Solutions Quiz 2

Find the general solution of the homogeneous equation $y^{iv} - y''' - y'' + y' = 0$. Find one solution of the inhomogeneous equation $y^{iv} - y''' - y'' + y' = 4\cos t$. Using the result from part b) find infinitely many solutions of $y^{iv} - y''' - y'' + y' = 4\cos t$ such that $\lim_{t\to\infty} |y(t)| \neq \infty$.

Solution. The characteristic polynomial $r^4 - r^3 + r^2 + r$ has roots $r_1 = 0$, $r_2 = r_3 = 1$ and $r_4 = -1$. The general solution is then

$$y(t) = c_1 + c_2 e^t + c_2 t e^t + c_4 e^{-t}, \;\; c_1, c_2, c_3, c_4 \in \mathbf{R}$$

We seek next a particular solution of the form $Y(t) = A\sin t + B\cos t$. Compute

$$\begin{aligned} Y'(t) &= A\cos t - B\sin t, \\ Y''(t) &= -A\sin t - B\cos t, \\ Y'''(t) &= -A\cos t + B\sin t, \\ Y^{iv}(t) &= A\sin t + B\cos t. \end{aligned}$$

Plug into the equation to get $2(A - B)\sin t + 2(A + B)\cos t = 4\cos t$ hold for all t. Hence $A + B = 1$ and $Y(t) = \sin t + \cos t$. The general solution for the inhomogeneous equation is then

$$y(t) = c_1 + c_2 e^t + c_2 t e^t + c_4 e^{-t} + \sin t + \cos t, \;\; c_1, c_2, c_3, c_4 \in \mathbf{R}.$$

If either of the constants c_1 or c_2 is non-zero then we have $\lim_{t\to\infty} |y(t)| = \infty$. However, any y of the form $y(t) = c_1 + c_4 e^{-t} + \sin t + \cos t$ stays bounded. Since c_1 and c_4 are arbitrary, we exhibited infinitely many such solutions.