture: all finitely generated projective R-modules are free. (Also, Quillen & Suslin.)

Remark 1.4. The finitely generated assumption is not necessary in the $R = \mathbb{Z}$ case/example. It may or may not be necessary in the $R = k[X_1, \ldots, X_n]$ case, and it's not easier to assess if we fix $k = \mathbb{C}$.

Definition 1.11. $W \in \operatorname{Ob} \mathcal{A}$ is called injective if r_W is exact.

Example 1.18. In the category of abelian groups, W is injective if and only if it is a divisible group. That is, for every $w \in W$, for every $n \in \mathbb{Z} - 0$, there exists $w' \in W$, such that $nw' = w$.

Exercie 1.9. Prove it.

Definition 1.12. $\mathcal{A}$ has enough projectives if any $X \in \operatorname{Ob} \mathcal{A}$ can be covered by a projective object P, i.e. there exists β such that

$$P \xrightarrow{\beta} X \longrightarrow 0 \text{ with } \operatorname{Coker} \beta = 0$$

Likewise, $\mathcal{A}$ has enough injectives if any module X can be embedded in an injective module I, i.e. there exists α such that

$$X \xhookrightarrow{\alpha} I$$

Exercie 1.10 (Exercise-Theorem). The category of R-modules has enough injectives.

Definition 1.13. A right module M over a ring R is called flat if the functor $X \mapsto M \otimes_R X$ is exact.

Note that flatness is not a categorical notion. Yet, knowing that we have enough flats will be important to us later when we talk about derived $\otimes$ structure, where neither is $\otimes$ a categorical notion.

Exercie 1.11. Free and projective = flat.

Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor. The naive definition of the (right) derived functor $R^i F$ of F is as follows. Given a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

there exists a long exact (at every term) sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow R^1F(X) \longrightarrow R^1F(Y) \longrightarrow R^1F(Z) \longrightarrow \cdots$$

The non-naive construction will be to package (all?) the $R^i F$ into a single functor between the derived categories of $\mathcal{A}$ and $\mathcal{B}$.

Example 1.19. Take $F = l_W$ from before. $R^i F(X) = \operatorname{Ext}^i(W, X)$.

1.6 Cohomology

Fix $\mathcal{A} = \mathsf{Ab}$.

Definition 1.14. A (co-chain) complex consists of objects K^i and maps $d_i : K^i \to K^{i+1}$ such that $d_{i+1} \circ d_i = 0$ for all i. Set $\mathcal{C}(\mathcal{A})$ to be the category of all complexes over $\mathcal{A}$. Then morphisms of $\mathcal{C}$ are

$$\left\{ f^i : K^i \to L^i \,\middle|\, \begin{array}{ccc} K^i & \longrightarrow & K^{i+1} \\ {\scriptstyle f^i}\downarrow & & \downarrow{\scriptstyle f^{i+1}} \\ L^i & \longrightarrow & L^{i+1} \end{array} \right\}$$

Exercie 1.12. Show that $\mathcal{C}$ is an abelian category. (“Useless, easy.")

Definition 1.15. Say $K^\bullet$ is bounded below if $K^i = 0$ for all i sufficiently small. Denote by $\mathcal{C}^\pm$ the categories of all complexes bounded below, above. By $\mathcal{C}^b = \mathcal{C}^+ \cap \mathcal{C}^-$ the cat of all bounded complexes.

Definition 1.16. For $K^\bullet \in \mathcal{C}$ the cohomology of $K^\bullet$ is the object $H^i(K^\bullet) \in \operatorname{Ob}\mathcal{A}$,

$$H^i(K^\bullet) = \frac{\operatorname{Ker} d_i}{\operatorname{Im} d_{i-1}}$$

Each H^i is an additive functor $\mathcal{C} \to \mathcal{A}$ that's neither left nor right exact.

Definition 1.17. A map of complexes $f^\bullet : K^\bullet \to L^\bullet$ is a quasi-isomorphism if $H^i(f^\bullet)$ is an isomorphism for all i.

Exercie 1.13. Give examples of quasi-isomorphisms that aren't isomorphisms.

Example 1.20. Let $\mathcal{A}$ be the category of k vector spaces for k a field. The same applies to any semisimple abelian category. Consider the complex $(K^\bullet, 0)$ with $0 = d_\bullet$ differential. Then $H^\bullet(K^\bullet)$ and $K^\bullet$ are quasi-isomorphic in the strongest possible sense. In either direction. (What does it mean? Quasi-isomorphism in either direction is isomorphism, isn't it?)

Chapter 2

Derived Categories

2.1 Main idea of derived categories

Want to study complexes up to quasi-isomorphism, i.e. to define a new category $\mathcal{D}(\mathcal{A})$ which we will call the derived category of $\mathcal{A}$ whose objects are complexes (still) but whose morphisms will come from morphisms of $\mathcal{A}$, with quasi-isomorphisms becoming isomorphisms (i.e., invertible).

For example, in the derived category, our example

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 \\ & & & \Big\Downarrow & & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

should have an inverse.

Theorem 2.1. *There exists a unique (up to unique equivalence) category $\mathcal{D}(\mathcal{A})$ together with a functor $Q : \mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ such that*

1. *If f is a quasi-isomorphism in $\mathcal{A}$, then $Q(f)$ is an isomorphism in $\mathcal{D}(\mathcal{A})$.*
2. *(Universality.) For every $F : \mathcal{C} \to \mathcal{D}'$ such that whenever f is a quasi-isomorphism, $F(f)$ is an isomorphism, there exists a unique $G : \mathcal{D} \to \mathcal{D}'$ such that $F = G \circ Q$.*

In general, given a cat $\mathcal{C}$ and a class of morphisms S stable under composition, we can produce a new (universal) category $\mathcal{D}$ where all elements of S become isomorphisms. How to define such a thing? $\operatorname{Ob}\mathcal{D} = \operatorname{Ob}\mathcal{C}$. The roof

$$\begin{array}{ccccc} & & Z & & \\ & \overset{f}{\swarrow} & & \overset{g}{\searrow} & \\ X & & \dashrightarrow & & Y \end{array}$$

should give a morphism in $\mathcal{D}$.

Exercie 2.1. When do such diagrams define the same morphism in $\mathcal{D}$?

Lecture 3 **[23.02.2016]**

Last Time

Let $\mathcal{A}$ be an abelian category and $\mathcal{C}(\mathcal{A})$ be the category of complexes on $\mathcal{A}$. $f : K^\bullet \to L^\bullet$ is a quasi–isomorphism if $\forall i \in \mathbb{Z}$, $H^i(f) : H^i(K^\bullet) \to H^i(L^\bullet)$ is an isomorphism. Then $\mathcal{D}(\mathcal{A})$, the derived cateogry of $\mathcal{A}$ is the universal category with a functor $Q : \mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ such that $Q(f)$ is invertible if f is a quasi–isomorphism.

Analogy

Let R be an associative ring and $S \subset R$ be a multiplicative subset not containing 0. Then $\exists S^{-1}R$ and a map $R \to S^{-1}R$. $S^{-1}R$ always exists but it is almost impossible to describe it.

Ore Condition

The right Ore condition is $sR \cap aS \neq \varnothing$ for all $s \in S$ and $a \in R$. Equivalently, $s^{-1}a$ can be written as bt^{-1} for some $t \in S$. If this condition (or the equivalent left version) is satisfied, then it becomes possible to describe the localisation.

We would like to find an analog of the Ore condition for a class of morphisms in a category and then discuss to what extent the class of quasi–isomorphisms satisfies this condition.

2.2 Structures on Derived Categories

1. Derived categories are additive categories.
2. There exists a shift functor $K^\bullet \to K^\bullet[n]$ for any $n \in \mathbb{Z}$ where $K^\bullet[q]^i = K^{i+1}$ and in general $K^\bullet[n] = K^{i+n}$.
3. For any i, there exist cohomology functors $H^i : \mathcal{D}(\mathcal{A}) \to \mathcal{A}$.
4. Distinguished triangles (to be defined below) exist.

Let $K^\bullet, L^\bullet \in \mathcal{C}(\mathcal{A})$ and let $f : K^\bullet \to L^\bullet$. Define a new complex $\operatorname{Cone}(f)$ such that if $\operatorname{Ker} f = 0$ then $\operatorname{Cone}(f)$ is quasi–isomorphic to $L^\bullet/K^\bullet$ and such that

$$\to K^\bullet \to L^\bullet \to \operatorname{Cone}(f) \to K^\bullet[1] \to L^\bullet[1] \to \operatorname{Cone}(f)[1] \to \cdots$$
$$\to H^i(K^\bullet) \to H^i(L^\bullet) \to H^i(\operatorname{Cone}(f)) \to H^{i+1}(K^\bullet) \to H^{i+1}(L^\bullet) \to \cdots$$

is a long exact sequence.

Corollary 2.1. *Assume $K^\bullet$ is a subcomplex of $L^\bullet$. Then there exists a long exact sequence*

$$H^i(K^\bullet) \to H^i(L^\bullet) \to H^i(L^\bullet/K^\bullet) \to H^{i+1}(K^\bullet) \to H^{i+1}(L^\bullet) \to H^{i+1}(L^\bullet/K^\bullet)$$

Definition 2.1. $\operatorname{Cone}(f)$ is $K^\bullet[1] \oplus L^\bullet$ if we forget about the differential so $\operatorname{Cone}(f)^i = K^{i+1} \oplus L^i$. The differential is given by $d_{\operatorname{Cone}(f)} = (-d_K, f + d_L)$; ie.

$$d(k^{i+1}, \ell^i) = (-d_K k^{i+1}, f(k^{i+1}) + d_L \ell^i) \in K^{i+2} \oplus L^{i+2}$$

Note that

$$\begin{aligned} d^2(k^{i+1}, \ell^i) &= d(-d_K k^{i+1}, f(k^{i+1}) + d_L \ell^i) \\ &= (0, -f(d_K k^{i+1}) + d_L f(k^{i+1})) \\ &= (0, 0) \end{aligned}$$

so this is a differential.

In the sequence,

$$K^\bullet \to L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1] \to L^\bullet[1] \to \mathrm{Cone}(f)[1] \to \cdots$$

it is easy to check that the composition of any two arrows is zero in $H^\bullet$ and hence in the sequence

$$\cdots \to H^i(K^\bullet) \to H^i(L^\bullet) \to H^i(\mathrm{Cone}(f)) \to H^{i+1}(K^\bullet) \to \cdots$$

the composition of any two maps is zero.

Assume that $\mathrm{Ker}(f) = 0$. We always have the obvious map $\mathrm{Cone}(f) \to L^\bullet/f(K^\bullet)$ which is a quasi–isomorphism in the case that $\mathrm{Ker}(f) = 0$. For example, to show it is surjective on cohomology, we need to show that $H^i(\mathrm{Cone}(f)) \to H^i(L^\bullet/f(K^\bullet))$ is surjective. Cocycles in $L^\bullet/f(K^\bullet)$ come from $\ell^i \in L^i$ such that $d_L(\ell^i) = f(k^i)$ so

$$f d_K(k^{i+1}) = d_L(f(k^{i+1})) = d_L^2(\ell^i) = 0$$

and so $d_K k^{i+1} = 0$. But then $d(-k^{i+1}, \ell^i) = 0$ and so the projection to $L^\bullet/f(K^\bullet)$ is equal to the projection of ℓ^i.

Exercie 2.2. Show injectivity.

We now have $K^\bullet \to L^\bullet \to \mathrm{Cone}(f)$ and $\mathrm{Cone}(f) \sim L^\bullet/f(K^\bullet)$ if $\mathrm{Ker}(f) = 0$.

Definition 2.2. The **cylinder** of f is $\mathrm{Cyl}(f)^i = K^i \oplus K^{i+1} \oplus L^i$ with the differential

$$d : (k^i, k^{i+1}, \ell^i) \mapsto (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) + d_L \ell^i).$$

Lemma 2.1.

(i) $L \to \mathrm{Cyl}(f)$ *is a quasi–isomorphism.*

(ii) $K^\bullet \to \mathrm{Cyl}(f) : k^i \mapsto (k^i, 0, 0)$ *is injective.*

(iii) $K \to \mathrm{Cyl}(f) \to \mathrm{Cone}(f)$ *is a short exact sequence of complexes.*

Proposition 2.1. $K^\bullet \to L^\bullet \to \mathrm{Cone}(f)$ *gives rise to a long exact sequence in cohomology.*

Remark 2.1. It is enough to prove exactness in the previous proposition when $\mathrm{Ker}(f) = 0$.

Definition 2.3. A **distinguished triangle** $X \to Y \to Z \to X[1]$ for $X, Y, Z \in \mathrm{Ob}\,\mathcal{D}(\mathcal{A})$ is an image in $\mathcal{D}(\mathcal{A})$ of $K^\bullet \xrightarrow{f} L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1]$ in $\mathcal{C}(\mathcal{A})$.

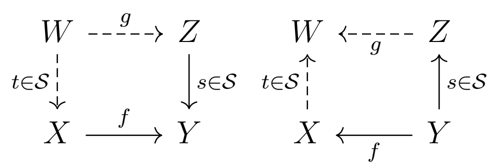
Warning: Any $X \xrightarrow{f} Y$ can be completed to a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ but **not** canonically.

2.3 Explicit Description of Derived Categories

Let $\mathcal{C}$ be a category and let $\mathcal{S}$ be a class of morphisms which is multiplicative (ie. closed under composition.)

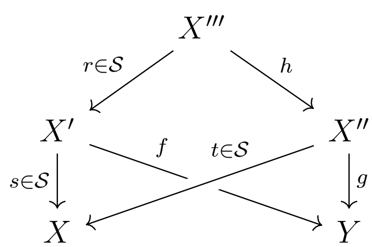
Definition 2.4. $\mathcal{S}$ is a **localizable class** if

1. For all $s : B \to C$ and $t : A \to B$, $s \circ t \in \mathcal{S}$ (multiplicativity)

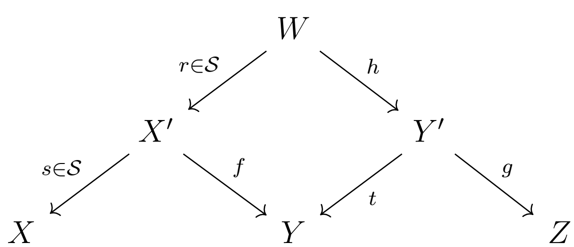
2. The following two diagrams commute: 

3. Let $f, g : X \to Y$. There exists $s \in \mathcal{S}$ such that $sf = sg$ iff there exists $t \in \mathcal{S}$ with $ft = gt$.

If $\mathcal{S}$ is localizable then $\mathcal{C}[\mathcal{S}^{-1}]$ has a simple description. Namely, $\mathrm{Ob}\,\mathcal{C}[\mathcal{S}^{-1}] = \mathrm{Ob}\,\mathcal{C}$ and morphisms are **roofs** $X \xleftarrow{s\in\mathcal{S}} X' \xrightarrow{f} Y$ modulo the equivalence relation where $X \xleftarrow{s\in\mathcal{S}} X' \xrightarrow{f} Y$ is equivalent to $X \xleftarrow{t\in\mathcal{S}} X'' \xrightarrow{g} Y$ if there exists X''' and a commuting diagram



Composable morphisms are those fitting into a commutative diagram of the form



and the composition of these two morphisms is $X \xleftarrow{s\circ r\in\mathcal{S}} W \xrightarrow{g\circ h} Z$

Proposition 2.2. *If $\mathcal{S}$ is a localizable class, then*

1. *This is $\mathcal{C}[S^{-1}]$.*

2. *If $\mathcal{C}$ is an additive category then so is $\mathcal{C}[S^{-1}]$*

Let $\mathcal{A}$ be an abelian category.
Question: Do quasi–isomorphisms form a localizable class?
No: Let $K^\bullet$ be $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to \cdots$ (nonzero in degree -1 and 0) and let $L^\bullet$ be $\cdots 0 \to \mathbb{Z}/\mathbb{Z} \to 0 \to \cdots$. These are quasi–isomorphic. Let $f : K^\bullet \to K^\bullet$ be multiplication by 2 and let $g : K^\bullet \to K^\bullet$ be 0. Then $sf = 0 = sg$ but there does not exist a quasi–ismorphism such that $ft = 0$ since for $t : L \to K$, $t(L^0) \neq 0$ so $2t(L^\bullet) \neq 0$.
Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes.

Definition 2.5. f is **homotopic** to 0 if for any i there exists $h_i : K^i \to L^{i-1}$ such that $f = dh + hd$.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & K^{i-1} & \longrightarrow & K^i & \longrightarrow & K^{i+1} & \longrightarrow & \cdots \\
 & & \downarrow f^{i-1} & \swarrow h^i & \downarrow f^i & \swarrow h^{i+1} & \downarrow f^{i+1} & & \\
\cdots & \longrightarrow & L^{i-1} & \longrightarrow & L^i & \longrightarrow & L^{i+1} & \longrightarrow & \cdots
\end{array}$$

For $f, g : K^\bullet \to L^\bullet$, f and g are **homotopic** if $f - g$ is homotopic to 0.

Lemma 2.2. *If f is homotopic to 0, then f becomes 0 in $\mathcal{D}(\mathcal{A})$.*

Proof. We have $K^\bullet \xrightarrow{id_K} K^\bullet$ and $\mathrm{Cone}(id_K)$ is quasi–isomorphic to 0, and so becomes 0 in $\mathcal{D}(\mathcal{A})$. The result follows from this. □

Definition 2.6. The **homotopy category** $\mathcal{K}(\mathcal{A})$ has $\mathrm{Ob}\,\mathcal{K}(\mathcal{A}) = \mathrm{Ob}\,\mathcal{C}(\mathcal{A})$ and morphisms

$$Hom_{\mathcal{K}(\mathcal{A})}(K^\bullet, L^\bullet) = \overline{\mathrm{Hom}}_{\mathcal{C}(\mathcal{A})}(K^\bullet, L^\bullet)/(\text{maps homotopic to 0})$$

$\mathcal{K}(\mathcal{A})$ is obviously additive, and for $f = hd + dh$ and $d(k) = 0$, $f(k) = (hd + dh)(k) = d(h(k))$. Furthermore, H^i are well–defined in $\mathcal{K}(\mathcal{A})$ for all i and quasi–isomorphisms make sense.

