A Useful Model

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These slides and some notes expanding on them can be found at www.math.utoronto.ca/tall. Also there can be found references and/or preprints for:

1. Locally compact perfectly normal spaces may all be paracompact (with P. Larson).
2. On the hereditary paracompactness of locally compact hereditarily normal spaces (with P. Larson).
3. PFA(S)[S] and the Arhangel'skiĭ-Tall problem.
4. PFA(S)[S] and locally compact normal spaces.
5. PFA(S)[S]: mutually consistent topological consequences of PFA and $V = L$.

The following paper will be posted in the near future:

6. Forcing with a coherent Souslin tree and compact countably tight spaces (with A. Fischer and S. Todorcevic).

I. Introduction

For a set-theoretic topologist it is useful to have consequences of PFA and $V = L$ in the same model. Such a model was used to obtain, e.g.,

1. locally compact, perfectly normal spaces are paracompact,
2. (locally) compact spaces with hereditarily normal squares are metrizable, [LT02], [LT10].

The two most useful consequences are:

a) Normal spaces which are locally compact or first countable are collectionwise Hausdorff (CWH),

b) $\sum$: Let X be compact and countably tight. Then locally countable subspaces of size $\aleph_1$ are σ-discrete. More particularly, if X is compact and countably tight, and $Z \subseteq X$ is such that $|Z| \leq \aleph_1$ and there exists a collection $\mathcal{V}$ of open sets, $|\mathcal{V}| \leq \aleph_1$, and a collection $\mathcal{U} = \{U_V : V \in \mathcal{V}\}$ of open sets, such that $Z \subseteq \bigcup \mathcal{V}$, and for each $V \in \mathcal{V}$, there is a $U_V \in \mathcal{U}$ such that $V \subseteq \overline{V} \subseteq U_V$, and $|U_V \cap Z| \leq \aleph_0$, then Z is σ-closed discrete in $\bigcup \mathcal{V}$.

Other consequences include:

a) Axiom R (defined below),
b) P-ideal Dichotomy (defined below).
c) compact countably tight spaces are sequential,
d) every compact countably tight space has a point of countable character.
e) first countable hereditarily Lindelöf spaces are hereditarily separable,
f) $\mathfrak{b} = \aleph_2$, $\mathfrak{p} = \aleph_1$,
g) the Open Coloring Axiom,
h) every Aronszajn tree is special,

Treating this model as a "black box", we have:

Theorem 1

In this model, locally compact hereditarily normal spaces satisfying the countable chain condition are hereditarily Lindelöf and hereditarily separable.

Proof.

X^* is compact T_5, so hereditarily CWH. $s(X^*) = t(X^*) = \aleph_0$. If not hL, has right-separated ($\beta < \alpha \to \alpha \notin U_\beta$) subspace of size $\aleph_1$. Then by $\sum$ it has uncountable discrete subspace, contradiction. X^* is hL so first countable. But there are no first countable L-spaces, so X^* is hereditarily separable. $\square$

Corollary 2

In this model, there are no locally compact separable hereditarily normal Dowker spaces.

Corollary 3

In this model, compact homogeneous hereditarily normal spaces are first countable.

Proof.

Juhász, Nyikos, Szentmiklóssy [JNS05] proved:

Lemma 4

Homogeneous compacta which are T_5 and hereditarily $\aleph_1$-CWH are countably tight.

Combine this with d) to prove Corollary 3. □

The proof of d) is difficult; an easier proof of Corollary 3 can be obtained by quoting [JNS05] and proving open Lindelöf subspaces have hereditarily Lindelöf closures via the methods we shall be using here.

II. The Model

Let S be a *coherent* (defined later) Souslin tree (obtainable from $\diamondsuit$ or by adjoining a Cohen real). As in the consistency proof for PFA (which requires a supercompact cardinal), iterate proper posets which preserve S. This yields PFA(S): PFA restricted to posets which preserve S. This iteration preserves S [Miy93]. Now force with S. Any model obtained by forcing with a coherent S over PFA(S) is called a **model of PFA**(S)[S]. We write **PFA**(S)[S] **implies** Φ to mean Φ holds in every model of PFA(S)[S].

