Final Exam Helper

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This document is meant to help you review the material for the ODE exam. If you know the answer to the questions below and if you've been solving the assigned homework problems all long then you are well prepared to take the exam.

1 First Order Equations

Find $y : I \subset \mathbf{R} \to \mathbf{R}$ which solves $y'(t) = f(t, y(t))$. Sometimes we specify the initial condition $y(t_0) = y_0$. There are special cases

1. Linear, inhomogeneous $y'(t) + a(t)y(t) = g(t)$. Multiply by the integrating factor $e^{\int a(t)dt}$ to get an equation of the form $z'(t) = f(t)$ which can be solved by integration.

2. Separable variables: $M(x) + N(y)y'(x) = 0$. Rewrite as $M(x)dx = -N(y)dy$ and integrate both sides.

3. Exact equations $N(x, y) + M(x, y)y'(x) = 0$, when $N_y = M_x$. We can find a function F from integrating in x: $F(x, y) = \int N(x, y)dx + c(y)$ then determine $c(y)$ from compatibility condition $\frac{\partial N}{\partial y}(x, y) = \frac{\partial M}{\partial x}(x, y)L$.

4. Inexact equations $N(x, y) + M(x, y)y'(x) = 0$, when $N_y \neq M_x$ but which admit an integrating factor $\mu(x, y)$. They transform into exact $\mu(x, y)N(x, y) + \mu(x, y)M(x, y)y'(x) = 0$. You need to remember (or to deduce) the equations for $\mu = \mu(x)$ or $\mu = \mu(y)$ (when possible). This reduces to the previous case Here we will only consider one of these two cases.

When getting an equation, we need to check first in which situation doest it fall? What method to use? What is the unknown and what is the argument of the unknown? Integration with respect to which variable?

More questions. (Here you need to understand the concept not to memorize a definition) What is a general solution? a special solution? How do you find a special solution from initial data? What is a nonlinear problem (last two cases in the list above)? What is the difference between homogeneous and inhomogeneous equation?

2 Modelling questions

How do I model a mixing problem? Let Q be the mass of salt in a container of volume V.

$$\frac{dQ(t)}{dt} = \text{rate in} - \text{rate out}$$

If the debit of the flow in = debit of the flow out then the volume of the mix in the tank does not change. If unequal then $V = V(t)$ itself can be a function of time and then the concentration of the salt coming out is $Q(t)/V(t)$.

How do I model an interest rate problems? Recall the ones in the midterm.

3 Linear second order equations

$L[y] \equiv y'' + p(t)y' + q(t) = g(t)$. Where is the solution of the equation defined (where p and q are continuous). What is the difference between homogeneous and inhomogeneous? What is the Wronskian? What does Abel's formula says? When are two solutions linearly independent? When are two general functions linearly independent? What;s the connection with the Wronskian? What is a general solution? How many conditions we need in the initial value problem? How do you find such a solutions (**it is unique !**)? Given a particular solution, can you find the other (reduction of order)? Given two linearly independent homogeneous how do you find a special solution of the inhomogeneous equation (variation of parameter)? Do I know how to find special solutions when the right hand side is of the form $e^{\alpha t}P_n(t)$ with P_n a polynomial of degree 1 or 2? What if α is complex valued (bring in sin and cos)? What if α is a root of the characteristic polynomial?

4 n-th order ODE

We consider two cases.

1. of the type $x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + ... + a_n y = f(x)$, which by the substitution in the dependent variable $x = e^t$ reduces to the case of constant coefficients. Can you handle the chain rule for reduction above in the case $n = 3$?

2. In the constant coefficients case $y^{(n)} + a1^y(n-1) + ... + a_n y = f(t)$ we assume $y(t) = e^{rt}$. What is the characteristic equation? How do you find n linearly

independent solutions? For which $f(t)$ can you find a particular solution of a special form? How do you check linear independence here? What is the $\sqrt{i}$? What are the values of $(i)^{1/3}$?(Hint: Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$). Revise the homework questions.

5 Linear Systems of ODE

Given an $n \times n$ matrix A with real entries, consider the linear system $y' = Ay$. Here y is a functin valued vector $y(t) = (y_1(t), ..., y_2(t))$. Can you write down the linear system which models the concentration of salt in the situation of two interconnected tanks?

Look for solutions of the form $x = \xi e^{rt}$. What is the characteristic equation of A? What is r for this equation? What is the vector ξ for r? What is an eigenvalue of A? What is an eigenvector corresponding to an eigenvalue? How do you find independent solutions? What is the general solution once you know n linearly independent solutions? Given n mutually distinct eigenvalues of A: $\lambda_1, .., \lambda_n$, how do you find n linearly independent solutions? How do you use this to find the exponential of A? How do you find e^{At}? What is the Wronskian of n functions? What's the relation between the Wronskian of n solutions of a linear system and their linear dependence? What is the characteristic equation? How do you use the eigenvalues to construct linearly independent solutions? How do you handle the repeated roots case? What is a generalized eigenvector? How do you handle (simple) complex eigenvalues? How do you find a particular solution for the inhomogeneous case?

The case of a 2×2 system. What is the phase space, a trajectory? In which case the origin is a sink, source or a saddle point? Consider the system $y' = Ay$, where

$$A = \begin{pmatrix} \alpha & 1 \\ -1 & 1 \end{pmatrix}$$

Describe the values of α for which the origin is a sink,source or a saddle point. For which values od α is $(0,0)$ is a stable/unstable solution?