Theorem 2.2. *In $\mathcal{K}(\mathcal{A})$ quasi–isomorphisms form a localizable class.*

Recall the example $K^\bullet = 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to \cdots$ and $f : K^\bullet \to K^\bullet$ multiplication by 2. Let $h^0 = id : \mathbb{Z} \to \mathbb{Z}$ (this is the only necessary map to define a homotopy in this example.) $hd + dh$ is multiplication by 2 so f is homotopic to zero and our previous counterexample no longer works.
This yields the desired explicit description of the homotopy category:

$$\mathcal{D}(\mathcal{A}) = \mathcal{K}(\mathcal{A})[S^{-1}].$$

In particular this implies that $\mathcal{D}(\mathcal{A})$ is additive.

2.4 The Ext Functors

Let $X, Y \in \mathcal{A} \hookrightarrow \mathcal{C}(\mathcal{A}) \xrightarrow{Q} \mathcal{D}(\mathcal{A})$.

Definition 2.7. The **i–th Ext functor** is given by

$$\mathrm{Ext}^i(X, Y) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[i])$$

Remark 2.2. $\mathcal{D}^+(\mathcal{A}), \mathcal{D}^-(\mathcal{A}), \mathcal{D}^b(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$ are all full subcategories.

Assume that $\mathcal{A}$ has enough projectives (ie. for all $X \in \mathcal{A}$, there exists P a projective object in $\mathcal{A}$ and a surjective map $P \to X$.

Definition 2.8. A **projective resolution** of X is a complex

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to 0$$

with each P^{-i} projective and

$$H^i(P^\bullet) = \begin{cases} 0 & i \neq 0 \\ X & i = 0 \end{cases}$$

Lemma 2.3. *Every X has a projective resolution.*

Proof. Have

$$P^0 \xrightarrow{\alpha_0} X \to 0.$$

Then there exists

$$P^{-1} \xrightarrow{\alpha_1} \operatorname{Ker} \alpha_0 \to 0.$$

Repeating, there exists

$$P^{-2} \xrightarrow{\alpha_2} \operatorname{Ker} \alpha_1 \to 0.$$

and so on. This yields the projective resolution. □

Theorem 2.3. *Let $P^\bullet$ be a projective resolution of X and define the complex*

$$\overline{\operatorname{Hom}}(P^\bullet, Y)^i = \operatorname{Hom}_{\mathcal{A}}(P^{-i}, Y).$$

This is a complex since for all i, $P^{-i-1} \to P^{-i}$ induces a map $\overline{\operatorname{Hom}}(P^{-i}, Y) \to \overline{\operatorname{Hom}}(P^{-i-1}, y)$. Then

$$\operatorname{Ext}^i(X, Y) = H^i(\overline{\operatorname{Hom}}(P^\bullet, Y)).$$

There is a dual version of this. If $\mathcal{A}$ has enough injectives, then any Y has an injective resolution

$$0 \to I^0 \to I^1 \to I^2 \to \cdots$$

with each I^j injective and

$$H^i(I^\bullet) = \begin{cases} 0 & i \neq 0 \\ Y & i = 0 \end{cases}$$

The resolution is constructed as before. Have

$$0 \to Y \xrightarrow{\beta_0} I^0.$$

$$0 \to \operatorname{Coker} \beta_0 \xrightarrow{\beta_1} I^1$$

$$0 \to \operatorname{Coker} \beta_1 \xrightarrow{\beta_2} I^2$$

etc.

Theorem 2.4. *If $I^\bullet$ is an injective resolution of Y then*

$$\operatorname{Ext}^i(X, Y) = H^i(\overline{\operatorname{Hom}}(X, I^\bullet)).$$

Lecture 4 **[01.03.2016]**

Warning: Missed lecture and only had partial notes to ge by so some sections may be incomplete/incorrect for this lecture.

Exercie 2.3. Let $\mathcal{A}$ be an abelian category. Then $Q : \mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is an equivalence if and ony if $\mathcal{A}$ is semisimple (ie. every short exact sequence splits.)

Let $\mathcal{A}$ be an abelian category. Given any exact sequence

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{A}$, if two of the tarms are in $\mathcal{B}$ theno is the third and furthermore it is closed under subquotients. In part, $\mathcal{B}$ is itself therefore an abelian category and we have a functor

$$\mathcal{D}(\mathcal{B}) \to \mathcal{D}_{\mathcal{B}}(\mathcal{A})$$

where $\mathcal{D}_{\mathcal{B}}(\mathcal{A})$ is the full subcategory of $\mathcal{D}(\mathcal{A})$ consisting of objects whose cohomology is in $\mathcal{B}$. There is no reason for this functor to be an equivalence.

Example 2.1.

$$\begin{aligned}
&\mathfrak{g} \text{ a complex simple Lie algebra} \\
&\mathcal{A} = \text{modules over } \mathfrak{g} \\
&\mathcal{B} = \text{finite dimensional modules over } \mathfrak{g}\text{(semisimple)}
\end{aligned}$$

Exercie 2.4. Show that $\mathcal{D}(\mathcal{B}) \to \mathcal{D}_{\mathcal{B}}(\mathcal{A})$ is NOT an equivalence.

Let $X, Y \in \mathrm{Ob}(\mathcal{A})$. Then $\mathrm{Ext}^i_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[i])$ but

$$\mathrm{Ext}^i_{\mathcal{B}}(X,Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X,Y)$$

is often not an isomorphism.

Let $\mathcal{A}$ be an abelian category, $K^\bullet, L^\bullet \in \mathrm{Ob}(\mathcal{C}(\mathcal{A}))$, $Q : \mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$. If $f, g : K^\bullet \to L^\bullet$ are two homotopic maps, then $Q(f) = Q(g)$ and we have a diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & L^\bullet & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & K^\bullet[1] \\
 & & & & \big\downarrow{\scriptstyle \alpha} & & \big\| & & \\
0 & \longrightarrow & K^\bullet & \longrightarrow & \mathrm{Cyl}(f) & \longrightarrow & \mathrm{Cone}[f] & & \\
 & & \big\| & & \big\downarrow{\scriptstyle \beta} & & & & \\
 & & K^\bullet & \xrightarrow{f} & L^\bullet & & & &
\end{array}$$

where $\beta\alpha = id$ and $\alpha\beta$ is homotopic to id. Recall

$$\mathrm{Cyl}(f)^i = K^i \oplus K^{i+1} \oplus L^i$$

with differential

$$d : (k^i, k^{i+1}, \ell^i) \mapsto (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) + d_L \ell^i)$$

$$\begin{aligned}
&\alpha\beta \text{ is homotopic to the identity via } h \\
&\text{means there exists an } h : \mathrm{Cyl}^i(f) \to \mathrm{Cyl}^{i-1}(f) \qquad (\star) \\
&\text{such that } \alpha\beta = hd + dh + id.
\end{aligned}$$

Lemma 2.4 (5-Lemma)**.** *Suppose*

$$\begin{array}{ccccccccc} A & \twoheadrightarrow & B & \longrightarrow & C & \twoheadrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

is exact and assume $\alpha, \beta, \delta, \varepsilon$ are isomorphisms. Then γ is also an isomorphism.

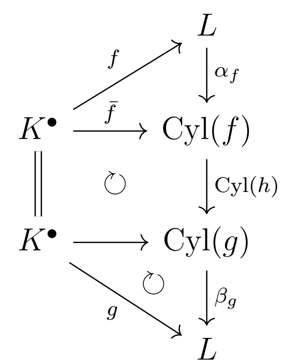
Proof. Exercise □

Corollary 2.2. *Given distinguished triangles*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

if a, b are isomorphisms (in $\mathcal{D}(\mathcal{A})$) then so is c.

If $f, g : K^\bullet \to L^\bullet$ are h–homotopic, ie. $f - g = dh + hd$ then there is a diagram



where $\bar{f}(k^i) = (k^i, 0, 0)$.

Note 2.1. $\alpha_f f \neq \bar{f}$ (the top triangle does not commute.) However, we will show that it commutes in $\mathcal{D}(\mathcal{A})$.

$$\mathrm{Cyl}(h)(k^i, k^{i+1}, \ell^i) = (k^i, k^{i+1}, \ell^i + h(k^{i+1}))$$

and $Q : \mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$. By $(\star)$ $Q(\alpha_f)$ is inverse to $Q(\beta_f)$ and $f = \beta_f \circ \bar{f}$ so $Qf = Q(\beta_f) \circ Q(\bar{f})$. So the diagram

$$\begin{array}{ccc} \mathcal{C}(\mathcal{A}) & \longrightarrow & \mathcal{K}(\mathcal{A}) \\ & \searrow & \vdots\downarrow \\ & & \mathcal{D}(\mathcal{A}) \end{array}$$

factorizes (on the level of objects this was clear. We just proved this factors also on the level of morphisms.)

Reference:

Chapter 4 of Gelfand, Manin: Methods of Homological Algebra

Theorem 2.5. *In $\mathcal{K}(\mathcal{A})$ quasi–ismorphisms form a localizing class.*

Proof. Omitted. □

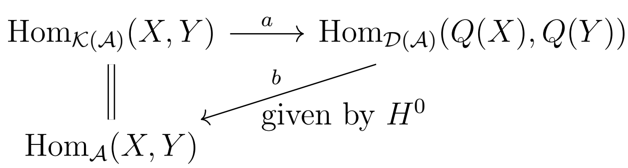
Corollary 2.3. *$\mathcal{D}(\mathcal{A})$ is additive.*

Corollary 2.4. *Define $\mathcal{D}_0(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$ to have objects X such that $H^i(X) = 0$ for all $i \neq 0$. Define $\mathcal{C}(\mathcal{A})$ similarly. Then the composition*

$$\mathcal{A} \longrightarrow \mathcal{C}_0(\mathcal{A}) \longrightarrow \mathcal{D}_0(\mathcal{A})$$

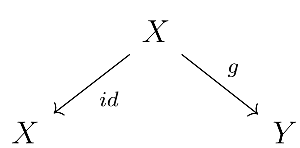
is an equivalence of categories.

Proof. $\mathcal{A} \to \mathcal{K}(\mathcal{A})$ is fully faithfu since if two complexes sit in degree 0 then there are no non–trivial homotopies. Let $X, Y \in \mathcal{A}$. Have

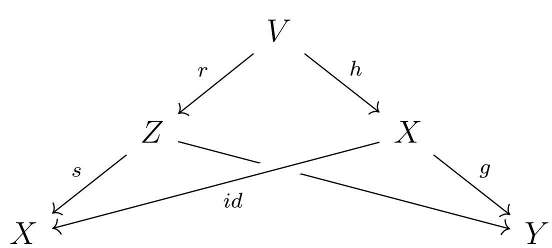


Want: a to be an isomorphism
Know: $b \circ a = id$ is clear
Not Obvious: $a \circ b = id$

Start with an elementary $\mathrm{Hom}(Q(X), Q(Y))$ given by f in $\mathcal{K}(\mathcal{A})$. Define $g : X \to Y$ by $g := H^0(f) \circ H^0(S)^{-1}$. Then $(a \circ b)(\varphi)$ comes from g:



Need: These roofs to be equivalent, ie.



Define $V^\bullet$ by

$$V^i = \begin{cases} Z^i & i < 0 \\ \mathrm{Ker}\, d_Z^0 & i = 0 \\ 0 & i > 0 \end{cases}$$

and d_V is induced by d_Z. Then $H^i(V) = H^i(Z)$ for $i \leq 0$ and $H^i(V) = 0$ for $i > 0$. $r : V \to Z$ is thus a quasi–isomorphism since $H^i(Z)$ only lives in $i = 0$.

Define $h : V \to X$ as follows: we know $H^0(V) = H^0(Z) = H^0(X) = X$ (the second last equality is via S.) Therefore $V^0 / \mathrm{Im}\, d_Z^{-1} = X$. Thus we have a map $V^0 \to X$ which defines h. □

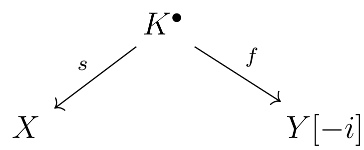
Let $X, Y \in \mathcal{A}$. Have $\operatorname{Ext}^i(X,Y) = \operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[i])$. Know that for any7 $X, Y, Z \in \operatorname{Ob}(\mathcal{A})$, the map

$$\operatorname{Ext}^i(X,Y) \times \operatorname{Ext}^j(Y,Z) \to \operatorname{Ext}^{i+j}(X,Z)$$

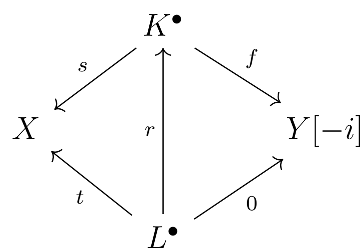
is associative, eg. $\oplus_i \operatorname{Ext}^i(X,X)$ is a graded associative algebra.

Theorem 2.6. *Let $X, Y \in \operatorname{Ob}(\mathcal{A})$. Then $\operatorname{Ext}^i(X,Y) = 0$ for all $i < 0$.*

Proof. Take $i > 0$, $\varphi : X \to Y[-i]$. A roof:



If we find $L^\bullet$ fitting into the diagram



where r, s, t are quasi–isomorphisms, then $\varphi = 0$. Define

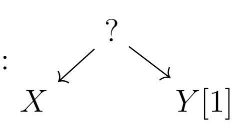
$$L^\bullet = \begin{cases} K^j & j < i-1 \\ \operatorname{Ker} d_k^{i-1} & j = i-1 \\ 0 & j \geq i \end{cases}.$$

Then $L^\bullet \to K^\bullet$ induces an isomorphism on H^j with $j \leq i-1$ and is 0 on H^j for $j \geq 1$. Hence, $L^\bullet \to K^\bullet$ is a quasi–isomorphism which we take to be r.

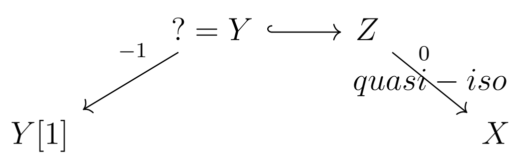
Since $L^\bullet \hookrightarrow K^\bullet$, can let t be the quasi–isomorphism $t = s|_L^\bullet$. Since $Y[-i]^j = 0$ for all $j < i$, we have no non–zero maps from $L^\bullet \to Y[-i]$ and therefore the diagram above must commute. □

2.5 Yoneda Extensions

<u>$i = 1$:</u> $$0 \to Y \to ? \to X \to 0$$

Want to define $K^\bullet \rightsquigarrow y(K^\bullet) \in \operatorname{Ext}^i(X,Y)$:  . For $i = 1$,

$$0 \to Y \to Z \to X \to 0 :$$



$\rightsquigarrow$ $X \to Y[1]$ in $\mathcal{D}(\mathcal{A})$.

Exercie 2.5. Gives an isomorphism between $\mathrm{Ext}^i(X, Y)$ and isomorphism classes of extensions.

Warning 2.1. If $K^\bullet, L^\bullet \in \mathcal{D}(\mathcal{A})$ and $\varphi : K^\bullet \to L^\bullet$ it might happen that $H^i(\varphi) = 0$ for all i but $\varphi \neq 0$. For general i,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y = K^{-i} & \longrightarrow & K^{-i-1} & \longrightarrow & \cdots & \longrightarrow & K^0 & \longrightarrow & 0 \\ & & \downarrow & & & & & & & & \\ & & Y[i] & & & & & & & & \end{array}$$

and $H^j(X) = 0$ for $j \neq 0$ while $H^0(K^\bullet) = X, K^\bullet \to X$.

Given such a $K^\bullet$, you can define $y(K^\bullet)$

Assume that $\mathcal{A}$ has enough projectives s othat for all $X \in \mathrm{Ob}(\mathcal{A})$ there is a projective resolution , ie. a sequence

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to 0$$

of projective objects with $H^0 = X$ and $H^i(P^\bullet) = 0$ for $i \neq 0$.

Lemma 2.5. *Let $P^\bullet$ and $Q^\bullet$ be projective resolutions of X and Y. Given $f : X \to Y$, there exists a unique $\tilde{f}$ fitting in the diagram*

$$\begin{array}{ccc} P^\bullet & \longrightarrow & X \\ {\scriptstyle \exists! \tilde{f}} \downarrow & & \downarrow {\scriptstyle f} \\ Q^\bullet & \longrightarrow & Y \end{array}$$

Proof.

$$\begin{array}{ccccccc} & & P^0 & \xrightarrow{\varepsilon_X} & X & \longrightarrow & 0 \\ & \swarrow & \downarrow {\scriptstyle \tilde{f}_0} & \searrow & \downarrow {\scriptstyle f} & & \\ Q^{-1} & \xrightarrow[d_Q^{-1}]{} & Q^0 & \xrightarrow[\varepsilon_Y]{} & Y & \longrightarrow & 0 \end{array} .$$

P^0 projective implies there exists $\tilde{f}_0 : P^0 \to Q^0$ in the above diagram continue building $\tilde{f}$ inductively.