A. More powerful models of PFA$(S)[S]$

1. Start with a model for which $\diamondsuit$ (for stationary systems) holds for regular $\lambda \geq \aleph_2$, e.g. adjoin λ^+ Cohen subsets of λ for all regular λ. These $\diamondsuit$ principles will be preserved for $\lambda \geq \aleph_2$ by the rest of the forcing, so that if the PFA$(S)[S]$ forcing establishes, e.g., locally compact normal $\rightarrow \aleph_1$-CWH, we will get full CWH.
2. Instead of PFA(S), obtain MM(S) or PFA$^+(S)$. The resulting model for PFA$(S)[S]$ will satisfy **Axiom R**

B. Models for weaker versions of PFA$(S)[S]$

Just as with PFA $\rightarrow$ there are no S-spaces, one can take care as to the posets one is iterating, so as to establish consistency results not requiring large cardinals. This usually works for statements about $\aleph_1$.

III. Topological Consequences

Theorem 5

PFA$(S)[S]$ implies that if X is a locally compact normal space with $L(X) \leq \aleph_1$, and X includes no perfect pre-image of ω_1, then X is paracompact.

Corollary 6

PFA$(S)[S]$ implies every locally compact normal space of size $\leq \aleph_1$ with a G_δ-diagonal is metrizable.

The corollary follows easily. For the theorem, recall:

Lemma 7 ([Bal83])

If X is locally compact, $t(X^*) = \aleph_0$ iff X does not include a perfect pre-image of ω_1.

Also note that:

Lemma 8

$\sum$ implies that if X is locally compact, includes no perfect pre-image of ω_1, $L(X) \leq \aleph_1$, and $Y \subseteq X$, $|Y| = \aleph_1$ is such that each point in X has a neighbourhood meeting only countably many y, then Y is σ-closed-discrete.

We also need some results of Nyikos:

Definition 9

A space X is of **Type I** if $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where each X_α is open, $\alpha < \beta$ implies $\overline{X}_\alpha \subseteq X_\beta$, and each $\overline{X}_\alpha$ is Lindelöf. $\{X_\alpha : \alpha < \omega_1\}$ is **canonical** if for limit α, $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$.

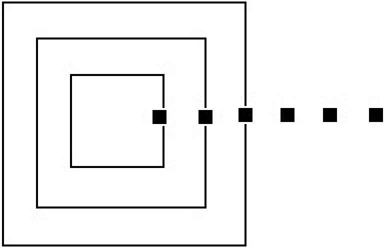
Lemma 10 ([Nyi03])

If X is locally compact, $L(X) \leq \aleph_1$, and every Lindelöf subset of X has Lindelöf closure, then X is of Type I, with a canonical sequence.

Lemma 11 ([Nyi83])

If X is of Type I, then X is paracompact if and only if $\{\alpha : \overline{X}_\alpha - X_\alpha \neq 0\}$ is non-stationary.

Once we establish (in a minute) that Lindelöf subsets of X have Lindelöf closure, the rest of the proof of Theorem 5 is easy. If X is paracompact, this is straightforward. Suppose X were not paracompact. X is of Type I so we may pick a canonical sequence and we may pick a stationary $S \subseteq \omega_1$ and $x_\alpha \in \overline{X}_\alpha - X_\alpha$, for each $\alpha \in S$. $\{x_\alpha : \alpha \in S\}$ is σ-closed-discrete, so there is a stationary set of limit ordinals $S' \subseteq S$ such that $\{x_\alpha : \alpha \in S'\}$ is closed discrete. Let $\{U_\alpha : \alpha \in S'\}$ be a discrete collection of open sets expanding it. Pressing down yields an uncountable closed discrete subspace of some X_α, contradiction.



To see that closures of Lindelöf subspaces are Lindelöf, if $Y \subseteq X$ is Lindelöf, but $\overline{Y}$ isn't, $\overline{Y}$ includes a set Z of size $\aleph_1$ with no complete accumulation point. Since $L(\overline{Y}) \leq \aleph_1$, by $\sum$ Z is σ-closed-discrete. Take an uncountable discrete $Z' \subseteq Z$ and expand it to obtain an uncountable discrete collection of open sets. Their traces on Y violate its Lindelöfness.

