Solutions Midterm Exam, MAT244

1. Consider the equation $y'' + 4y' + 4y = 0$. Find its general solution. Find the solution which also satisfies $y(0) = \lambda$, $y'(0) = 0$, where $\lambda \in \mathbf{R}$ is a parameter. Find one value of λ for which the corresponding solution from part b) satisfies $|y(t)| \leq 1$ for all $t \in \mathbf{R}$.

Solution: The characteristic equation $r^2 + 4r + 4 = 0$ has a double root $r = -2$. Hence e^{-2t} and te^{-2t} are both solutions. They have a non-zero Wronskian hence the general solution is $y(t) = c_1 + e^{-2t} + c_2 te^{-2t}$, $c_1, c_2 \in \mathbf{R}$. The initial conditions $y(0) = c_1 = \lambda$ and $y'(0) = -2c_1 + c_2 = 0$. Therefore the solution of the initial value problem is $y(t) = \lambda e^{-2t} + 2\lambda te^{-2t}$. For $\lambda = 0$ we have $y \equiv 0$ satisfies $|y(t)| \leq 1$.

2. Let $L[y](t) \equiv y''(t) + y'(t) - 2y(t)$, for $t \in \mathbf{R}$. Find the general solution of the homogeneous equation $L[y] = 0$. Find a particular solution of the inhomogeneous equation $L[y] = e^t$. Solve the initial value problem $L[y](t) = te^t$, $y(0) = y'(0) = 0$.

Solution: The characteristic equation $r^2 + r - 2 = 0$ has solutions $r_1 = 1$ and $r_2 = -2$. Hence $y_0(t) = c_1 e^t + c_2 e^{-2t}$, $c_1, c_2 \in \mathbf{R}$,is the general solution of the homogeneous equation. We look for a particular solution $Y(t)$ of the inhomogeneous equation of the form $Y(t) = (At^2 + Bt)e^t$. Calculate:

$Y'(t) = (At^2 + (2A + B)t + B)e^t$

$Y''(t) = (At^2 + (4A + B)t + 2A + 2B)e^t$.

Plug into the equation to get that $6At + 2A + 3B = 9t$ must hold for all $t \in \mathbf{R}$. Therefore $A = 1/6$ and $B = -1/9$. The general solution of the inhomogeneous equation is then $y(t) = c_1 e^t + c_2 e^{-2t} + t^2/6 - t/9)e^t$. The initial condition imply $y(0) = c_1 + c_2 = 0$ and $y'(0) = -2c_1 + c_2 = 1/9$. Solve to get $c_1 = -1/27$ and $c_2 = 1/27$.

3. A fresh college graduate thinks of buying a condominium. A bank is offering him credit at an interest of 8% APR (annual percentage rate). He expects to make fixed payments of \$1200 each month. Also he wants to pay off his loan in 25 years. Set up an equation for $S(t)$, his debt after t months. Write down the solution which satisfies the initial condition $S(0) = S_0$. How big of an amount can he borrow from the bank? (you may need $1 - e^{-2} \simeq 0.8647$.)

Solution: Let $S(t)$ denote the amount of money he owes the bank after t months (hence t is measured in months). The rate of interest is $r = 8/12\% = 1/150$. The **rate of change** of his debt after t months increases because of the interest by $rS(t)$ and decreases with the payment. The equation is $S'(t) = rS(t) - 1200$. The integrating factor is $e^{-rt} = e^{-t/150}$. We get $\frac{d}{dt}\left(e^{-rt}S(t)\right) = -1200e^{-rt}$. Integrating, we have $e^{-rt}S(t) = \frac{1200}{r}e^{-rt} + c$. Using that $S(0) = S_0$ we identify the constant and write the solution as $S(t) = \frac{1200}{r} + \left(S(0) - \frac{1200}{r}\right)e^{rt}$. Since 25 years have 300 months, we want $S(300) = 0$ (the debt is paid off after 300 months). This gives $S(0) = 180000(1 - e^{-2}) \simeq 155646$.

4. Consider a second order, linear, homogeneous equation $y''(t) + p(t)y'(t) + q(t)y(t) = 0$, with p and q continuous maps on $\mathbf{R}$. Can $y(t) = (t-1)^2$ be its solution on the interval $(0, 2)$?

Solution: **NO**. Otherwise, both $(t-1)^2$ and $y \equiv 0$ would be solutions of the initial value problem $y''(t) + p(t)y'(t) + q(t)y(t) = 0$, with $y(1) = y'(1) = 0$. We know that such a problem has a unique solution.