Given another lift $\tilde{f}'^0$ lift at the zeroth level,

$$\begin{aligned} &\varepsilon_Y \circ (\tilde{f}^0 - \tilde{f}'^0) = 0 \\ \Longrightarrow\ &\tilde{f}^0 - \tilde{f}'^0 : P^0 \to \mathrm{Im}(d_Q^{-1}) \\ \Longrightarrow\ &\exists h^0 : P^0 \to Q^{-1}, \tilde{f}^0 - \tilde{f}'^0 = d_Q^{-1} \circ h^0 \end{aligned}$$

The homotopy h extends to the full complex inductively.

□

Theorem 2.7.

1. *Given $X, Y \in \mathrm{Ob}(\mathcal{A})$ and a projective resolution $P^\bullet \to X$,*

$$\mathrm{Ext}^i(X, Y) = H^i \, \mathrm{Hom}(P^\bullet, Y))$$

where $H^\bullet \, \mathrm{Hom}(P^\bullet, Y))$ is the homology of the sequence

$$\mathrm{Hom}(P^0, Y) \to \mathrm{Hom}(P^{-1}, Y) \to \cdots .$$

2. *If $P^\bullet \to X$ and $Q^\bullet \to Y$ are projective resolutions, then*

$$\mathrm{Ext}^i(X, Y) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P^\bullet, Q^\bullet[i])$$

3.

$$\mathcal{D}^{-1}(\mathcal{A}) \xleftarrow{\simeq} \textit{Projective complexes(bounded above)/homotopy}$$

(Similar statements also hold for injective objects)

Example 2.2. Let k be a field, $\mathcal{A}$ be $k[x]$–modules. $\mathrm{Ext}^i_k(k,k) = ?$

1. There is a projective resolution
$$0 \to k[x] \xrightarrow{\cdot x} k[x] \to 0$$
of k. Apply $\mathrm{Hom}_{k[x]}(\cdot, k)$ to get $k \xrightarrow{0} k$ (in degree 0 and 1). Hence
$$\mathrm{Ext}^0(k,k) = k$$
$$\mathrm{Ext}^1(k,k) = k$$
(use the sequence $0 \to k \to k[x]/(x^2) \to k \to 0$ for Ext^1.)

2. Have a projective resolution
$$P^\bullet : \quad \cdots \to k[x]/(x^2) \xrightarrow{\cdot x} k[x]/(x^2) \xrightarrow{\cdot x} k[x]/(x^2) \to k.$$
Apply $\mathrm{Hom}_{k[x]/(x^2)}(P^\bullet, k)$ to get $0 \xrightarrow{\cdot 0} k \xrightarrow{\cdot 0}$ (in degrees $0, 1, 2, \cdots$) Hence,
$$\mathrm{Ext}^i_{k[x]/(x^2)}(k,k) = k$$
for all $i \geq 0$.
In fact, as an algebra
$$\bigoplus_{i \geq 0} \mathrm{Ext}^i(k,k) \cong k[t]$$
("Koszul duality.") This is an example where $\mathcal{D}_{\mathcal{B}}(\mathcal{A}) \longleftarrow \mathcal{D}(\mathcal{B})$ is not equal. The map kills Ext^i for i large.

Definition 2.9. Let $X \in \mathrm{Ob}(\mathcal{A})$.

i) The **projective dimension** of X is
$$\mathrm{pdim} X = \max\{i : \exists Y \text{ with } \mathrm{Ext}^i(X,Y) \neq 0\}$$
(may be ∞.)

ii) The **injective dimension** of X is
$$\mathrm{idim} X = \max\{i : \exists Y \text{ with } \mathrm{Ext}^i(Y,X) \neq 0\}$$

iii) The **homological dimension** of $\mathcal{A}$ is
$$\begin{aligned}\mathrm{hdim}\mathcal{A} &= \max\{\mathrm{pdim} X : X \in \mathrm{Ob}\,\mathcal{A}\}\\ &= \max\{\mathrm{idim} X : X \in \mathrm{Ob}\,\mathcal{A}\}\end{aligned}$$

Lemma 2.6. *Assume that $\mathcal{A}$ has enough projectives. Then*

$$\text{pdim}X = \min\{i : \exists \textit{ projective resolution } 0 \to P^{-i} \to P^{-i+1} \to \cdots \to P^{-1} \to P^0 \textit{ of } X\}$$

(it is clear that $i \geq \text{pdim}X$.)

Example 2.3. $\text{pdim}X = 0 \iff X$ is projective. If $\text{pdim}X = 0$, then have an exact sequence $0 \to Y \to P \to X$ with P projective, $Y = \text{Ker}(P \to X)$. $\text{Ext}^1(X, Y) = 0$ implies that $P \simeq X \oplus Y$ so X is a direct summand of P. Thus X is projective.

Claim 2.1. Let $0 \to Y_1 \to Y_2 \to Y_3 \to 0$ be a short–exact sequence and let $X \in \text{Ob}(\mathcal{A})$. Then

$$\cdots \to \text{Ext}^i(X, Y_1) \to \text{Ext}^i(X, Y_2) \to \text{Ext}^i(X, Y_3) \to \text{Ext}^{i+1}(X, Y_1) \to \cdots$$

is long exact. Similarly, for $X, Y \in \mathcal{D}(\mathcal{A})$, $\text{Ext}^i(X, Y) = \text{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[i])$.

More generally, given $X \in \mathcal{D}(\mathcal{A})$ and a distinguisheed triangle $Y_1 \to Y_2 \to Y_3 \to Y_1[1]$, there is a long exact sequence of Ext's. (Similar statements hold for $X_1 \to X_2 \to X_3 \to X_1[1]$.)

We will assume this claim without proof.

Proof. (of Lemma (2.6).) Use induction on $\text{pdim}(X)$. Suppose $\text{pdim}X = n$. Want a projective reolution of length n. Since $\mathcal{A}$ has enough projectives we have a sequence

$$0 \to Y \to P \to X \to 0$$

where P is projective.

Claim 2.2. $\text{pdim}Y \leq n - 1$

By the previous claim, get an exact sequence

$$\text{Ext}^i(P, Z) \to \text{Ext}^i(Y, Z) \to \text{Ext}^{i+1}(X, Z)$$

so by induction Y has a projective resolution of length less than or equal to $n - 1$. Composing with $Y \hookrightarrow P$, we get the projective resolution

$$\to P^{-(n-1)} \to \cdots \to P^{-2} \to P^{-1} \hookrightarrow P \to X \to 0$$

of length less than or equal to n. □

Let R be a ring.

Definition 2.10. $\text{hdim}R := \text{hdim}(R\text{–Mod})$

Claim 2.3. $\text{hdim}R[x] = \text{hdim}R + 1$

Proof. Next Time. □

Corollary 2.5. $\text{hdim}k[x_1, \cdots, x_n] = n$

Theorem 2.8 (Serre)**.** *Let R be a commutative Noetherian ring with unit.* $\text{hdim}R < \infty \iff$ *R is regular and in this case it is the Krull-dimension of R.*

Let $P^\bullet$ be a projective resolution of X.

1. $\text{Ext}^i(X, Y) = H^i(\text{Hom}(P, Y))$.

2. P,Q projective resolutions of X, Y. Then

$$\mathrm{Ext}^i(X,Y) = \mathrm{Hom}(P, Q[i])/\text{homotopy}.$$

Look at $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P^\bullet, Y[i])$.

$$\begin{array}{ccccccc}
P^{-i-1} & \longrightarrow & P^{-1} & \longrightarrow & P^{-i+1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

$\varphi : P^{-i} \to Y$ defines a complex map if and only if $\varphi \circ d_P^{i-1} = 0$ $(\star)$. In $\mathcal{K}(\mathcal{A})$, $\varphi \mapsto \varphi + h \cdot d^{-i}$ $(\star\star)$.

$$(\star) \iff \varphi \text{ is an } i\text{–cocycle in } \mathrm{Hom}(P^\bullet, Y)$$
$$(\star\star) \iff \text{allowed to change by coboundaries.}$$

Thus, proving

$$\mathrm{Ext}^i(X,Y) = H^i(\mathrm{Hom}(P^\bullet, Y))$$

is equivalent to proving

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[i]) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P^\bullet, Y[i]).$$

Given $K^\bullet \to P^\bullet$ a quasi–isomorphism we can find a subcomplex $L^\bullet \hookrightarrow K^\bullet$ where $L^\bullet \to K^\bullet$ is a quasi–isomorphism. Then there exists $P^\bullet \to L^\bullet$ inverse to s in the diagram

$$\begin{array}{ccc}
 & K^\bullet & \\
{\scriptstyle s}\swarrow & & \searrow{\scriptstyle f} \\
P^\bullet & & Y[i]
\end{array}.$$

Hence we can assume $K^i = 0$. Finish next time.

Lecture 5 **[15.03.2016]**

Let $\mathcal{A}$ and abelian category and let $\mathcal{D}(\mathcal{A})$ be the derived category. Given $X, Y \in \mathrm{Ob}\,\mathcal{A}$, we defined $\mathrm{Ext}^i(X,Y) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(x, y[I])$. Let $P^\bullet$ be a projective resolution of X. Then

$$\mathrm{Ext}^i(X,Y) = H^i(\overline{\mathrm{Hom}}(P^0, Y).$$

Proof. For $X_1, X_2 \in \mathrm{Ob}\,\mathcal{A}$ with corresponding projective resolution P_1, P_2, a morphism $\varphi : X_1 \to X_2$ is equivalent to a morphism $P_1^0 \to P_2^0$ unique up to homotopy. In fact the same is true assuming only that P_1^0 is projective. Need a quasi-isomorphism s fitting into the roof

$$\begin{array}{ccc}
 & K^\bullet & \\
{\scriptstyle s}\swarrow & & \searrow{\scriptstyle f} \\
P^\bullet & & Y[i]
\end{array}$$

Can assume that $K^i = 0$ for $i > 0$. Replace $K^\bullet$ by $\tilde{K}^\bullet \subset K^\bullet$ with $\tilde{K}^i = 0$ for any $i > 0$ and $\tilde{K}^i = K^i$ for $i < 0$ and $\tilde{K}^0 = \mathrm{Ker}(K^0 \to L^1)$.

Then both $K^\bullet$ and $P^\bullet$ are resolutions of X so their exists a uniqyue up to homotopy map $P^\bullet \to K^\bullet$ which induces and isomorphism on H^0 which is inverse to $H^0(s)$. Thus we get $P^\bullet \to Y[i]$ a map of complexes which is well–defined up to homotopy. $\square$

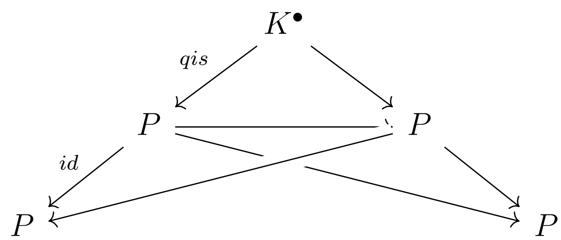
Conclusion

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P^\bullet, Y[i]) \to \overline{\mathrm{Hom}}_{\mathcal{D}(\mathcal{A})}(X, Y[i])$$

is surjective.

$$\begin{array}{ccccc} P^{-i-1} & \xrightarrow{d} & P^{-i} & \xrightarrow{d} & P^{-i+1} \\ \downarrow & & \downarrow\varphi & & \downarrow \\ 0 & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

$\varphi \circ d = 0 \iff \varphi$ is a cocycle in $\overline{\mathrm{Hom}}(P^\bullet, Y)$ and $h : P^{-i+1} \to Y$ is 0 $\iff$ h is a cocylce in $\mathrm{Hom}(P^{-i}, Y)$ (here $\varphi \mapsto \varphi + hd$. So homotopies are just adding a coboundary in $\overline{\mathrm{Hom}}(P^\bullet, Y)$.



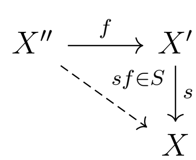
Theorem 2.9. *Let $\mathcal{I}$ denote the full subcategory of injective objects in $\mathcal{A}$ and let $\mathcal{K}^+(\mathcal{A})$ be the homotopy category of injective complexes bounded below. Then*

1. *$\mathcal{K}^+(\mathcal{I})$ is a full subcategory of $\mathcal{D}^+(\mathcal{A})$*
2. *If $\mathcal{A}$ has enough injectives then $\mathcal{K}^+(\mathcal{A}) \cong \mathcal{D}^+(\mathcal{A})$.*

Proposition 2.3. *Let $\mathcal{C}$ be a category and $\mathcal{S}$ a localizing class of morphisms. Let $\mathcal{B} \subset \mathcal{C}$ be a full subcategory and $\mathcal{S}_\mathcal{B} = S \cap B$ so we have a functor $F : \mathcal{B}[S_\mathcal{B}^{-1}] \to \mathcal{C}[\mathcal{S}^{-1}]$. Assume that (a) or either (b_1) or (b_2) are satisfied. Then F is fully faithful. Here*

(a) $\mathcal{S}_\mathcal{B}$ is a localizing class in $\mathcal{B}$.

(b1) For all $s : X' \to X$, $s \in \mathcal{S}$, $X \in \mathrm{Ob}\,\mathcal{B}$ there exists $f : X'' \to X'$, $X'' \in \mathrm{Ob}\,\mathcal{B}$, $sf \in S$ such that



(b_2) Same as (b_1) with all arrows reversed

Proof. Exercise. □

Remark 2.3. If we know (1) then (2) just means any (bounded below) complex is quasi–isomorphic to an injective complex.

Sketch of proof of (1). Let $\mathcal{C} = \mathcal{K}^+(\mathcal{A})$, $\mathcal{B} = \mathcal{K}^+(\mathcal{I})$ and let $\mathcal{S}$ be quasi–isomorphisms. (a) Is proven in the same way as for $\mathcal{K}^+(\mathcal{A})$ itself and (b2) is satisfied.

Lemma 2.7. *Let $I^\bullet \to K^\bullet$ be a quasi–isomorphism in $\mathcal{C}^+(\mathcal{A})$ with $I^\bullet$ injective. Then there exists $t : K^bullet \to I^\bullet$ such that $t \circ s$ is homotopic to the identity. (This implies that any quasi–isomorphism between injective complexes is invertible up to homotopy.)*

Proof. Have a sequence

$$I \xrightarrow{s} K \to \text{Cone}(s) \to I^\bullet[1]$$

and if s is a quasi–isomorphism then Cone(s) as acyclic.

Claim 2.4. Any map from an acyclic complex $C^\bullet$ to an injective complex $I^\bullet$ (in $\mathcal{C}^+(\mathcal{A})$ is homotopic to 0.

Recall that Cone$(s) = K \oplus I[1]$. The idea is that $h_{K^\bullet} : K^\bullet \to I^\bullet$ is a map of complexes which satisfies the conditions of the lemma (check this as an exercise.)

Assume that $C^i, I^i = 0$ for $i < 0$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \cdots \\ & & \downarrow & \swarrow & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

The dashed arrow in the above diagram exists since $C^0 \to C^1$ is injective. Continue constructing the homotopy by induection. □

Lemma 2.8. *For any $K^\bullet \in \mathcal{C}^+(\mathcal{A})$ there exists $f : K^\bullet \to I^\bullet$ with $I^\bullet$ injective and f a quasi–isomorphism (we require that $\mathcal{A}$ has enough injectives as well.)*

Proof. Exercise □

The result follows if we can prove these two lemmas.

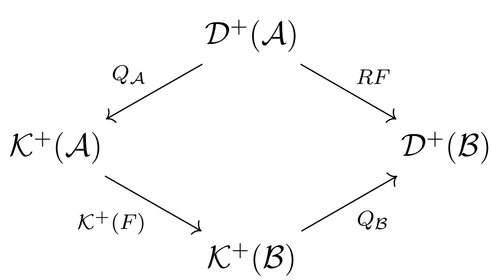
Remark 2.4. We really need to be in $\mathcal{C}^+(\mathcal{A})$ since we need to start the inductive arguments somewhere.

□

2.6 Derived Functors

Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor.

Definition 2.11. The **right derived functor** of F is an exact functor (ie. exact triangles map to exact triangles) $RF : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ and a morphism $\varepsilon_F : Q_{\mathcal{B}} \circ \mathcal{K}^+(F) \to RF \circ Q_{\mathcal{A}}$ such that



is universal in the sanse that for any exact functor $G : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ and any $\varepsilon : Q_{\mathcal{B}} \cdot \mathcal{K}^+(F) \to G \circ Q_{\mathcal{A}}$ there exists a unique $\eta : RF \to G$ which makes the diagram

$$RF \circ Q_{\mathcal{A}} \xrightarrow[\eta \circ Q_{\mathcal{A}}]{} G \circ Q_{\mathcal{A}}$$

Remark 2.5. If RF exists it is defined uniquely by the universal condition.

Claim 2.5. If RF exists define $R^iF : \mathcal{A} \to \mathcal{B}$ by $R^iF(X) = H^i(RF(X))$. Then $R^iF = 0$ for $i < 0$ and $R^0F = F$. Moreover, given an exact sequence

$$0 \to X \to Y \to Z \to 0$$

we get an exact sequence

$$0 \to F(X) \to F(Y) \to F(Z) \to R^1F(X) \to R^1(Y) \to R^1F(Z) \to R^2F(X) \to \cdots$$

Example 2.4. $\mathrm{Ext}^*(X, Y) = R\ell_X(Y) = Rr_Y(X)$

Example 2.5. Let R be an associative ring with unit, let M be a right R-module and let N be a left R–module. Then we can form the tensor product $M \otimes_R N$ and define the functor F_M : left modules $\to$ Abelian Groups by $F_M(N) = M \otimes_R N$. Then F_M has a left derived functor

$$LF_M(N) =: \mathrm{Tor}(M, N)$$

and we define

$$\mathrm{Tor}_i(M, N) := H^{-i}(LF_M(N))$$

Questions

1. When are we guaranteed that RF (or LF) exists?
2. How to compute it?

Definition 2.12. Let F be a functor. A class of objects $\mathcal{R} \subset \mathrm{Ob}(\mathcal{A})$ is called **Adapted to** F if

1. F maps acyclic complexes (in $\mathcal{C}^+(\mathcal{R})$) to acyclic complexes.
2. Any $X \in \mathrm{Ob}(\mathcal{A})$ embeds into $Y \in \mathcal{R}$.