Definition 12

$\Gamma \subseteq [X]^{<\kappa}$ is **tight** if whenever $\{C_\alpha : \alpha < \delta\}$ is an increasing sequence from Γ and $\omega < \mathrm{cf}(\delta) < \kappa$, then $\bigcup\{C_\alpha : \alpha < \delta\} \in \Gamma$.

Axiom R: If $\Sigma \subseteq [X]^\omega$ is stationary and $\Gamma \subseteq [X]^{<\omega_2}$ is tight unbounded, then $\exists Y \in \Gamma$ such that $\mathcal{P}(Y) \cap \Sigma$ is stationary in $[Y]^\omega$.

Using Axiom R, we now get

Theorem 13 ([Tala])

PFA(S)[S] implies a locally compact normal space X is paracompact and countably tight iff it includes no perfect pre-image of ω_1 and the closure of each Lindelöf subspace in Lindelöf.

Proof: $\rightarrow$ is easy; for $\leftarrow$, reduce to the previous lemma. We need:

Lemma 14 ([Bal02])

Axiom R implies that if X is a locally Lindelöf, regular, countably tight space such that every open Y with $L(Y) \leq \aleph_1$ has $L(\overline{Y}) \leq \aleph_1$, then if X is not paracompact, it has a clopen non-paracompact subspace Z with $L(Z) \leq \aleph_1$.

Lemma 15 ([Bal02])

Axiom R implies that if X is locally Lindelöf, regular, countably tight, and not paracompact, then X has an open subspace Y with $L(Y) \leq \aleph_1$, such that Y is not paracompact.

Lemma 16

If Y is a subset of a locally Lindelöf space of countable tightness in which closures of Lindelöf subspaces are Lindelöf, then if $L(Y) \leq \aleph_1$, then $L(\overline{Y}) \leq \aleph_1$.

We can improve Theorem 5 to obtain:

Theorem 17 ([Tala])

PFA(S)[S] implies a locally compact normal space X is paracompact and countably tight iff it includes no perfect pre-image of ω_1, and the closure of each countable subspace is Lindelöf.

Corollary 18 ([LT])

PFA(S)[S] implies locally compact hereditarily normal spaces are (hereditarily) paracompact and countably tight iff they do not include perfect pre-images of ω_1.

Corollary 19 ([LT10])

PFA(S)[S] implies locally compact perfectly normal spaces are paracompact.

Corollary 20 ([LT10])

PFA(S)[S] implies if X is locally compact T_5 and has a G_δ-diagonal, then X is metrizable.

Corollary 21 ([LT02], [LT10])

PFA(S)[S] implies if X is locally compact and X^2 is T_5, then X is metrizable.

The first corollary follows because the closure of a countable subspace is a locally compact, separable, T_5 space, so is hereditarily Lindelöf. The second because perfectly normal spaces do not include perfect pre-images of ω_1. For the third, X is locally metrizable and paracompact. For the fourth, we first show X is paracompact because it does not include a perfect pre-image P of ω_1, for if it did, since P^2 is T_5 and P is countably compact, by a theorem of Katetov, P would be perfectly normal, contradiction. To show X is metrizable, a compact neighborhood N in X would have T_5 square, so N would be perfectly normal by Katětov. Then N is hL and hs, so N^2 is separable. N^2 then is hL and perfectly normal so N has a G_δ-diagonal and thus is metrizable.

To prove Theorem 17, we use P-**ideal Dichotomy**:

Definition 22

A collection $\mathcal{I}$ of countable subsets of a set X is a **P-ideal** if each subset of a member of $\mathcal{I}$ is in $\mathcal{I}$, finite unions of members of $\mathcal{I}$ are in $\mathcal{I}$, and whenever $\{I_n : n \in \omega\} \subseteq \mathcal{I}$, there is a $J \in \mathcal{I}$ such that $I_n - J$ is finite for all n.

PID: For every P-ideal $\mathcal{I}$ on a set X, either

i) there is an uncountable $A \subseteq X$ such that $[A]^{\leq\omega} \subseteq \mathcal{I}$

or ii) $X = \bigcup_{n<\omega} B_n$ such that for each n, $B_n \cap I$ is finite, for all $I \in \mathcal{I}$.

Lemma 23 ([EN09])

PID *implies that if* X *is a locally compact space, then either*

a) X *is the union of countably many* ω*-bounded subspaces,*

or b) X *does not have countable extent,*

or c) X *has a separable closed subspace which is not Lindelöf.*

Recall a space is **ω-bounded** if every countable subspace has compact closure. ω-bounded spaces are obviously countably compact.

From [Gru77] we have:

Lemma 24

An ω-bounded space is either compact or includes a perfect pre-image of ω_1.

We can now prove Theorem 17.

The forward direction is routine. To prove the other direction, it suffices to show that if Y is a Lindelöf subspace of our space X, then $\overline{Y}$ is Lindelöf. Applying **PID**, we see that by Lemmas 23 and 7, $\overline{Y}$ will be σ-compact if we can exclude alternatives b) and c). c) is excluded by hypothesis, so it suffices to show that $\overline{Y}$ has countable extent. But that is easily established, since $\overline{Y}$ is locally compact normal and hence $\aleph_1$-**CWH**. A closed discrete subspace of size $\aleph_1$ in $\overline{Y}$ could thus be fattened to a *discrete* collection of open sets. Their traces in Y would contradict its Lindelöfness. $\square$.

Corollary 25

There is a model of form $\mathrm{PFA}(S)[S]$ *in which a locally compact space is metrizable if and only if it is normal, has a* G_δ*-diagonal, and every separable closed subspace is Lindelöf.*

Proof.

Theorem 17 applies, since spaces with G_δ-diagonals do not include perfect pre-images of ω_1. □

Some Preliminaries

Lemma 26 ([Miy93])

$\mathcal{P}$ is proper and preserves S if for all sufficiently large regular θ and for a closed unbounded family $\mathcal{C}$ (in $[H_\theta]^{\aleph_0}$) of countable elementary submodels M of H_θ with $\mathcal{P}, S \in M$, letting $\delta = M \cap \omega_1$, for every $p \in \mathcal{P} \cap M$, there is a $q \leq p$ such that for all $s \in S$ of height δ, $\langle q, s \rangle$ is $(\mathcal{P} \times S, M)$-generic.

The overall strategy for using Miyamoto's lemma is the same as in the proof that PFA implies there are no S-spaces, and many other proofs as well: "copy" the "growth" of a condition into an elementary submodel by a finite induction, using elementarity at each step.

A method of proof

For many more topological applications, see [Tala]. The fixation on locally compact normal spaces reflects what I have done and does not imply this is the only area of application. The remainder of the workshop will concentrate on one method of proof, which yields PFA(S)[S] *implies both* $\sum$ and *locally compact normal spaces are* $\aleph_1$*-CWH*.

An old idea (Kunen): destroy an S-space by forcing an uncountable left-separated subspace through a right-separated subspace of size $\aleph_1$. Balogh: we can get the right-separated subspace to be the union of countably many such discrete subspaces. Additional topological hypotheses (compact countably tight) are needed to ensure the forcing is c.c.c. Todorcevic: Don't need extra hypotheses to get proper, but current proofs need such extra hypotheses in PFA(S)[S] context.

Todorcevic's proof [Tod] that $\mathrm{PFA}(S)[S]$ implies there are no compact S-spaces depends on showing that such spaces are sequential. This allows him to reduce an uncountable amount of information down to a countable amount, which Souslin tree forcing can handle. Our proof that PFA(S)[S] implies locally compact normal spaces are $\aleph_1$-CWH is along the same lines: we in effect use the fact that any countably infinite subset of an uncountable closed discrete subspace in a locally compact normal space has a discrete expansion by compact G_δ's which converges to the point at infinity in the one-point compactification of the space.