Usually we assume $\mathcal{R}$ is stable under direct sums also.

Lemma 2.9. *Assume that $\mathcal{A}$ has enough injectives. Then $\mathcal{I}$ (the class of injective objects) is adapted to any F (left exact!).*

Theorem 2.10.

0. *If F has an adapted class then RF exists.*
1. *$\mathcal{D}^+(\mathcal{A}) \cong \mathcal{K}^+_{\mathcal{R}}[S_{\mathcal{R}}^{-1}]$ where $S_{\mathcal{R}}$ are the quasi–isomorphisms in $\mathcal{K}^+_{\mathcal{R}}$, the homotopy category of complexes in $\mathcal{R}$.*
2. *If $K^\bullet \in \mathcal{C}^+(\mathcal{R})$ then*
$$RF(Q_{\mathcal{A}}(K^\bullet)) = Q_{\mathcal{B}}(\mathcal{K}^+F(K^\bullet))$$

Example 2.6.

Definition 2.13. A left R–module N is called **flat** if the functor $M \mapsto M \otimes N$ is exact.

Lemma 2.10. *If $K^\bullet$ is a complex (bounded above) of flat,acyclic R–modules then $M \otimes K^\bullet$ is also acyclic.*

Projective modules are flat but the opposite is not true ($\mathbb{Q}$ is a flat $\mathbb{Z}$ module but not a projective one.) Flat modules form an adapted class to F_M.

Think of $\mathbb{Q}/\mathbb{Z}$ as the complex $\mathbb{Z} \to \mathbb{Q}$ where $\mathbb{Z}$ is in degree -1. Tensoring this sequence with $\mathbb{Q}/\mathbb{Z}$ gives $\mathbb{Q}/\mathbb{Z} \to 0$. Hence,

$$\mathrm{Tor}_0(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = 0$$
$$\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$$

If $I^\bullet$ is an acyclic injective complex which is bounded below, then $F(I^\bullet)$ is also acyclic. $I^\bullet \xrightarrow{id} I^\bullet$ is a quasi–isomorphism implies that id is homotopic to 0 so there exists $h : I \to I[-1]$ with $id = dh + hd$. Then

$$id_{F(I^\bullet)} = F(d)F(h) + F(h)F(d)$$

so $id_{F(I)}$ homotopic to 0 implies acyclicity.

Proof. (Of Part (1) of the theorem) To construct RF,

$$\mathcal{D}^+(\mathcal{A}) \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathcal{K}^+_{\mathcal{R}}[\mathcal{S}^{-1}_{\mathcal{R}}]$$

We have $\overline{F} : \mathcal{K}^+_{\mathcal{R}}[\mathcal{S}^{-1}_{\mathcal{R}}] \to \mathcal{D}^+(\mathcal{B})$. $K^\bullet \xrightarrow{s} L^\bullet$ with s a quasi–isomorphism and $K^\bullet, L^\bullet \in \mathcal{K}^+_{\mathcal{R}}$ implies that $F(s)$ is a quasi–isomorphism and s is a quasi–isomorphism $\iff$ Cone(s) is acyclic. We have

$$F(K^\bullet) \xrightarrow{F(s)} F(L^\bullet) \to \mathrm{Cone}(F(s)) = F(\mathrm{Cone}(s))$$

and $\mathrm{Cone}(s) = L \oplus K[1]$ also in $\mathcal{K}^+_{\mathcal{R}}$ implies $F(\mathrm{Cone}(s))$ is also acyclic which is equivalent to $F(s)$ being a quasi–isomorphism.

Get a functor $\mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ which is $F \circ \phi$. This functor is exact and universal (Exercise: Prove universality) with ε_F which is given by the map $Q_{\mathcal{B}}(F(K^\bullet)) \to Q_{\mathcal{B}}(F(L^\bullet))$. □

Let F be a left exact functor for which there exists an adapted class. $R^iF = 0$ for $i < 0$. For any $X \in \mathcal{A}$, X has a resolution $K^\bullet$ such that $K^i = 0$ for $i < 0$, $K^i \in \mathcal{R}$ and

$$H^i(K^\bullet) = \begin{cases} X & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

There exists $\alpha_0 : X \hookrightarrow K^0 \in \mathcal{R}$ so $\mathrm{Coker}(\alpha_0) \hookrightarrow K^1$. Get

$$K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} K^2$$

There exists $\mathrm{Coker}(d^0) \hookrightarrow K^2 \in \mathcal{R}$ etc.

Hence, $RF(X)$ is represented by $F(K^\bullet)$ which lives in degrees ≥ 0. $R^0F(X) = \mathrm{Ker}(F(K^0) \to F(K^1))$ and F is leeft exact imply that

$$\mathrm{Ker}(F(K^0) \to F(K^1)) = F(\mathrm{Ker}(K^0 \to K^1)) = F(X).$$

Let $0 \to X \to Y \to Z \to 0$ be a short exacct sequence. Then get

$$RF(X) \to RF(Y) \to RF(Z) \to RF(X)[1]$$

A distinguished triangle yields a long exact sequence of cohomology so get a long exact sequence

$$\cdots \to R^iF(X) \to R^iF(Y) \to R^iF(Z) \to R^{i+1}F(x)[1] \to \cdots$$

Remark 2.6. $\{R^iF\}_{i\geq 0}$ form a δ–functor if $R^0F = F$ and the sequence

$$\cdots \to R^iF(X) \to R^iF(Y) \to R^iF(Z) \xrightarrow{\delta_i} R^{i+1}F(x)[1] \to \cdots$$

is exact. Then you can define $\{R^iF\}_{i\geq 0}$ as a universal δ–functor with $R^0F = F$.

Corollary 2.6. $\operatorname{Ext}^i(X,Y) = R^i\ell_X(Y)$ *if* $\mathcal{A}$ *has enough injectives and* $\operatorname{Ext}^i(X,Y) = R^ir_Y(X)$ *if* $\mathcal{A}$ *has enough projectives.*

Remark 2.7. Assume that $\mathcal{A}$ is k–linear for some commutative ring k. Think of $\ell_X : \mathcal{A} \to k - mod$. Then $R\ell_X(Y) \in \mathcal{D}^+(k - mod)$. In the case where there are not enough injectives, usually define $R^i\ell_X(Y) = R\,\overline{\operatorname{Hom}}(X,Y) \in \mathcal{D}^+(\mathcal{A}b)$ (or $\mathcal{D}^+(k - mod)$ if $\mathcal{A}$ is k–linear.

Lecture 6 **[22.03.2016]**

2.7 Derived Functors of Composition

Suppose we have functors

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

between abelian categories where F and G are exact functors. Then $G \circ F$ is also left exact.

Theorem 2.11. *Assume there exists* $\mathcal{R}_\mathcal{A} \subset \operatorname{Ob}\mathcal{A}$ *which is an adapted class to* F *and there exists* $\mathcal{R}_\mathcal{B} \subset \operatorname{Ob}\mathcal{B}$ *which is an adapted class to* G*. Assume also that* $F(\mathcal{R}_\mathcal{A}) \subset \mathcal{R}_\mathcal{B}$*. Then* $R(G \circ F)$ *exists and there is a natural isomorpism* $RG \circ RF \cong R(G \circ F)$ *of functors* $\mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{C})$*.*

Remark 2.8. Can also consider $R^pF \circ R^qG$ but the theorem says nothing about the relation between these and $R^i(G \circ F)$.

Example 2.7 (Group Cohomology). Let Γ be a finite group and let $\mathcal{A}$ be the category of Γ-modules (so these are modules over $\mathbb{Z}\Gamma$.) Given $M \in \Gamma$-mod we can define the cohomology in two ways:

$$H^i(\Gamma, M) = \operatorname{Ext}^i_{\Gamma - mod}(\mathbb{Z}, M) = R^iF(M)$$

where $F : \Gamma - mod \to \mathcal{A}b$ is defined by $F(M) = M^\Gamma = \operatorname{Hom}_\Gamma(\mathbb{Z}, M)$.

Let $1 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 1$ be a short exact sequence. Let $\mathcal{A}$ be Γ_2-modules, $\mathcal{B}$ be Γ_3-modules and $\mathcal{C} = \mathcal{A}b$ so that $F(M) = M^{\Gamma_1}$ is a functor $\mathcal{A} \to \mathcal{B}$ and $G(M) = M^{\Gamma_3}$ is a functor $\mathcal{B} \to \mathcal{C}$.

Exercie 2.6. The functors above satisfy the conditions of the theorem.

If M is a Γ_2-module we will show there is a relationship $H^q(\Gamma_3, H^p(\Gamma_1, M))$ and $H^i(\Gamma_2, M)$. In genereal we will later prove a relation ship between higher derived functors and composition which strengthens the following.

Weak Statement

$R^i(G \circ F)$ is a subquotient of $\bigoplus_{p+q=i} R^qG \circ R^pF$.

To compute the group cohomology, we need to compute an injective resolution of the Γ-module $\mathbb{Z}$. Can define the resolution

$$\cdots \to \mathbb{Z}\Gamma \otimes \mathbb{Z}\Gamma \otimes \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma \otimes \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma \to \mathbb{Z}$$

where the map $\mathbb{Z}\Gamma \to \mathbb{Z}$ is defined by $\{\sum_{\gamma\in\Gamma} a_\gamma \cdot \gamma\} \to \sum a_\gamma$.

By Lemma 2.11, $\mathbb{Z} \otimes \mathbb{Z}$ is a free module and it has basis given by pairs (γ_1, γ_2). Define the map $\mathbb{Z}\Gamma \otimes \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma$ by $(\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2 - \gamma_1$.

Similarly, $\mathbb{Z}\Gamma \otimes \mathbb{Z}\Gamma \otimes \mathbb{Z}\Gamma$ has a basis given by triples $(\gamma_1, \gamma_2, \gamma_3)$ and the map $\mathbb{Z}\Gamma \otimes \mathbb{Z}\Gamma \otimes \mathbb{Z}\Gamma$ is defined by

$$(\gamma_1, \gamma_2, \gamma_3) \mapsto (\gamma_1\gamma_2, \gamma_3) - (\gamma_1, \gamma_2\gamma_3) + (\gamma_1, \gamma_2)$$

In general, the map $(\mathbb{Z}\Gamma)^{\otimes^{n+1}} \to (\mathbb{Z}\Gamma)^{\otimes^n}$ is defined by

$$(\gamma_0, \cdots, \gamma_n \xrightarrow{d^n}) \sum_i (-1)^{i+1}(\gamma_0, \gamma_1, \cdots, \gamma_i\gamma_{i+1}, \gamma_{i+1}, \cdots, \gamma_n) + (-1)^n(\gamma_0, \cdots, \gamma_{n-1}).$$

Exercie 2.7. $d^{n-1} \circ d^n = 0$ so this is a free resolution of $\mathbb{Z}$.

This resolution is called the **bar complex** and is denoted by $B^\bullet$.

Lemma 2.11. *Let M be any Γ-module. Then $\mathbb{Z} \otimes M$ with the diagonal action is a free module over Γ.*

Let M be a Γ-module. To compute $H^n(\Gamma, M)$ we have $\operatorname{Hom}(B^{-n}, M)$ which is all maps $\Gamma^n \to M$. Note that $1 \otimes \gamma_1 \otimes \cdots \otimes \gamma_{n+1} \in B^{-n-1}$ gets mapped to $\sum_{i=1}^n (-1)^i(1, \gamma_1, \cdots, \gamma_i\gamma_{i+1}, \cdots) + (-1)^{n+1}(1, \gamma_1, \cdots, \gamma_n) + (\gamma_1 \cdots, \gamma_{n+1})$.

So, given $f \in \operatorname{Hom}(B^{-n}, M)$ we get

$$df(\gamma_1, \cdots, \gamma_{n+1}) = \gamma_1 f(\gamma_2 \cdots, \gamma_{n+1}) + \sum_{i=1}^n (-1)^i f(\gamma_1, \cdots, \gamma_i\gamma_{i+1}, \cdots, \gamma_{n+1} + (-1)^n f(\gamma_1, \cdots, \gamma_n)$$

In low degrees, we get

$$M \xrightarrow{d_0} \{f : \Gamma \to M\} \xrightarrow{d_1} \{\Gamma \times \Gamma \to M\}$$

where $m \mapsto \gamma(m) - m$ and for $f : \Gamma \to M$ $d_1 f = g$ where $g(\gamma_1, \gamma_2) = \gamma_1 f(\gamma(2) - f(\gamma_1\gamma_2) + f(\gamma_1)$.

So 1–cocycles are functions $f : \Gamma \to M$ which satisfy $f(\gamma_1\gamma_2) = \gamma_1 f(\gamma_2)) + f(\gamma_1)$, called a **skew-homomorphism**.

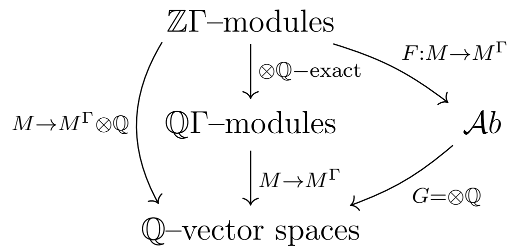
Remark 2.9. Assume that Γ acts trivially on M. Then we get $f(\gamma_1\gamma_2) = f(\gamma_1) + f(\gamma_2)$ so $f : \Gamma \to M$ is a homomorphism of groups.

Conclusion: If Γ acts trivially on M then $H^1(\Gamma, M) = \overline{\operatorname{Hom}}(\Gamma, M)$.

If Γ is a finite group then the category of $\mathbb{Q}\Gamma$–modules is semi–simple.

Corollary 2.7. *If Γ is finite then $H^i(\Gamma, M)$ $(i > 0)$ is a torsion group for any M. More precisely, $H^i(\Gamma, M)$ is killed by $\#\Gamma$.*

Proof.



Since Γ is finite the composition $M \to (M \otimes \mathbb{Q})^\Gamma$. $R(G \circ F) \cong RG \circ RF$ and $R^i(G \circ F)(M) = H^i(\Gamma, M) \otimes \mathbb{Q}) = 0$ for $i > 0$ since $G \circ F$ is exact.

The same thing works when we replace $\mathbb{Q}$ by $\mathbb{Z}[(\#\Gamma)^{-1}]$. □

Example 2.8 (Hochschild Cohomology). Let A be any associative ring (which is an algebra over a field k) with 1 (so all tensor products will be over k.) Let $\mathcal{A}$ be the category of A–bimodules (ie. modules over $A \otimes A^{op}$. A is naturally a bimodule over itself. If M is any other bimodule, define

$$HH^i(M) = \mathrm{Ext}^i_{\mathcal{A}}(A, M).$$

$\mathrm{Hom}(A, M) = \{m \in M \mid am = ma \text{ for all } a \in A\}$ so if $M = A$ for example then $\overline{\mathrm{Hom}}_{\mathcal{A}}(A, A) = Z(A)$ is the centre. To compute this cohomology group it is enough to find a resolution of A by free bimodules.

$$\cdots \to A \otimes A^2 \otimes A^{op} \to A \otimes A \otimes A^{op} \to A \otimes A \xrightarrow{d} A$$

The map $A \otimes A^{op} \to A$ is defined by $a \otimes b \mapsto ab$. In general the map $A \otimes A^{\otimes^n} \otimes A^{op}$ is defined by

$$d(a_0 \otimes a_1 \cdots \otimes a_n \otimes a_{n+1}) = \sum (-1)^i (a_0, a_1, \cdots, a_i a_{i+1}, a_{i+1}, \cdots, a_{n+1})$$

Lemma 2.12. *This is a resolution of A by free bimodules.*

$HH^n(M)$ is computed by the complex which in degree n has $\overline{\mathrm{Hom}}_k(A^{\otimes n}, M)$. In low degrees get

$$\begin{array}{ccccc}
0 & & 1 & & 2 \\
M & \longrightarrow & \{f : A \to M\} & \longrightarrow & g : A^{\otimes 2} \to M \\
m & \longmapsto & f(a) = am - ma, f : A \to M & \longmapsto & g(a_1, a_2) = a_1 f(a_2) - f(a_1 a_2) + f(a_1) a_2
\end{array}$$

So, 1–cocycles are functions $f : A \to M$ which satisfy $f(a_1 a_2) = a_1 f(a_2) + f(a_1) a_2$, ie. they are derivations. So these are related to vector fields in certain contexts.

There is a dual theory of Hochschild homology, which is the derived functor of the functor from Bimodules to k-Vect defined by $M \mapsto M/\mathrm{Span}(am - ma, a \in A, m \in M)$. Then the first homology is related to differential forms in certain contexts.

2.8 Spectral Sequences

Let $\mathcal{A}$ be an abelian category. A spectral sequence is given by $E_r^{p,q} \in \operatorname{Ob}\mathcal{A}$, $E^n \in \operatorname{Ob}\mathcal{A}$ plus

(a) For all r differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q+1}$ $(p, q \in \mathbb{Z}, r \geq 1)$ such that $d_r^2 = 0$

(b) $H^{p,q}(E_r^*) \cong E_{r+1}^{p,q}$

(c) For all p, q there exists $r_0 \geq 1$ such that for all $r \geq 0$, $d_r^{p,q} \circ d_r^{p+r,q+1} = 0$. This ensures that $\lim_{r\to\infty} E_r^{p,q}$ exists. Call this limit $E_\infty^{p,q}$.

(d) There exists a regular filtration $F^P E^n \supset F^{p+1}E^n \supset \cdots$ on E^n such that $E_\infty^{p,q} \cong F^pE^{p+q}/F^{p+1}E^{p+q}$.

The associated graded $grE^n = \oplus_p F^pE^n/F^{p+1}E^n$ is a subquotient of $\oplus_{p+q=n}E_1^{p,q}$.

Spectral Sequence of a Filtration

This generally appears in the following way. Let $K^\bullet$ be a complex in $\mathcal{C}(\mathcal{A})$ and assume we have a filtration on $K^\bullet$, ie. for each n we have a filtration $\cdots \supset F^pK^n \subset F^{p+1}K^n \supset \cdots$ such that $d(F^pK^n) \subset F^pK^{n+1}$.

Example 2.9 (Canonical Filtration).

$$(F^pK^\bullet)^n = \begin{cases} K^n & n \leq -p \\ \operatorname{Ker} d^{-p} & n = -p \\ 0 & n > -p \end{cases}$$

This has cohomology

$$H^n(F^pK^\bullet) = \begin{cases} H^n(K) & n \leq p \\ 0 & \text{otherwise} \end{cases}$$

Example 2.10 (Stupid Filtration).

$$\tilde{F}^p(K) = \begin{cases} 0 & n < p \\ K^n & n \geq p \end{cases}$$

This has cohomology

$$H^n(\tilde{F}^pK^\bullet) = \begin{cases} 0 & n < p \\ H^n(K) & n > p \\ \operatorname{Ker} d^p & n = p \end{cases}$$

Claim 2.6. Given a filtered complex $K^\bullet$ (under the finiteness condition that for a given n, $F^pK^n = 0$ for $p >> 0$ although this can often be relaxed) we can construct a spectral sequence with $E_1 = gr_F K^\bullet$. Ie. $E_1^{p,q} = F^pK^{p+q}/F^{p+1}K^{p+q}$, $E^n = H^n(K^\bullet)$

Theorem 2.12. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories with enough injectives and suppose we have functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ with F, G left exact and $F(I_\mathcal{A}) \subset \mathcal{R}_B$ ($\mathcal{R}_B$ is an adapted class to G. Then for all $X \in \operatorname{Ob}\mathcal{A}$ there exists a spectral sequence $E_r^{p,q}$ such that*

$$E_1^{p,q} = R^pG(R^qF(X))$$
$$E^n = R^n(G \circ F)(X).$$

E^n has a filtration such that $grE^n = \oplus_{p+q=n}R^pG(R^qF(X))$.

Corollary 2.8. *Assume that there exists i, j with*

$$R^q F = 0 \text{ for } q > i$$
$$R^p G = 0 \text{ for } p > j.$$

Then $R^n(G \circ F) = 0$ for $n > i + j$.

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor and let $K^\bullet \in \mathcal{D}^+(\mathcal{A})$. Then we can consider either $H^n(RF(K^\bullet))$ or $H^p(R^q F(K^\bullet))$.

Claim 2.7. Assume $\mathcal{A}$ has enough injectives. There exists a spectral sequence with

$$E_1^{p,q} = H^p(R^q(F(K^\bullet)))$$
$$E^n = H^n(RF(K^\bullet))$$

Moreover, in this case the filtration on E^n is trivial.

In general, $H^n(F(K^\bullet))$ is a subquotient of $\oplus_{p+q=n} R^p F(H^q(K^\bullet))$.

Chapter 3

Sheaves

3.1 Sheaf Cohomology

Let X be a topological space.

Definition 3.1. A **presheaf** $\mathcal{F}$ (of sets) on X consists of a set $\mathcal{F}(U)$ for overey open subset of X and for every inclusion $I \subset V$ a restriction map $r_{UV} : \mathcal{F}(V) \to \mathcal{F}(U)$ such that for any $U \subset V \subset W$ $r_{UV} \circ r_{VW} = r_{UW}$.

Remark 3.1. Can define sheaves in any category by replacing the set $\mathcal{F}(U)$ by an abject in the category and the restriction maps by morphisms in the category.

Definition 3.2. A presheaf $\mathcal{F}$ is a **Sheaf** if for all open covers $U = \bigcup_{i \in I} U_i$ the map

$$\mathcal{F}(U) \to \{s_i \in \mathcal{F}(U_i) : s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}\}$$

is an isomorphism.

There is functor $Sh : PSh(X) \to Sh(X)$ which is a left adjoint of the forgetful functor $F : Sh(X) \to PSh(X)$ so that

$$\overline{\text{Hom}}_{Psh}(\mathfrak{g}, F(\mathcal{F}) \cong \text{Hom}_{Sh}(Sh(\mathfrak{g}), \mathcal{F})$$

$Sh(\mathfrak{g})$ can be explicitly constructed as a direct limit over all open coverings $U = \bigcup_{i \in I} U_i$ of $\{s_i \in \mathcal{F}(U_i) : s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}\}$

Sh and PSh are abelian categories.

Warning 3.1. In Psh Ker and Coker are the obvious ones but given $CF(U) \xrightarrow{\varphi_U} \mathfrak{g}(U)$. $\text{Ker}\,\varphi_U$ and $\text{Coker}\,\varphi_U$ are presehaves but $\{\text{Coker}\,\varphi_U\}$ does not form a sheaf. In Sh the Cokernel is defined as $Sh(\text{Coker}_{Psh})$.

Example 3.1. Let A be an abelian group. The constant sheaf A_X on X defined by A is given by $A_X(U)" = "A$. Assigning A to every open set only gives a presheaf though so we define $(A_X)_{Sh} = Sh((A_X)_P Sh)$.

There is a functor $\Gamma : Sh(X) \to \mathcal{A}b$ defined by $\mathcal{F} \mapsto \mathcal{F}(X) = \overline{\text{Hom}}(\mathbb{Z}_X, \mathcal{F})$ called the **Global Sections Functor**. It is a left exact functor.

Lemma 3.1. *Sh has enough injectives so all left exact functors have right derived functors.*

Definition 3.3. $H^i(X, \mathcal{F}) = R^i\Gamma(\mathcal{F})$. In particular, $H^i(X, A) = R^i\Gamma(A_X)$.

Example 3.2 (Hypercohomology). Let $\mathcal{F}$ be a complex of sheaves in $\mathcal{D}^+(Sh)$). Then we can define $H^i(X, \mathcal{F}) = H^i(R\Gamma(\mathcal{F})$. There exists a spectral sequence $E_r^{p,q}$ with

$$E_1^{p,q} = H^p(X, H^q(\mathcal{F}))$$
$$E_n = H^n(X, \mathcal{F})$$

Lecture 7 [29.03.2016]

We will compare sheaf cohomology to the singular and de Rham cohomoloy theories. Let X be a topological space and let $\mathcal{SA}b(X)$ be the category of sheaves of abelian groups.

Lemma 3.2. *$\mathcal{SA}b(X)$ has enough injectives.*

There is a functor $\Gamma : \mathcal{SA}b(X) \to \mathcal{A}b$ defined by $\Gamma(\mathcal{F}) = \mathcal{F}(X)$ for a sheaf $\mathcal{F}$. The cohomology is then defined to be

$$H^i(X, \mathcal{F}) = R^i(\mathcal{F}).$$

For A an abelian group let A_X be the constant sheaf. Then

$$H^i_{Sh}(X, A) = H^i(X, A_X)$$

Singular Cohomology

Define

$$C^n(X, A) = \{\varphi : \{\text{maps } \Delta^n \to X\} \to A\}$$

where Δ^n is the n-simplex. There exists a map $d : C^n(X, A) \to C^{n+1}(X, A)$ and we define

$$H^i_{sing}(X, A) = H^i(C^\bullet(X, A)).$$

Theorem 3.1. *Assume that X is locally contractible. Then*

$$H^i_{Sh}(X, A) \cong H^i_{sing}(X, A)$$

for all A.

X is contractible implie sthat $H^i_{sing} = 0$ for $i > 0$. For the theorem, we need that for any $x \in X$, there exists a base of neighbourhoods U such that $H^i_{sing}(U, A) = 0$ for all $i > 0$.

de Rham Cohomology

Assume now that X is a (C^∞) manifold. Let $\Omega^\bullet(X)$ be the de Rham complex. Then

$$H^i_{dR}(X) = H^i(\Omega^\bullet(X)).$$

Theorem 3.2.

$$H^i_{dR}(X) = H^i_{Sh}(X, \mathbb{R})$$

Let Ω^i_X denote the sheaf of i-differentials on X. Then $d_{dR} : \Omega^i_X \to \Omega^{i+1}_X$ is a map of sheaves.

Lemma 3.3 (Poincare)**.**

$$H^i(\Omega_X^\bullet) = \begin{cases} 0 & i > 0 \\ \mathbb{R}_X & i = 0 \end{cases}$$

To prove Theorem 3.2 it is enough to show that the sheaves Ω_X^i belong to some class of sheaves adapted to Γ which is true if and only if $H^q(X, \Omega_X^i) = 0$ for all $q > 0$. The same thing is going to be true for the sheaf of C^∞ sections of any C^∞ vector bundle on X.

Let $\mathcal{C}^i(X, A)$ be the sheaf of singular cochains. We would like $\mathcal{C}^i(X, A)(U)$ to be singular cochains on U but this only defines a presheaf so we need to take the sheafification of this presheaf. Then d is a well-defined map of sheaves.

If X is locally contractible, then

$$H^i(\mathcal{C}^\bullet(X, A)) = \begin{cases} 0 & i > 0 \\ A_x & i = 0 \end{cases}$$

so Theorem 3.1 would follow if we knew that $\mathcal{C}^i(X, A)$ are acyclic with respect to Γ.

$$\Gamma(\mathcal{C}^i(X, A)) = C^i(X, A) \Big/ \{\text{interiors} = 0 \text{ locally}\}.$$

Definition 3.4. A sheaf $\mathcal{F}$ is called **flabby** if for all $U \subset X$ open, the map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. This implies that for all $U \subset V$, the mape $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective due to the commuting diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & \nearrow & \\ \mathcal{F}(X) & & \end{array}$$

Lemma 3.4. *i The class of flabby sheaves is adapted to* Γ*.*

ii Any injective sheaf is flabby.

Definition 3.5. For $\mathcal{F}$ a sheaf over X, the **stalk** of $\mathcal{F}$ at $x \in X$ is

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U).$$

Given any sheaf $\mathcal{F}$ we can construct a sheaf $\tilde{\mathcal{F}}(U)$ by defining

$$\tilde{\mathcal{F}}(U) = \prod_{x \in U} \mathcal{F}_x.$$

This sheaf is flabby, and there is an injective map $\mathcal{F} \to \tilde{\mathcal{F}}$.

Exercie 3.1. Show that if we have an exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

and if $\mathcal{F}_1$ is flabby, then $\Gamma(\mathcal{F}_2) \to \Gamma(\mathcal{F}_3)$ is surjective.

This exercise implies that an acycic complex of flabby sheaves remains acyclic under Γ.

Observation: $\mathcal{C}^i(X, A)$ is flabby.

Pick some $s \in C^i(X, A)$

Soft and Fine Sheaves

Definition 3.6. Let $f : Y \to X$ be a map of topological spaces and suppose $\mathcal{F}$ is a sheaf on X. The **inverse image sheaf** $f^*(\mathcal{F})$ is the sheaf on Y defined by

$$f^*(\mathcal{F})(U) = Sh\left(\varinjlim_{V \supset f(U)} \mathcal{F}(V)\right)$$

where the sets V must be open.

For Y a closed subset of X, $f^*(\mathcal{F}) = \mathcal{F}|_Y$ and we have a map $\mathcal{F}(X) \to \mathcal{F}|_Y(Y)$ for any closed Y.

Definition 3.7. $\mathcal{F}$ is called **soft** if the above map is surjective for every closed set Y.

Lemma 3.5. *Any flabby sheaf on a paracompact X is soft.*

Proof. Exercise. □

Lemma 3.6. *Assume that X is paracompact and Hausdorff. Then soft sheaves are acyclic for Γ (ie. soft sheaves form an adapted class for Γ.)*

Definition 3.8. Let $\mathcal{F}, \mathcal{G}$ be two sheaves on X. The **inner hom** is the sheaf on X defined by

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{Sheaves}(\mathcal{F}|_U, \mathcal{G}|_U)$$

Definition 3.9. Let $\mathcal{F}$ be a sheaf on X and let $s \in \mathcal{F}(X)$. The **support** of s is

$$\mathrm{supp}(s) = \text{minimal closed subset } Y \text{ of } X \text{ such that } s|_{X \setminus Y} = 0.$$

Definition 3.10. Assume that X is paracompact $\mathcal{F}$ is called **fine** if one of the following equivalent conditions is satisfied:

1. $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F})$ is soft.
2. For any $A, B \subset X$ closed with $A \cap B \neq \varnothing$ there exists $\alpha : \mathcal{F} \to \mathcal{F}$ such that $\alpha|_A = id$ and $\alpha|_B = 0$.
3. There exists a sheaf of rings $\mathcal{A}$ acting on $\mathcal{F}$ (so there is a map $\mathcal{A} \to \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F})$ such that for any locally finite covering $\{U_i\}$ of X there exists $a_i \in \mathcal{A}(X)$ such that $\mathrm{supp}(a_i) \subset U_i$ and $1 = \sum a_i$.

Corollary 3.1. *Assume that X is a manifold, $E \to X$ is a C^∞ vector bundle and $\mathcal{F}_E$ is the sheaf of C^∞ sections of E. Then $\mathcal{F}_E$ is soft.*

Proof. Let $\mathcal{A} = \mathcal{C}^\infty(X) =$ sheaf of C^∞–functions. $C^\infty(X)$ acts on every $\mathcal{F}_E$. □

Claim 3.1. If X is paracompact and $\mathcal{F}$ is fine then $H^p(X, \mathcal{F}) = 0$.

Proof. Let $\mathcal{A}$–mod be the category of sheaves of $\mathcal{A}$–modules. For $\phi : A - -mod \to \mathcal{A}b$ have $\phi_X : \mathcal{A} - mod \to \mathcal{SA}b(X)$ and $A = \Gamma(\mathcal{A})$. $\Gamma : \mathcal{A} - mod \to \mathcal{A} - mod$ and $\Gamma_{\mathcal{A}} : \mathcal{A} - mod \to A - mod$. Then

$$R\phi \circ R\Gamma_{\mathcal{A}} \cong RF \circ R\phi_X$$

Step 1 We have enough sheaves of $\mathcal{A}$-modules which are acyclic both as sheaves of $\mathcal{A}$ modules and as abstract sheaves (with respect to Γ.)

1. $\mathcal{A}$–mod has enough injectives.
2. Any injective $\mathcal{A}$–module is flabby. For $\mathcal{F} \in \mathcal{A}$–mod, we defined the map $\mathcal{F} \hookrightarrow \tilde{\mathcal{F}}$ into the flabby sheaf $\tilde{\mathcal{F}}$. $\tilde{\mathcal{F}}$ is also a sheaf of $\mathcal{A}$–modules. Since $\mathcal{F}$ is a direct summand of $\tilde{\mathcal{F}}$, $\mathcal{F}$ is flabby.

Step 2 Given $\mathcal{F}$ we can find a resolution

$$0 \to \mathcal{F} \to I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \to \cdots$$

where I^p is a sheaf of $\mathcal{A}$-modules acyclic with respect to Γ. For $p > 0$ and $s \in \Gamma(X, I^p)$, $ds = 0$. There exists a locally finite covering $X = \cup U_i$ such that for all i, $s_i \in I^{q-1}(U_i)$ and $ds_i = s|_{U_i}$. Choose a_i's with $\text{supp}(a_i) \subset U_i$ and $\sum a_i = 1$.

$a_i s_i \in I^{q+1}(X)$ and $X = U_i \cup X \setminus \text{supp}(a_i)$ so can define $t = \sum a_i s_i$ and then locally $dt = s$.

□

Definition 3.11. Let $f : X \to Y$. The direct image sheaf is the sheaf $f_*\mathcal{F}$ on Y defined by

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)).$$

Lemma 3.7.

1. *f_* is right adjoint to f^*.*
2. *f_* is left exact and f^* is exact.*
3. *For $f : X \to pt$, $f_* = \Gamma$.*

Variant

Change of notation: We will write $f^\bullet$ instead of f^* for the inverse image. Let $(X, \mathcal{R}_X)$ be a ringed space. A morphism

$$(X, \mathcal{R}_X) \xrightarrow{f} (Y, \mathcal{R}_Y)$$

consists of a continuous map $f : X \to Y$ plus a morphism of sheaves of rings $f^\bullet \mathcal{R}_Y \to \mathcal{R}_X$. The usual direct image can be naturally though of as $f_* : \mathcal{R}_X - mod \to \mathcal{R}_Y - mod$ ($\mathcal{R}_X(f^{-1}(U))$ acts on $\mathcal{F}(f^{-1(U)r})$ and we have $\mathcal{R}_Y(U) \to f^\bullet \mathcal{R}_Y(f^{-1}(U)) \to \mathcal{R}_X(f^{-1}(U))$ so $\mathcal{R}_Y(U)$ also acts.)

Let $\mathcal{F} \in \mathcal{R}_Y - mod$. $f^\bullet \mathcal{F}$ has an action of $f^\bullet \mathcal{R}_Y$ so can define $f^*\mathcal{F} = \mathcal{R}_X \otimes_{f^\bullet \mathcal{R}_Y} f^\bullet \mathcal{F}$. We have a pair of adjoint functors

$$\mathcal{R}_X - mod \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathcal{R}_Y - mod$$

where f_* is the left adjoint and f^* is the right adjoint.