Large portions of PFA(S)[S] proofs are independent of the particular problem we are working on, but instead involve general properties of Souslin trees, in particular, coherent ones. To emphasize this and to render the technology more accessible we have organized much of the proofs of a weak version of $\sum$ and of the CWH theorem as a sequence of lemmas and notation having nothing to do with topology.

Intuitively, what coherence does for us is it deals with the following problem: in trying to go from a PFA proof to a $\mathrm{PFA}(S)[S]$ proof, we have much less control over what the $\mathcal{P}$-generic S-name becomes when we force with S, than we would have over simply an object — rather than a name — we construct with PFA. A coherent Souslin tree, however, has — up to automorphism — only one generic branch. Therefore the possible interpretations of a name will be "isomorphic," i.e. although there are many possible objects to deal with, they are all essentially the same. We do not yet, however, have a clear understanding of under which circumstances this intuition leads to a $\mathrm{PFA}(S)[S]$ proof from a PFA proof.

It is somewhat easier to force an uncountable discrete subspace than it is to make the latter σ-discrete; however, once one has figured out the right notation, it is not too much more difficult to do the σ-discrete version. Moreover, it is a useful technique I have not found written anywhere other than in my recent as yet unpublished papers, although it is due to Todorcevic. So we will do σ-proofs.

Lemma 27

Let S be a Souslin tree and N a countable elementary submodel of some H_θ containing S. Suppose $A \subseteq S$, $A \in N$, $t \in A - N$. Suppose there is an $s \in S \cap N$, s below t. Then there is a $u \in [s, t) \cap N$ such that A is dense above u.

Proofs of this lemma and others that I omit here can be found either in the Useful Model notes which will be posted on my website, or in my "Mutually Consistent" paper posted there.

Definition 28

An **m-chain with possible repetitions** is an m-tuple $\langle a_1, \ldots, a_m \rangle$, each $a_i \in S$, such that a_{i+1} extends a_i. We admit the possibility that $a_{i+1} = a_i$.

Definition 29

Let $\mathcal{A}$ be a family of chains with possible repetitions of a Souslin tree S. $\mathcal{A}$ is **dense above** $s \in S$ if for each s' extending s, there is an $A \in \mathcal{A}$ such that $\min A$ extends s'. We shall use "s' above s" and "s' extends s" synonymously, and admit the possibility that $s' = s$.

Corollary 30

Let S be a Souslin tree and N a countable elementary submodel of some H_θ containing S. Suppose $\mathcal{A}$ is a family of chains with possible repetitions of S, $\mathcal{A} \in N$, and suppose there is an $A_0 \in \mathcal{A}$, $\min A_0 \notin N$. Suppose $s \in S \cap N$, s below $t = \min A_0$. Then there is a $u \in S \cap N$, $u \in [s, t)$, such that $\mathcal{A}$ is dense above u.

How to force an uncountable discrete subspace of a space X in a model of PFA(S)[S].

Find a $\mathcal{P}$ that is proper and preserves S, such that the branches of S code discrete subspaces of X. Then after forcing with S, the generic branch of S will code an uncountable discrete subspace of X. Of course X itself is also not “known” until S is forced with. Intuitively, coherence assures us that all generic branches are isomorphic, so that the possible X's are homeomorphic.

Definition 31

A **coherent** tree is a downward closed subtree S of ${}^{<\omega_1}\omega$ with the property that $\{\vec{\xi} \in \text{dom } s \cap \text{dom } t : s(\vec{\xi}) \neq t(\vec{\xi})\}$ is finite for all $s, t \in S$. A **coherent Souslin tree** is a Souslin tree given by a coherent family of functions in ${}^{<\omega_1}\omega$ closed under finite modifications.