Lemma 3.8. *Let $f : X \to Y$ be a continuous map and $\mathcal{F} \in \mathcal{SA}b(X)$. Then $R^i f_*(\mathcal{F})$ is the sheafification of the presheaf*

$$U \to H^i(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)})$$

Corollary 3.2. *Flabby sheaves are acyclic with respect to f_*. Or, on paracompact spaces fine sheaves are acyclic with respect to f_*.*

For $\mathcal{F} \in \mathcal{R}_X - mod$ the two notions of $Rf_*(\mathcal{F})$ coincide.

So far, we have three functors $f_*, f^*, \underline{\mathrm{Hom}}(-,-)$ out of the 6. From now on assume all topological spaces are assumed to be locally compact. There is another functor $f_! : \mathcal{SA}b(X) \to \mathcal{SA}b(Y)$. Define

$$f_!(\mathcal{F})(U) = \left\{ s \in \mathcal{F}(f^{-1}(U)) | \operatorname{supp}(s) \to U \text{ is proper} \right\}$$

(recall that $f : X \to Y$ is **proper** (assuming all spaces are locally compact and Hausdorff) if it is universally closed, ie. for every $g : Z \to Y$ the map $X \times_Y Z \to Z$ is closed. Equivalently, the preimage of compact sets is compact.) $f_!(\mathcal{F})$ is a subsheaf of $f_*(\mathcal{F})$ and is again left exact.

Lemma 3.9. *Let $f : X \to Y$, $\mathcal{F} \in \mathcal{SA}b(X)$. Then*

$$f_!(\mathcal{F}) = \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$$

(where the c subscript denotes compat support.) $\Gamma_c = f_!$ *when* $f : X \to pt$.

More generally, given

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\tilde{g}} & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

we have $g^* f_! = \tilde{f}_! \tilde{g}^*$ (this is called **base change**.) Note that this is wrong if we use f_* instead of $f_!$.

We also want to consider $Rf_!$. In particular, $H^i_c(X, \mathcal{F}) = R^i\Gamma_c(\mathcal{F})$. If $\mathcal{F} = A_X$ then we can define $H^i_c(X, A_X) = H^i_c(X, A)$, the **cohomology with compact support**. If X is a (Hausdorff) manifold, sheaves of the form $\mathcal{F}_E$ can still be used for computing H^i_c (acyclic with respect to Γ_c.

Example 3.3. Let $X = \mathbb{R}^n$. Then

$$H^i_c(\mathbb{R}^n, \mathbb{R}) = \begin{cases} \mathbb{R} & i = n \\ 0 & \text{otherwise} \end{cases}$$

Exercie 3.2. Let $X = \{(x, y) \in \mathbb{R}^2 | \; xy = 0\}$. What is $H^i_c(X, \mathbb{R})$?

If f is proper, then $f_! = f_*$. Assume that f is an open embedding. Then f_* is right adjoint to f^* and $f_!$ is left adjoint to f^*. Hence, there is a morphism $f_* f^* \leftarrow \mathcal{F}$. $f : X \hookrightarrow Y$ is an open embedding and so we need a map

$$\mathcal{F}(Y) \to f_* f^*(\mathcal{F})(Y) = f^*(\mathcal{F})(X) = \mathcal{F}(X)$$

which is the obvious map.

For the other adjoint relation, we have

$$f_! f^*(\mathcal{F}) = \{ s \in \mathcal{F}(X) | \; \operatorname{supp}(s) \text{ is closed in } Y \}$$

Given such an s we can construct a section of $\mathcal{F}$ on all of Y to produce this map.

Under some assumptions on X and Y we'll show that $Rf_!$ always has a right adjoint $f^!$ so the complete list of 6 functors is $f_*, f^*, f_!, \underline{\mathrm{Hom}}(-,-), f^!$ and $\mathbb{D}$ which comes from Verdier duality.

Lecture 8 [05.04.2016]

3.2 The Functors $f_!$ and $Rf_!$

Assume all spaces are locally compact and Hausdorff. For $f : X \to pt$, $RF_!(\cdot) = H_c^*(\cdot)$ is cohomology with compact support. Suppose that $f : X \to Y$ is locally compact. We have

$$H_c^i(\mathbb{R}^n, \mathbb{R}) = \begin{cases} 0 & i \neq n \\ \mathbb{R} & i = n \end{cases}$$

and

$$H^i(\mathbb{R}, \mathbb{Z}) = \begin{cases} 0 & i \neq 1 \\ \mathbb{Z} & i = 1 \end{cases}$$

(where here $X = \mathbb{R}$ and $\mathbb{Z} = \mathbb{Z}_X$.) This follows since we have a short exact sequence of sheaves

$$0 \to \mathbb{Z}_{\mathbb{R}} \to C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}, S^1) \to 0$$

$$\{f : \mathbb{R} \to \mathbb{R} |\operatorname{supp} \text{ is compact}\} \xrightarrow{\alpha} \{\varphi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} |\operatorname{supp} \varphi \text{ is compact}\}$$

Exercie 3.3. Define a map from the second set above to $\mathbb{Z}$ which produces an isomorphism with $\operatorname{Coker} \alpha$.

Definition 3.12. $\dim X \leq n$ if for any sheaf $\mathcal{F}$ of abelian groups, $H_c^i(X, \mathcal{F}) = 0$ for $i > n$. $\dim X = n$ if $\dim X \leq n$ but $\dim X \nleq n - 1$.

Remark 3.2. If $H_c^{n+1}(X, \mathcal{F}) = 0$ for all $\mathcal{F}$ then $\dim X \leq n$ by induction on i. There exists

$$0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{G} \to 0$$

where $\mathcal{H}$ is injective so on cohomology,

$$0 \to H_c^i(X, \mathcal{G}) \xrightarrow{\sim} H_c^i(X, \mathcal{F}) \to 0.$$

Proposition 3.1.

1. $\dim \mathbb{R}^n = n$ *and for any* X*,* $\dim(X \times \mathbb{R}) = \dim X + 1$.
2. *For any* $Y \hookrightarrow X$ *which is locally closed,* $\dim Y \leq \dim X$*.*
3. $\dim X$ *is local; ie. if any point* $x \in X$ *has a neighbourhood* U_x *then* $\dim U_x \leq n$ *implies* $\dim X \leq n$*.*

Proof.

1. Will prove $\dim \mathbb{R} = 1$. It is clear that $\dim \mathcal{R} \geq 1$. Assume that there exists $\alpha \in H_c^2(\mathbb{R}, \mathcal{F})$ with $\alpha \neq 0$. For a closed subset $i : Z \hookrightarrow X$, $\mathcal{F}_Z = i^*\mathcal{F}$ and by adjointness

$$\mathcal{F} \to i_* i^* \mathcal{F} = i_! i^* \mathcal{F} = i_!(\mathcal{F}_Z)$$

Given $\alpha \in H_c^*(X, \mathcal{F})$ let $\alpha|_Z$ be the image via the map $H_c^*(X, \mathcal{F}) \to H_c^*(X, \mathcal{F}_Z)$.

Let Z be a minimial closed subset of $\mathbb{R}$ such that $\alpha|_Z \neq 0$ and let $z \in Z$ be a point which seperates two other points of Z. Define $Z_- = \{y \in Z | y \leq z\}$ and $Z_+ = \{y \in Z | y \geq z\}$. Then $\alpha|_{Z_-} = 0$ and $\alpha|_{Z_+} = 0$ so $\alpha|_{Z_- \cap Z_+} = 0$.

Let X be a space, X_-, X_+ be closed subsets and $\mathcal{F}$ be a sheaf. Then

$$H_c^i(X, \mathcal{F}) \to H_c^i(X_-, \mathcal{F}|_{X_-}) \oplus H_c^i(X_+, \mathcal{F}|_{X_+}) \to H_c^i(X_- \cap X_+, \mathcal{F}|_{X_- \cap X_+}) \to H_c^{i+1}(\cdots)$$

is exact since

$$0 \to \mathcal{F} \to i_{-*}i_-^* \oplus i_{+*}i_+^*\mathcal{F} \to j_*j^*\mathcal{F} \to 0$$

is a short exact sequence of sheaves (here $i_\pm : X_\pm \to X$ and $j : X_- \cap X_+ \to X$.)

In our previous situation, $Z_- \cap Z_+ = \{z\}$ so for any sheaf $\mathcal{G}$ on Z (in particular for $\mathcal{G} = \mathcal{F}|_Z$),

$$\begin{array}{ccc} H_c^1(Z_- \cap Z_+, \mathcal{G}|_{Z_- \cap Z_+}) \longrightarrow H_c^2(Z, \mathcal{G}) & & \ni \alpha|_Z \\ \downarrow & & \downarrow \\ H_c^2(Z_-, \mathcal{G}|_{Z_-}) \oplus H_c^2(Z_+, \mathcal{G}|_{Z_+}) & & \alpha_{Z_-} = 0, \alpha_{Z_+} = 0 \end{array}$$

a contradiction.

2. $i : Y \hookrightarrow X$ closed, $\mathcal{F} \in \mathcal{SA}b$. $i_! = i_*$ so

$$H_c^i(Y, \mathcal{F}) = H_c^i(X, i_!\mathcal{F})$$

3. Exercise

□

Proposition 3.2. *If $i : Y \hookrightarrow X$ is an open embedding then $i_!$ is still exact and for any sheaf $\mathcal{F}$, $i_!\mathcal{F} = R_{i_!}\mathcal{F}$.*

Proof. $i_!F_Y = \mathcal{F}$ since the stalks of $i_!\mathcal{F}$ on $X \backslash Y$ are all 0.

Warning 3.2. This is NOT true for i_*.

□

Corollary 3.3. *If $\mathcal{F} \to \mathcal{G}$ is surjective then $i_!\mathcal{F} \to i_!\mathcal{G}$ is surjective.*

Remark 3.3. For $\alpha : X \backslash Y \to X$, $\alpha^* i_!\mathcal{F} = 0$. $i_!\mathcal{F}$ is called the extension of $\mathcal{F}$ by zero.

Theorem 3.3. *Let X,Y be locally compact, Hausdorff finite dimensional topological spaces. Let $f : X \to Y$ be a continuous map. Then $Rf_! : \mathcal{D}^b(X) \to CD^b(Y)$ (or $Rf_! : D^+(X) \to D^+(Y)$) has a right adjoint $f^!$, ie.*

$$\mathrm{H\overline{om}}(RF_!\mathcal{F}, \mathcal{G}) \cong \mathrm{H\overline{om}}(\mathcal{F}, f^!\mathcal{G})$$

Let $D(X)$ be the derived category of sheaves of abelian groups on X. $RF_! : D^+(X) \to D^+(Y)$ but for finite dimensional spaces $Rf_! : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$.

Remark 3.4. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ and if $f^!, g^!$ exist then $(g \circ f)^!$ also exists and $(g \circ f)^! = g^! \circ f^!$.

Example 3.4. Lets replace abelian groups by $\mathbb{Q}$–vector spaces in this example. Let $Y = pt$, X a compact oriented, connected manifold of dimension n, $\mathcal{F} = \mathbb{Q}_X[i]$ for $i \in \mathbb{Z}$ and $\mathcal{G} = \mathbb{Q}$. Then the adjointness relation becomes

$$\overline{\mathrm{Hom}}(H^i(X, \mathbb{Q}), \mathbb{Q}) = H^i(X, \mathbb{Q})^* \cong \mathrm{Hom}_{\mathcal{D}(X)}(\mathbb{Q}[i], f^!\mathcal{G})i = H^{-i}(X, f^!\mathcal{G})$$

By Poincaré duality, $H^i(X, \mathbb{Q})^* \cong H^{n-i}(X, \mathbb{Q})$ so we should define $f^!\mathcal{G} = \mathbb{Q}_X[n]$.

For X (possibly) not compact but oriented, Poincaré duality says

$$H^i_c(X, \mathbb{Q})^* \cong H^{n-i}(X, \mathbb{Q}).$$

If X is any (possibly not orientable) manifold of dimension n, let ω_X be the orientation sheaf given by

$$U \mapsto H^n_c(U, \mathbb{Z}).$$

This is a locally constant sheaf so a similar analysis shows that if $f^!$ exists for $f : X \to pt$ where X is a connected manifold then $f^!A = \omega_X \otimes A[\dim X]$. This generalizes Poincaré duality.

Definition 3.13. The **dualizing sheaf** on X is $\mathcal{D}_X = f^!\mathbb{Z}$ where $f : X \to pt$.

Our previous example show that for X a manifold, $\mathcal{D}_X = \omega_X[\dim X]$.

Explicit Descriptions of $f^!$

Case 1: f is an open embedding. Then $f^! = f^*$. If $\mathcal{F} \in \mathcal{SA}b(X)$ and $\mathcal{G} \in \mathcal{SA}b(Y)$ then

$$\mathrm{Hom}(f_!\mathcal{F}, \mathcal{G}) = \overline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}|_X)$$

(the map arising from $f_!\mathcal{F}|_X = \mathcal{F}$ is easily seen to be injective on stalks and the map arising from restricting $\alpha : \mathcal{F} \to \mathcal{G}_X$ to $s \in f_!\mathcal{F}(U)$ for any U is surjective.)

Case 2: f is a closed embedding. $f_* = f_!$ and f^* is left adjoint to $f_* = f_!$ but not right adjoint.

Claim 3.2. $f^! =$ sections with support on X.

Define a functor $f^? : \mathcal{SA}b(Y) \to \mathcal{SA}b(X)$ by

$$f^?\mathcal{F}(U) = \varinjlim_{V \text{ open}, V \cap X = U} \{s \in \mathcal{F}(V) |\, \mathrm{supp}(s) \subset U\}$$

Remark 3.5. It is enough to take some particular V.

$f^?$ is left exact but not right exact. Given $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ with $\mathrm{Ker}(\alpha) = 0$ it follows directly from the definition that $\mathrm{Ker}\, f^?(\alpha) = 0$

Claim 3.3. $f^! = Rf^?$

Proof. It is enough to show that $f_!$ is left adjoint to $f^?$.

$$\overline{\mathrm{Hom}}_{\mathcal{SA}b(Y)^\bullet}(f_!\mathcal{F}, \mathcal{G}) \cong \overline{\mathrm{Hom}}_{\mathcal{SA}b(X)}(\mathcal{F}, f^?\mathcal{G}).$$

For any $V \subset Y$, a map $\alpha : f_!\mathcal{F} \to \mathcal{G}$ is given by $f_!\mathcal{F}(V) \xrightarrow{\alpha_V} \mathcal{G}(V)$. But $f_!\mathcal{F}(V) = \mathcal{F}(V \cap X)$ so $\mathrm{Im}\,\alpha_V$ is contained in sections supported on $V \cap X$. This gives a map from left to right and the other direction is obvious. □

Let Y be an oriented manifold and $X = \{y\}$. For any $f : X \to Y$, $f^! \mathcal{D}_Y = \mathcal{D}_X$ so $f^! \mathbb{Z}_Y[\dim Y] = \mathbb{Z}$ so define $f^! \mathbb{Z}_Y = \mathbb{Z}[-\dim Y]$.

Case 3: $f : Y \times M \to Y$ the projection and M a manifold.

Definition 3.14. Let Z, W be spaces, $\mathcal{F}$ a sheaf on Z and $\mathcal{G}$ a sheaf on W. Then define the sheaf $\mathcal{F} \boxtimes \mathcal{G}$ on $Z \times W$ as follows. Given $U \subset Z \times W$ define

$$\mathcal{F} \boxtimes \mathcal{G}(U) = \varinjlim_{\text{coverings of } U \text{ by } U_1^\alpha \times U_2^\alpha} \{s_\alpha \in \mathcal{F}(U_1^\alpha) \otimes \mathcal{G}(U_2^\alpha) \text{ compatible on intersections}\}$$

Then

$$f^! \mathcal{G} = \mathcal{G} \boxtimes \mathcal{D}_M = \mathcal{G} \boxtimes \omega_M[\dim M].$$

Note that $f^!$ only makes sense in the derived category.

Remark 3.6. Suppose that X, Y are quasi–projective varieties (over $\mathbb{R}$ or $\mathbb{C}$) and $f : X \to Y$ is an algebraic map. Then f is always a composition of maps of the form 1,2,3 so $f^!$ has an explicit description.

Theorem 3.4 (Base Change)**.** *Given*

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\tilde{g}} & X \\ \big\downarrow \tilde{f} & & \big\downarrow f \\ Y & \xrightarrow{g} & Z \end{array},$$

$g^* f_! \simeq \tilde{f}_! \tilde{g}^*$ *and* $g^! f_* \simeq \tilde{f}_* \tilde{g}^!$

Definition 3.15. Define the contravariant functor

$$\mathbb{D}F = R\,\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{D}_X).$$

We have constructed the functors $f_*, f^*, f_!, f^!, \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ and $\mathbb{D}$ between derived categories of sheaves.

We will need to define a nice subcategory of $\mathcal{D}^b(X)$ which is preserved by all six of these functors and such that $\mathcal{D}^2 = Id$ on this category. This will be the category of constructible sheaves. Then for $f : X \to Y$ we will have $f_! = \mathcal{D} f_* \mathcal{D}$ and $f^! = \mathcal{D} f^* \mathcal{D}$ and $\mathcal{D} f_! = f_* \mathcal{D}$

Note that $\mathcal{D}\mathbb{Z}_X = \mathcal{D}_X$. Given $f : X \to pt$ and $\mathcal{F}$ a sheaf on X, $\mathbb{D} f_!(\mathcal{F}) = f_*(\mathbb{D}(\mathcal{F})$. Working with $\mathbb{Q}$–vector spaces, we will get **Verdier duality**, ie. that

$$H^i_c(X, \mathcal{F})^* = H^{-i}(\mathbb{D}\mathcal{F}).$$

For X an oriented manifold of dimension $2n$, $\mathcal{F} = \mathbb{Z}_X[n]$, $\mathbb{D}\mathcal{F} \simeq \mathcal{F}$ so

$$H^i_c(X, \mathcal{F})^* = H^{-i}(X, \mathcal{F}).$$

Preview

When X is any complex algebraic variety, we'll define IC_X such that $\mathcal{D}IC_X \simeq IC_X$ and

$$IC_X|_{X^{smooth}} = \mathbb{Q}_{X^{smooth}}[\dim_{\mathbb{C}} X]$$

Moreover we'll define an abelian category $\mathrm{Perv}(X)$ of **perverse sheaves** on X such that $\mathbb{D} : \mathrm{Perv}(X) \to \mathrm{Perv}(Y)$ and $IC_X \in \mathrm{Perv}(X)$. Then $H^*(X, IC_X) = IH^*(X)$ is the **intersection cohomology** of X.

Exercie 3.4. Compute $\mathcal{D}_X$ for $X = x$.

Remark 3.7. For $f : Y \times M \to Y$ the projection, we saw that $f^!(\mathcal{F}) = \mathcal{F} \boxtimes \mathcal{D}_M$. Suppose that X, Y are algebraic varities over $\mathbb{C}$ (or smooth manifolds) and that $f : X \to Y$ is a smooth map (or a submersion). Then

$$f^!(\mathcal{F}) = f^*[2(\dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y)]$$

Lecture 9 **[12.04.2016]**

Recall

From now on, all topological spaces will be complex algebraic varieties and all maps will be algebraic (quasi–projective.) Further, we'll work iwth sheaves of vector spaces over $/BC$.

Given $f : X \to Y$, we have $f_*, f_! : D^b(Sh_X) \to D^b(Sh_Y)$, and $f^!, f^*$ in the opposite direction. There is an isomorphism $f_! \to f_*$ if f is proper and f_* is right adjoint to f^* and $f_!$ is left adjoint to $f^!$.

Given a diagram

$$\begin{array}{ccc} X \times Z & \xrightarrow{\tilde{g}} & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y, \end{array}$$

$g^* f_! \simeq \tilde{f}_! \tilde{g}^*$, and $g^! f_* \simeq \tilde{f}_* \tilde{g}^!$.

Assume that f is a smooth morphism (equivalently, locally on X f looks like a product with a smooth manifold.) Then $f^! = f^* p[(\dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y)]$.

Given $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$ with U open, let $Z = X \backslash U$ and $\mathcal{F} \in D^b(Sh_X)$. Then $i_! i^! \mathcal{F} \to \mathcal{F} \to j_* j_* \mathcal{F}$ and $j_* j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}$ are both exact triangles. This follows since we have $\mathcal{K} \to \mathcal{F} \to j_* j^* \mathcal{F}$ ($\mathcal{K}$ is the cone) and $\mathcal{K}$ is supported on Z. But from $i^! \mathcal{K} \tilde{\to} i^! \mathcal{F} \to i^* j_* j^* \mathcal{F}$ we know $\mathcal{K}$ is supported on Z if and only if $K = i_! i^! K$.

Given $f : X \to pt$ we defined $D_X = f^! \mathbb{C}$ and

$$\mathbb{D}(\mathcal{F}) = \underline{\mathrm{Rhom}}(\mathcal{F}, D_X).$$

For X smooth, $D_X = \mathbb{C}_X[2 \dim X]$ and there is a canonical morphism $Id \to \mathbb{D}^2$.

3.3 Constructible Sheaves

Let X be smooth and $\mathbb{C}_X$ constructible. More generally, any locally constant sheaf with finite dimensional stalks will be constructible.

Remark 3.8. Assume that X is connected. Then locally constant sheavese are equivalent to representations (finite dimensional since our sheaf has finite dimensional stalks) of $\pi_1(X, x)$. These are sometimes also called **local systems.** In fact, these are also equivalent to representations of the fundamental groupoid (ie. for any path $\gamma : [0,1] \to X$ we have an isomorphism $\mathcal{F}_{\gamma(0)} \tilde{\to} \mathcal{F}_{\gamma(1)}$ which depends only on the homotopy class of γ and such that these isomorphism are compatible with composition.) It is clear that a locally constant sheaf gives such a representation by choosing an open cover of γ with each open set small enough that $\mathcal{F}$ is constant in it. In the other direction given a representation V, take the universal cover $\tilde{X} \xrightarrow{\pi} X$. Then $X = \tilde{X}/\pi_1(X)$. On $f_! V_{\tilde{X}}$ there is an action of $\pi_1(X)$ and we can take $\mathcal{F} = (f_! V_{\tilde{X}})^{\pi_1(X)}$.

Definition 3.16. The sheaf $\mathcal{F}$ on X is called **constructible** if any of the following equivalent conditions are satisfied:

1. For any $Y \hookrightarrow X$ locally closed there exists an open $U \subset Y$ such that $\mathcal{F}|_U$ is locally constant.
2. There exists a stratification $X = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_n$ such that X_i is closed and $\mathcal{F}|_{S_i}$ is locally constant where $S_i = X_i \backslash X_{i+1}$.

Example 3.5. For $\dim X = 1$, $\mathcal{F}$ is constructible if and only if X has an open dense subset U such that $\mathcal{F}|_U$ is locally constant and $\mathcal{F}$ has finite dimensionals stalks on $X \backslash U$.

Let $D^b_c(X)$ be the full subcategory of $D^b(Sh_X)$ consisting of complexes whose cohomology is constructible.

Theorem 3.5.

1. *$f_*, f^*, f_!, f^!$ and* <u>Rhom</u> *map $D^b_C(-)$ to $D^b_C(\bullet)$.*
2. *on D^b_C, $\mathbb{D}f_* = f_!\mathbb{D}$, $\mathbb{D}f^* = f^!\mathbb{D}$ for $f : X \to Y$.*
3. *$\mathbb{D}^2 = Id$ on $D^b_c(X)$ and so $f_! = \mathbb{D}f_*\mathbb{D}$ and $f^! = \mathbb{D}f^*\mathbb{D}$.*

Proof.

3. We have the map $Id \to \mathbb{D}^2$. $\mathcal{F} \to \mathbb{D}(\mathbb{D}(\mathcal{F}))i$ is an isomorphism for all $\mathcal{F} \in D^b_C(\mathcal{F})$. The proof is by induction onf $\dim \operatorname{supp} \mathcal{F}$.

   If $\dim \operatorname{supp} \mathcal{F} = 0$ then the statement is equivalent to thee statement for vector spaces. Let U be a smooth dense open subset of X such that $\mathcal{F}|_U$ is locally constant. Have

   $$i_* i^* \mathcal{F} \to \mathcal{F} \to j_!(\mathcal{F}|_U)$$

   supported on $X \backslash U$. If we have a morphism between two exact triangles and two of the three maps are isomorphisms, then so is the third. Hence, it is enough to show that $\mathbb{D}^2(j_!\ (\mathcal{F}|_U)) = j_!(\mathcal{F}|_U))$.

   Suppose X is smooth and $\mathcal{F}$ is locally constant. Let $\mathcal{F}^\vee$ be the dual locally constant sheaf. Then $\mathbb{D}\mathcal{F} = \mathcal{F}^\vee[2 \dim X]$

   Corollary 3.4. *$\mathbb{D}^2 = Id$ for locally constant sheaves on asmooth varieties.*

It now follows that

$$\mathbb{D}j_!(\mathcal{F}|_U) = j_*\mathbb{D}(\mathcal{F}|_U) = \mathbb{D}(\mathbb{D}j_*(\mathcal{F}|_U)) = \mathbb{D}j_*\mathbb{D}(\mathcal{F}|_U) = j_!(\mathbb{D}^2(\mathcal{F}|_U)) = j_!(\mathcal{F}|_U)$$

1. Let $f : X \to Y$

f^*: It is clear for f^* since given $Y \supset Y_1 \supset \cdots \supset Y_n$, $\mathcal{F}|_{Y_i \setminus Y_{i+1}}$ is locally constant. Letting $X_i = f^{-1}(Y_i)$, $f^*\mathcal{F}|_{X_i \setminus X_{i+1}} = f^*(\mathcal{F}|_{Y_i \setminus Y_{i+1}})$ and the result is immediate.

$f_!$: Want to show $f_!\mathcal{F} \in D^b_c(Y)$ if $\mathcal{F} \in D^b_c(X)$. The proof is by induction on $\dim \operatorname{supp} \mathcal{F}$. For $\dim \operatorname{supp} \mathcal{F} = 0$ the result is obvious.

Assume $f_!\mathcal{G} \in D^b_c(Y)$ for all $\mathcal{G} \in D^b_c(X)$ with $\dim \operatorname{supp} \mathcal{G} < \dim \operatorname{supp} \mathcal{F}$. There exists dense open subsets $U \subset X$, $V \subset Y$ such that $\mathcal{F}|_U$ is locally constant and such that $f : U \to V$ is a locally trivial fibration (topologically.) Replace X by $\operatorname{supp} X$. We have $U \xrightarrow{j} X \xleftarrow{i} Z$ where Z is the complement of U and the maps are the inclusions, and hence we have the distinguished triangle

$$i_*i^*\mathcal{F} \to \mathcal{F} \to j_!(\mathcal{F}_V).$$

Applying f we have an exact triangle

$$f_!i_*i^*\mathcal{F} \to f_!\mathcal{F} \to f_!j_!(\mathcal{F}|_U).$$

By induction $f_!i_*i^*\mathcal{F}D^b_c$ so by the following exercise

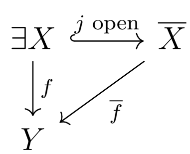
Exercie 3.5. Let $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ be a distinguished triangle where $\mathcal{F}, \mathcal{G} \in D^B_c(X)$.. Then $\mathcal{H} \in D^b_c(X)$.

it is enough to prove $f_!j_!(\mathcal{F}|_U) \in D^b_c(X)$.

Writing $\alpha : V \hookrightarrow Y$, it is easy to see $f_!j_!(\mathcal{F}|_U) = \alpha_!((f|_U)_!(\mathcal{F}|_U))$. We want the RHS to be constructible which is equivalent to the constructibility of $(F|_U)_!(\mathcal{F}|_U)$. Hence, it is enough to assume $f : X \to Y$ is a topologically locally trivial fibration and $\mathcal{F}$ is locally constant.

Exercie 3.6. Show that in this case $f_!\mathcal{F}$ is also locally constant.

f_*: There exists an open $X \overset{j}{\hookrightarrow} \overline{X}$ which can be put in the diagram



where $\bar{f}$ is proper. We have $f_* = \overline{f}_* \circ j_*$ and $\overline{f}_* = \overline{f}_!$. It is enough to show f_* preserve constant sheaves for f an open embedding.

Let $j : X \hookrightarrow Y$, $\mathcal{F} \in D^b_c(X)$. There exists $\mathcal{F} \in D^b_c(Y)$ with $\mathcal{G}|_X = \mathcal{F}$ (eg. $\mathcal{G} = j_!\mathcal{F}$.) There is an exact triangle

$$i_!i^I\mathcal{G} \to \mathcal{G} \to j_*\mathcal{F}.$$

The result follows from a theorem of Deligne

Theorem 3.6 (Théorème de finitude (SGA $4\frac{1}{2}$))**.**

$f^!$: It is enough to prove the theorem for f being a locally closed embedding since you can factorize f as a product $f = g \circ h$ with h a closed embedding and g smooth.

□

Exercie 3.7. $\underline{\text{RHom}}(\mathcal{F}, \mathcal{G}) = \Delta^!(\mathbb{D}\mathcal{F} \boxtimes \mathcal{G})$ where $\Delta : X \to X \times X$ and $\mathcal{F} \otimes \mathcal{G} = \Delta^*(\mathcal{F} \boxtimes \mathcal{G})$. $\Delta : X \to X \times X$

Basic Problem

Let X be smooth and $\mathcal{F}$ a locally constant sheaf. Then $\mathbb{D}\mathcal{F}[\dim X] = \mathcal{F}^\vee[\dim X]$. Look for more general such things. In particular, we want some $IC_X \in D_c^n b(X)$ such that $IC_X|_{X^{\text{smooth}}} = \mathbb{C}_{X^{\text{smooth}}}[\dim X]$ and $\mathbb{D}IC_X = IC_X$. We'll constuct Perv $\subset D_c^b(X)$ which is abelian and stable under $\mathbb{D}$.

3.4 t-Structures and Triangulated Categories

Given $\mathcal{D}^b(\mathcal{A})$, how can we produce other big abelian categories inside?

Definition 3.17. A **Triangulated Category** is an additive category $\mathcal{D}$ with the structures

a) $T : D \to D$ an auto–quivalence called the **shift** and we write $T(X) = X[1]$. A **triangle** in $\mathcal{D}$ is then a sequence $X \to Y \to Z \to X[1]$.

b) A class of triangles called "distinguished triangles".

We require the following axioms be satisfied:

TR1: a) $X \to X \to 0$ is distinguished.

b) Any triangle is isomorphic to a distinguished triangle.

(c) Any $X \xrightarrow{u} Y$ ca be completed to a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Y \to X[1]$.

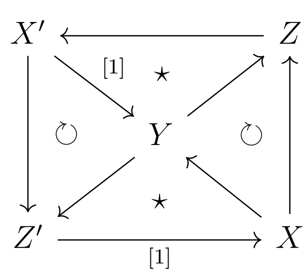
TR2: $X \to y \to Z$ is distinguishied if and only if $Y \to Z \to X[1] \to Y[1]$ is distinguisheed.

TR3:

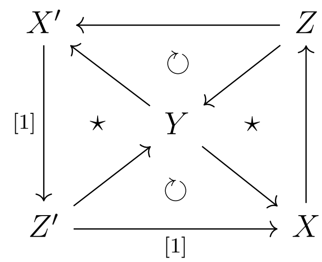
$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

There exists $H : Z \to Z'$ making the diagram commute.

TR4: (Octahedron axiom) Any upper cap can be completed to an octahedron. Here, given three objects X, Y, Z an upper cap is a diagram



and a lower cap is a diagram



where a triangle with a $\star$ in it indicates a distinguished triangle while a triangle with a $\circlearrowright$ in it indicates a commutative triangle in these diagrams. We say an upper cap can be completed to an octahedron if it can be completed to a lower cap.

Motivation for octahedron axiom.

For $\mathcal{A}$ abelian and $X \subset Y \subset Z$, $Z/X \supset Y/X$ and $(Z/X)/(Y/X) \simeq Z/Y$. Here we can think of $Z' = Y/X$ and $Y' = Z/X$ so that Z/Y is the upper cap and $X' = (Z/X)/(Y/X)$ is the lower cap.

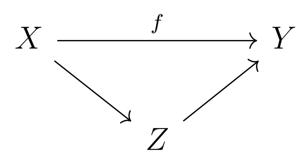
Example 3.6.

1. For $\mathcal{A}$ abelian, $\mathcal{D}(\mathcal{A}), \mathcal{D}^+(\mathcal{A}), \mathcal{D}^-(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A})$ are triangulated.

2. $\mathcal{K}(\mathcal{A})$ is a triangulated category.

3. For $\mathcal{B} \subset \mathcal{A}$ a full abelian subcategory stable under extension, $\mathcal{D}_{\mathcal{B}}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$, the complexes with cohomology in $\mathcal{B}$ is triangulated.

4. Let V be a finite dimensional vectors space over $\mathbb{C}$ and let $\wedge = \wedge(V)$ be the exterior algebra. Then $\mathcal{F} \subset \tilde{\mathcal{D}}$ where $\mathcal{F}$ are free modules and $\tilde{\mathcal{D}}$ are graded $\wedge$–modules. Let $\mathcal{D} = \tilde{\mathcal{D}}/\mathcal{F}$ so $\operatorname{Ob}\mathcal{D} = \operatorname{Ob}\tilde{\mathcal{D}}$ and for all $X, Y \in \operatorname{Ob}\tilde{\mathcal{D}}$ $\operatorname{Hom}_0(X,Y)$ is the collection of all maps f which factorize as



for $Z \in \operatorname{Ob}\mathcal{F}$. Then
$$\operatorname{Hom}_{\mathcal{D}}(X,Y) = \operatorname{H\overline{om}}_{\tilde{\mathcal{D}}}(X,Y)/\operatorname{Hom}_0(X,Y).$$
This is a triangulated category which is isomorphic to $D^b(\operatorname{Coh}_{\mathbb{P}(V)})$.

Lecture 10 **[19.04.2016]**

Definition 3.18. Let $\mathcal{D}$ be a triangulated category. A t–**structure** on $\mathcal{D}$ is a pair of full subcategories $\mathcal{D}^{\leq 0}$, $\mathcal{D} \geq 0$ (and therefore $\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n}$ for all n using the shift) such that

1. $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{leq1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$. This implies that for any $n \leq m$, $\mathcal{D}^{\leq n} \subset \mathcal{D}^{\leq m}$ and $\mathcal{D}^{\geq n} \supset \mathcal{D}^{\geq m}$.

2. $\operatorname{H\overline{om}}(X,Y) = 0$ for all $X \in \operatorname{Ob}\mathcal{D}^{\leq 0}$, $Y \in \operatorname{Ob}\mathcal{D}^{\geq 1}$.