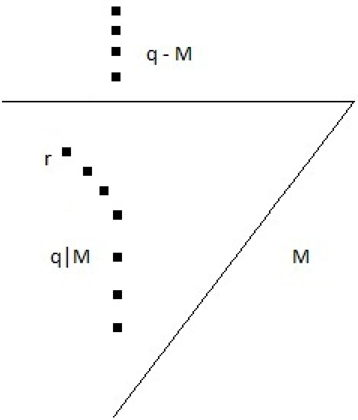
For S a coherent Souslin tree, and s, t on the same (ηth) level of S, there is a canonical isomorphism σ^S_{st} between the cones above (we think of our trees as growing upwards) s and t, defined by letting $\sigma^S_{st}(s')(\alpha)$ be $t(\alpha)$ if $\alpha < \eta$ and $s'(\alpha)$ otherwise, for each s' extending s. These isomorphisms are such that $\sigma^S_{su} = \sigma^S_{tu} \circ \sigma^S_{st}$ and $\sigma^S_{st} = (\sigma^S_{ts})^{-1}$.

We shall work in a model of PFA(S). We construct a $\mathcal{P}$ there and show it is proper and preserves S. Typically, the members of $\mathcal{P}$ look like $\langle f_p, \mathcal{N}_p \rangle$ where f_p is a finite partial function, and $\mathcal{N}_p$ is a finite $\in$-chain of elementary submodels of a sufficiently large H_κ. We think of $f_p^{-1}(\{n\})$ as a "level" and require that $\mathcal{N}_p$ *separates* each level in the sense that if $f_p(s) = f_p(s') = n$, then if $s \neq s'$, there is an $N \in \mathcal{N}_p$ such that $s \in N$ and $s' \notin N$.
Let M be a countable elementary submodel of H_θ containing everything relevant. (There will be a club of such M's.) Let $\delta = M \cap \omega_1$. Let $p \in \mathcal{P} \cap M$. Let $p^M = \langle f_p, \mathfrak{N}_p \cup \{M \cap H_\kappa\} \rangle$. Then, by a standard argument, $p^M \in \mathcal{P}$.

The advantage of the method of proper forcing with elementary submodels as side conditions is that you know what the generic condition should be, namely p^M. In the PFA(S)[S] situation, there is a natural variation on this.

Let t_M be an arbitrary node at the δth level of S. We will show $\langle p^M, t_M \rangle$ is generic. Let $\mathcal{D} \in M$ be a given dense open subset of $\mathcal{P} \times S$ and let $\langle q, t \rangle$ be a given extension of $\langle p^M, t_M \rangle$. We need to show $\langle q, t \rangle$ is compatible with some member of $\mathcal{D} \cap M$. Extending $\langle q, t \rangle$, we may assume that $\langle q, t \rangle \in \mathcal{D}$. Moreover, by extending further (since $\mathcal{D}$ is open), we may assume that t is not in the largest model of $\mathfrak{N}_q$, and that this model contains all the members of dom f_q. (Here and elsewhere I make assertions that require proofs a few lines long. If you can't work them out yourself, they can be found in my "Mutually Consistent" paper.)

Let $q_M = q \,|\, M$. What this means will depend on the partial order $\mathcal{P}$. E.g. if f_q is a single partial function, $q \,|\, M = \langle f_q \cap M, \mathcal{N}_q \cap M \rangle$. Since finite subsets of M are members of M, $q_M \in M$. If the partial order is at all reasonable, q_M will be in M and q will extend it. Here is the picture, temporarily ignoring the t's:

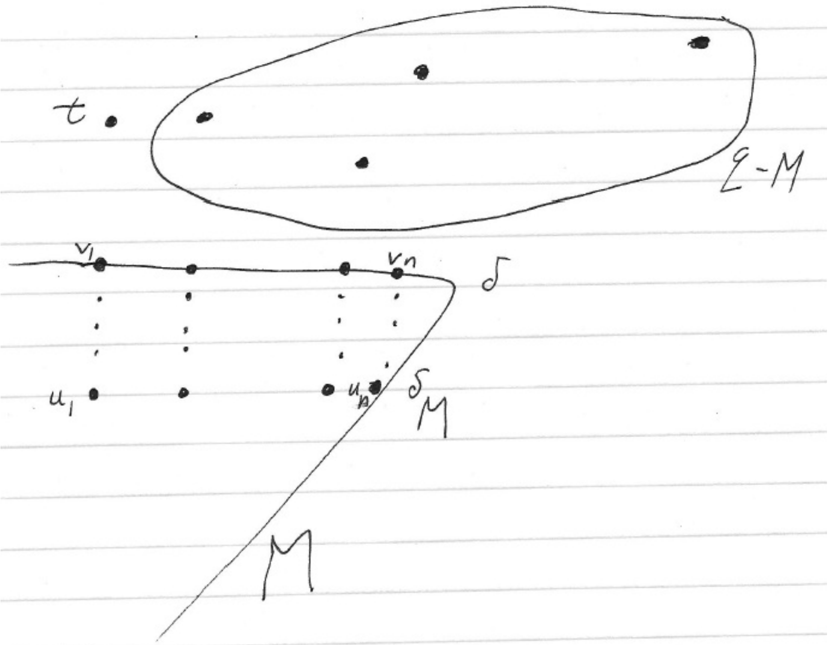


$\langle q, t\rangle \in \mathcal{D}$; we want to get an $\langle r, t_r\rangle \in M$, t_r below t, such that $\langle r, t_r\rangle$ "is just like $\langle q - M, t\rangle$" and in particular, $\langle$the common extension of r and $q|M, t_r\rangle$ is in $\mathcal{D}$. By elementarity this is not difficult to do; it is not difficult to take r sufficiently high up in M so that it is compatible with $q|M$. The hard part is to find such an r which is also compatible with $q - M$. This is accomplished by a finite induction, at each stage of which one decides one of the finitely many elements of r. Toward making that decision, one constructs $\aleph_1$ many possible candidates for that element. Then, finally using not just general machinery, but the particulars of the partial order and the assumptions of whatever result one is trying to prove, one argues that there are $\aleph_0$ of the possible candidates such that almost all of them are suitable – they don't conflict with $q - M$. This is the place where one uses e.g. easy topological facts about sequences like that if $x_n \to x \notin \overline{U}$, then almost all $x_n \notin \overline{U}$. This process is repeated finitely many times, winding up with an r such that $\langle$ the common extension of r and $q - M, t_r\rangle$ is in $\mathcal{D} \cap M$ and is compatible with $\langle q, t\rangle$.

The rather weak assumptions we have made about $\mathcal{P}$ now enable us to develop some machinery.

We may assume that the maximal model of $\mathfrak{N}_{q_M}$ contains all the members of dom f_{q_M} $(= \text{dom } f_q \cap M)$, else we could have extended $\mathfrak{N}_q$ to ensure this.

Let δ_M be the intersection of ω_1 with the maximal model of $\mathfrak{N}_{q_M}$. By taking the maximal model large enough, we may ensure that the projection of $(\mathrm{dom}\, f_q \cup \{t\}) - M$ on the δth level of S has the same size as its projection on the δ_Mth level.
Let $\{u_1, \ldots, u_n\}$, $\{v_1, \ldots, v_n\}$ respectively enumerate these projections on the δ_Mth and δth levels, such that $u_i = v_i \,|\, \delta_M$, $i \leq n$, and such that $u_1 = t \,|\, \delta_M$ and $v_1 = t \,|\, \delta$. For $1 \leq i, j \leq n$, let σ_{ij} be the canonical isomorphism which moves u_i to u_j. Note $\sigma_{ij}^{-1} = \sigma_{ji}$, and σ_{ii} is the identity isomorphism.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Let N_{q_M} be the maximal model of $\mathcal{N}_{q_M}$. For any $\langle r, t_r \rangle \in \mathcal{P} \times S$ that is $\leq \langle q_M, u_1 \rangle$, define:

$$F_r = \{x \in (\text{dom}\, f_r \cup \{t_r\}) - N_{q_M} : \quad x \,|\, \delta_M = \text{ some } u_{i_x} \text{ and some } \sigma_{1 i_x}(t) \text{ extends } x\}.$$

Then, considering t as t_q, claim:

$$F_q = \{x \in (\text{dom}\, f_q \cup \{t\}) - M : x \,|\, \delta = \text{ some } v_{i_x} \text{ and } \sigma_{1 i_x}(t) \text{ extends } x\}.$$

(Because of the way we shall define the partial order, it turns out that these are the only x's we will have to think about.)
We claim that if v_i and v_j are projections of elements of F_q, then $\sigma_{ij}(v_i) = v_j$.