3. For all $X \in \operatorname{Ob} \mathcal{D}$ there exists a distinguished triangle
$$A \to X \to B \to A[1]$$
with $A \in \operatorname{Ob} \mathcal{D}^{\leq 0}$ and $B \in \operatorname{Ob} \mathcal{D}^{\geq 1}$

Proposition 3.3 (Main Example)**.** *Let $\mathcal{A}$ be an abelian category and $\mathcal{D} - \mathcal{D}(\mathcal{A})$. Then*
$$\mathcal{D}^{\leq 0} = \{X \in \mathcal{D} | H^i(X) = 0 \quad i > 0\}$$
$$\mathcal{D}^{\geq 0} = \{X \in \mathcal{D} | H^i(X) = 0 \quad i < 0\}$$
is a t–structure.

Proof. Let C be a complex. Have the truncation $\tau_{\leq 0} : C \to C$
$$(\tau_{\leq 0} C)^i = \begin{cases} 0 & i > 0 \\ C^i & i < 0 \\ \operatorname{Ker}(d : C^0 \to C^1) & o = 0 \end{cases}$$
for which
$$H^i(\tau_{\leq 0} C) = \begin{cases} 0 & i > 0 \\ H^i(C) & i \leq 0 \end{cases}.$$
Part (i) of the definition of a t–structure then follows from the sequence
$$\tau_{\leq 0} C \to C \to C / \tau_{\leq 0} C.$$
The proof of part (ii) follows from the diagram □

Definition 3.19. The core of $\mathcal{A}$ of the t–structure is $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ (which is a full subcategory in $\mathcal{D}$.)

Theorem 3.7. *$\mathcal{A}$ (core) is an abelian category.*

Warning 3.3. $\mathcal{D}$ is not necessarily equivalent to $\mathcal{D}(\mathcal{A})$. Take $\mathcal{A} \supset \mathcal{B}$ abelian and stable under extensions. Then $\mathcal{D}(\mathcal{A}) \supset \mathcal{D}_{\mathcal{B}}(\mathcal{A})$ and $\mathcal{D}_{\mathcal{B}}(\mathcal{A})$ has a t–structure with core $\mathcal{B}$. Explicitly, let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, let $\mathcal{A}$ be $\mathfrak{g}$–modules and take $\mathcal{B}$ to be the finite dimensional $\mathfrak{g}$–modules. Then $\mathcal{B}$ is semisimple and $\operatorname{Ext}^3(\mathbb{C}, \mathbb{C}) \neq 0$.

Example 3.7 (dg-Algebras). Let A be a graded algebra, $A = \oplus_{i \in \mathbb{Z}} A_i$. A is a dg-algebra if it is endowed with a differential $d_A : A \to A$ such that $(d_A)_i : A_i \to A_{i+1}$ and $d_A(ab) = (d_A a)b + a d_A b$. Then $H^\bullet(A)$ is a graded algebra.

A dg–module over A is a graded A–module $M = \oplus_{i \in \mathbb{Z}} M_i$ such that $A_i M_j \subset M_{i+j}$ together with a differential $d_M : M \to M$ such that $(d_M)_i : M_i \to M_{i+1}$ and $d_M(am) = d_A(a)m + a d_M(m)$. Then $H^\bullet(A)$ is a module over $H^\bullet(A)$.

Let $DG - mod(A)$ be the category of dg–modules. Localizing by quasi–isomorphisms yields the derived category of DG–modules $\mathcal{D}(A)$.

Theorem 3.8.

1. *This is a triangulated category.*
2. *Assume $A_i = 0$ for all $i > 0$. Then the usual definition of $\mathcal{D}^{\geq 0}$ and $\mathcal{D}^{\geq 0}$ again gives a t–structure with core the modules over $H^0(A) = \operatorname{Coker}(A_{-1} \to A_0)$.*

For example, take $A = \mathbb{C}[t]/t^2$ where $\deg t = -1$ and take $d_A = 0$. Then $A_0 = H^0(A) = \mathbb{C}$. Letting $\mathcal{D}$ be the derived category of dg–modules, $\operatorname{Ext}^2_{\mathcal{D}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$ since the short exact sequence
$$0 \to \mathbb{C} \cdot t \to \mathbb{C}[t]/t^2 \to \mathbb{C} \to 0$$
is aa distinguished triangle (here think of $\mathbb{C} \cdot t$ as $\mathbb{C}[-1]$. Hence we have a map $\mathbb{C} \to \mathbb{C}[2]$ which gives a nonzero element in $\operatorname{Ext}^2(\mathbb{C}, \mathbb{C})$. In fact $\operatorname{Ext}^\bullet(\mathbb{C}, \mathbb{C}) = \mathbb{C}[x]$ where $\deg x = 2$.

3.5 Perverse Sheaves

Let X be an algebraic variety over $\mathbb{C}$ and $D = D_c^b(X)$. The goal is to define another t–structure such that $\mathbb{D}(\mathcal{D}^{\leq 0}) = \mathcal{D}^{\geq 0} \implies \mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. Then $\mathbb{D} : \mathcal{A} \to \mathcal{A}$ and $\mathcal{A}$ is the category of perverse sheaves.

If X is smooth, then $\mathbb{D}(\mathbb{C}_X) = \mathbb{C}_X[2\dim_{\mathbb{C}} X]$ and $\mathbb{D}(\mathbb{C}_X[\dim X]) = \mathbb{C}_X[\dim X]$. Let $i : Y \hookrightarrow X$ be a closed smooth subvariety. Then sheaves on Y can be thought of as sheaves on X by taking $\mathcal{F}$ to $i_*\mathcal{F}$. $\mathbb{D}(i_*\mathbb{C}_Y) = i_*\mathbb{C}_Y[2\dim Y]$ and $\mathbb{D}(i_*\mathbb{C}_Y[\dim Y]) = i^*\mathbb{C}_Y[\dim Y]$.

Want: If $i : Y \hookrightarrow X$ is a closed smooth subvariety and $\mathcal{E}$ is a locally constant sheaf on Y, then $i_*\mathcal{E}[\dim Y]$ should be perverse.

Definition 3.20 (Perverse t–structure)**.** $\mathcal{F} \in \mathrm{Ob}({}^cD^{\leq 0})$ if and only if for all k,dim supp $H^{-k}(\mathcal{F}) \leq k$ for $\mathcal{F} \in \mathcal{D}^{\leq 0}$. Equivalently, for $x \in X$ and $i_* : \{x\} \hookrightarrow X$, $\dim\{x | H^{-k}(i_x^*\mathcal{F}) \neq 0\} \leq k$.

Another equivalent way of stating the definition is given a stratification $X_0 \supset X_1 \supset X_2 \supset \cdots$, $S_i = X_i \backslash X_{i+1}$ smooth, and $\mathcal{F}|_{S_i}$ locally constant, $\mathcal{F}|_{S_i} \in \mathcal{D}^{\leq -\dim S_i}$.

$\mathcal{F} \in {}^p\mathcal{D}^{\geq 0} = \mathbb{D}({}^p\mathcal{D}^{\leq 0}) \iff \dim\{x | H^k(i_x^!\mathcal{F}) \neq 0\} \leq k$.

Theorem 3.9 (Beilinson–Bernstein–Deligne)**.**

1. *This is a t-structure.*
2. *Let X be smooth and let $\mathcal{E}$ be an irreducible local system on X. Then $\mathcal{E}[\dim X]$ is an irreducible perverse sheaf.*
3. *Any object of* Perv *has finite length.*

Definition 3.21. Let $\mathcal{A}$ be an abalian category. An object $X \in \mathrm{Ob}\,\mathcal{A}$ has **finite length** if there exists

$$0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$$

such that X_i/X_{i+1} is irreducible.

Definition 3.22. The category Perv of **Perverse sheaves** is the core of this t–structure.

Example 3.8. Let X be smooth, $\dim X = 1$. $\mathcal{U} = \mathbb{C}^* \overset{j}{\hookrightarrow} \mathbb{C} = X$.

Claim 3.4. $\mathcal{F} = j_*\mathbb{C}_{\mathcal{U}}[1] \in \mathcal{D}^{\geq 0}$ is perverse.

Let $i : \{0\} \hookrightarrow \mathbb{C}$.

To show something is in $\mathcal{D}^{\geq 0}$, need to show $H^k(i^!\mathcal{F}) = 0$, $k < 0$ and $i^!\mathcal{F} \in \mathcal{D} \geq 0$. But this is obvious for this sheaf since $i^!\mathcal{F} = 0$.

To show a sheaf is in $\mathcal{D}^{\leq 0}$ need to show $JH^k(i^*\mathcal{F}) = 0$, $k > 0$.

$$i^*j_*\mathbb{C}_{\mathcal{U}} = \varinjlim_{V} H^*(V \backslash 0, \mathbb{C})$$

for V neighbourhood of 0. But then

$$H^k(i^*j_*\mathbb{C}_{\mathcal{U}}[1]) = \begin{cases} 0 & k \neq 0,1 \\ \mathbb{C} & k = 0,-1 \end{cases}.$$

Perv $\subset \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -\dim X}$.

Have a short exact sequence

$$\mathbb{C}_X \to j^*\mathbb{C}_{\mathcal{U}} \to \delta_0[-1]$$

where δ_0 is the sky–scraper sheaf at 0, $i_*\mathbb{C}$ for $i : \{0\} \to \mathbb{C}$.

$$\mathbb{C}_X[1] \to j_*\mathbb{C}_{\mathcal{U}}[1] \to \delta_0$$

Let $j : U \hookrightarrow X$ be an open subset, $\mathcal{F} \in \operatorname{Perv}(U)$.

Theorem 3.10 (Perverse,Intermediate or Goresky–MacPherson extension)**.** *There exists a unique extension $j_{!*}\mathcal{F}$ of $\mathcal{F}$ to a perverse sheaf on X such that $j_{!*}\mathcal{F}$ has neither subobjects nor quotient objects supported on $X \backslash U$.*

$\mathbb{D} : \operatorname{Perv}(X) \to \operatorname{Perv}(X)$ is an exact contravariant functor with $\mathbb{D}^2 = Id$. The theorem implies that $\mathbb{D}j_{!*}(\mathcal{F}) = j_{!*}(\mathbb{D}\mathcal{F})$.

If X is irreducible and $j : U \hookrightarrow X$ is open, dense and smooth, then $j_{!*}\mathbb{C}_U[\dim X] = IC_X \in \operatorname{Perv}(X)$ and $\mathbb{D}(IC_X) = IC_X$.

Construction

Remark 3.9. Given $\mathcal{D}, \mathcal{D}^{\geq 0}$ and $\mathcal{D}^{\leq 0}$ we can talk about truncation functors $\tau_{\leq n} L\mathcal{D} \to \mathcal{D}^{\leq n}$ and $\tau_{\geq n} : \mathcal{D} \to \mathcal{D}^{\geq n}$. Further, $H^n = \tau_{\leq n}\tau_{\geq n} \simeq t_{\geq n}\tau_{\leq n}$.

Let $X \in \operatorname{Ob}\mathcal{D}$ and consider an exact triangle $A \to X \to B \to A[1]$ with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

Exercie 3.8. Show that A and B are canonically unique.

Given $\mathcal{F} \in D^b_C(X)$ and $n \in \mathbb{Z}$, then have ${}^pH^n(\mathcal{F}) \in \operatorname{Perv}(X)$.

Let $A \in D^{\leq 0}$ and $B \in D^{\geq 0}$. Any map $f : A \to B$ factorizes as

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow & & \uparrow \\ H^0(A) = \tau_{\geq 0}A & \longrightarrow & H^0(B) = \tau_{\leq 0}B \end{array}$$

We have a map ${}^p\mathcal{D}^{\leq 0} \ni j_!\mathcal{F} \to j_*\mathcal{F} \in {}^p\mathcal{D}^{\geq}$ so by the remark it factorizes as

$$\begin{array}{ccc} j_!\mathcal{F} & \longrightarrow & j_*\mathcal{F} \\ \downarrow & & \uparrow \\ {}^pH^0(j_!\mathcal{F}) & \xrightarrow{\alpha} & {}^pH^0(j_*\mathcal{F}) \end{array}$$

Define

$$j|_{!*}\mathcal{F} = \operatorname{Im}\alpha.$$

Let $\mathfrak{g}$ be a subobject of $j_{!*}\mathcal{F}$ supported on $X \backslash U$. This factorizes as

$$\mathfrak{g} \hookrightarrow j_{!*}\mathcal{F} \hookrightarrow {}^pH^0(j_*\mathcal{F}) \to j_*\mathcal{F}$$

$$\overline{\operatorname{Hom}}(\mathfrak{g}, j_*\mathcal{F}) = \overline{\operatorname{Hom}}(\mathfrak{g}|_U, \mathcal{F}) = 0$$

and also $\overline{\operatorname{Hom}}(\mathfrak{g}, {}^pH^0(j_*\mathcal{F}) = 0$ which implies $\mathfrak{g} = 0$. A similar argument works for quotients.

Assume that we have $K \in \operatorname{Perv}(X)$, $K|_U = \mathcal{F}$. K has neither subbjects nor quotients on $X \backslash U$. We want $K \to j_{!*}\mathcal{F}$.

The fact that $K|_U = \mathcal{F}$ implies we have a diagram

$$\to j_!\mathcal{F} \to K \to K \to J_*\mathcal{F}.$$

But then K sits in the sequence

$${}^pH^0(j_!\mathcal{F}) \to K \to {}^pH^0(j_*\mathcal{F})$$

and the first map is surjective while the second map is injective. Therefore $K = \operatorname{Im}\alpha$.

Lemma 3.10. *Let $\mathcal{F} \in \operatorname{Perv}(U)$ and assume that $\mathcal{F}$ is irreducible. Then $j_{!*}\mathcal{F}$ isi irreducible.*

Proof. It follows from the fact that $\mathfrak{g} \hookrightarrow j_{!*}\mathcal{F}$ and $\mathfrak{g}|_U \hookrightarrow \mathcal{F}$. □

Exercie 3.9. Show that any irreducible perverse sheaf on X has the form $j_{!*}\mathcal{E}[d]$ where $j : U \hookrightarrow$ is locally closed, U is smooth, connected and of dimension d and $\mathcal{E}$ is an irreducible local system on U.

In the situation

$$\begin{array}{c} V \xhookrightarrow{\;i\;} U \xhookrightarrow{\;j\;} X, \\ \underbrace{\qquad\qquad\qquad\quad}_{\alpha} \end{array}$$

Given $\mathcal{F} \in \operatorname{Perv}(V)$, $\alpha_{!*}\mathcal{F} = j_{!*}(i_{!*}\mathcal{F})$.

For $i : U \hookrightarrow X$ any smooth dense open subset, $IC_X = j_{!*}\mathbb{C}_U[\dim U]$, $\mathbb{D}IC_X = IC_X$ and $H^i(X, IC_X)^* = H_c^{-i}(X, IC_X)$.

How to compute IC_X

Example 3.9. Let $X = \{v \in \mathbb{C}^n | Q(v) = 0\}$ for some non–degenerate quadratic form Q.

$n = 2 :\ xy = 0$

$$IC_X = \mathbb{C}_{X_1}[1] \oplus \mathbb{C}_{X_2}[1]$$

where X_1 (X_2) is given by $z_2 = 0 (z_1 = 0)$.

$n = 3 :\ \{xy = z^2\} \simeq \mathbb{C}^2/\pm 1.$

Exercie 3.10. Let Γ be a finite group acting on a smooth variety Y. Let $X = Y/\Gamma$. Then $IC_X = \mathbb{C}_X[\dim X]$.

$n = 4 :$ Want a nice map $\pi : \tilde{X} \to X$ with $\tilde{X}$ smooth and π proper and generically an isomorphism. There exists a resolution such that π is an isomorphism away from 0 and such that $\pi^{-1}(0) \simeq \mathbb{P}^1$.

Claim 3.5. $IC_X = \pi_*\mathbb{C}_{\tilde{X}}[3]$.

To construct the resolution, identify $\mathbb{C}^4 = \operatorname{Mat}(2 \times 2, \mathbb{C})$ and identify Q with the determinant so that $X =$ degenerate 2×2 matrices. Take $\tilde{X} = \{x \in X, \ell \in \mathbb{P}^1 : x|_\ell = 0\}$. This clearly maps to $\mathbb{P}^1$ and in fact $\eta : \tilde{X} \to \mathbb{P}^1$ is a vector bundle of rank 1 since $\eta^{-1}(\ell) = \overline{\operatorname{Hom}}(\mathbb{C}^2/\ell, \mathbb{C}^2)$. There is also a map $\pi : \tilde{X} \to X$ with fibre $\mathbb{P}^1$.

Given $j : U \hookrightarrow X$ smooth and $\mathcal{E}$ a local system on on U, let $\mathcal{F} = j_{!*}\mathcal{E}$. Given a stratification $X = X_0 \supset X_1 \supset X_2 \supset \cdots$ with $S_i = X_i \backslash X_{i+1}$ smooth, $\mathcal{F}$ is locally constant on S_i's. Perversity implies $\mathcal{F}|_{S_i} \in D^{\leq -\dim S_i}$ and the dual condition.

Have

$$\begin{array}{c}\tilde{X}\\ \big\downarrow{\scriptstyle\pi}\\ X = X\backslash\{0\}\bigsqcup\{0\}\end{array}$$

$\pi_*\mathbb{C}_{\tilde{X}}[3]$ is self–dual and $\pi_*\mathbb{C}_{\tilde{X}}[3]|_{\{0\}}$ lives in degrees $-3,-1$ so

$$H^*(X, IC_X) = H^*(\tilde{X}, \mathbb{C})[3] = H^*(\mathbb{P}^1, \mathbb{C})[3].$$

Theorem 3.11 (Decomposition). *Let $\pi : X \to Y$ be a proper morphism and $\mathcal{F} \in \mathrm{Perv}(X)$ which is irreducible. Then $\pi_*\mathcal{F}$ is a semi–simple complex, ie. it is isomorphic to a direct sum of things of the form $\mathfrak{g}[i]$ where $\mathfrak{g}$ is an irreducible perverse sheaf on Y.*