For $x \in F_q$, let $\hat{x} = \sigma^{-1}_{1_{i_x}}(x)$. For an m-tuple $\vec{x} = \langle x_1, \ldots, x_m \rangle$, if $\hat{x}_j$ is defined for $1 \leq j \leq m$, let $\hat{\vec{x}} = \langle \hat{x}_1, \ldots, \hat{x}_m \rangle$. In particular, let $\hat{F}_q$ be the chain with possible repetitions of length $|F_q| : \langle \hat{x} : x \in F_q \rangle$. Similarly define F^l_q and $\hat{F}^l_q$ for $l \in L = \{l : \mathrm{dom}_l\, f_q \neq 0\}$. We can make analogous definitions of $\hat{F}_r$ etc. for an arbitrary $\langle r, t_r \rangle$ extending $\langle q_M, u_1 \rangle$. Let $c =$ length $\hat{F}_q = |F_q|$ and $c_l =$ length $\hat{F}^l_q = |F^l_q|$. We use sequence notation and chains with possible repetitions to avoid losing information when we pass from F_q to $\hat{F}_q$. The reason for these "hats" is that we need to prove more than "proper"; we need to prove "proper and preserves S". Thus we need to be able to move our conditions to positions below t. That is what coherence does for us.

The intent of the next three paragraphs is to define *in* M the set of all conditions in $\mathcal{D}$ that "look just like $\langle q, t \rangle$".
Let $\mathcal{D}_0 = \{\langle r, t_r \rangle \in \mathcal{D} : \langle r, t_r \rangle \leq \langle q_M, u_1 \rangle$ and

i) q_M is an *initial part* of r, i.e. for each l, $\mathrm{dom}_l\, f_{q_M}$ is an initial segment of $\mathrm{dom}_l\, f_r$, and $\mathfrak{N}_{q_M}$ is an initial segment of $\mathfrak{N}_r$,

ii) the height of each node in $F_r - F_q$ is $> \delta_M$,

iii) $L_r = L$, each $|F_r^l| = c_l$, $|F_r| = c$,

iv) $f_r(\text{the } j\text{th element of } F_r) = f_q(\text{the } j\text{th element of } F_q)$,

v) the height of t_r is greater than the height of any of the nodes in $\mathrm{dom}\, f_r\}$.

The above requirements will ensure that the natural correspondence between r and q induces a natural correspondence of F_r and $\hat{F}_r$ to F_q and $\hat{F}_q$ respectively.
Notice that the u_i's and hence the σ_{ij}'s are in M, and so $\mathcal{D}_0 \in M$ by definability.

$$\mathcal{F} = \left\{ F \in S^c : F = \hat{F}_r \text{ for some } \langle r, t_r \rangle \in \mathcal{D}_0 \right\},$$

and

$$\mathcal{F}_I = \left\{ F \in S^{c_I} : F = \hat{F}_r^I \text{ for some } \langle r, t_r \rangle \in \mathcal{D}_0 \right\}$$

are also in M and in H_κ as well.
Since $M \cap H_\kappa \in N$ for each $N \in \mathfrak{N}_q - M$, it follows that $\mathcal{F}$ and $\mathcal{F}_I \in N$, for all such N. Note that $\hat{F}_q \in \mathcal{F}$ and $\hat{F}_q^I \in \mathcal{F}_I$, since $\langle q, t \rangle \in \mathcal{D}_0$. Note also that the terms of $\hat{F}_q^I$ are separated by models of $\mathfrak{N}_q$.

Our plan is to reflect $\langle q, t\rangle$ to an $\langle r, t_r\rangle \in \mathcal{D}_0 \cap M$ by using elementarity to systematically reflect the members of F_q down into M. Our topological hypotheses will be used to obtain such a reflection which is also compatible with $\langle q, t\rangle$. Let $\mathcal{N}'_q$ be a minimal subchain of $\mathfrak{N}_q$ containing $M \cap H_\kappa$ at its bottom and separating $\hat{F}^l_q$ for each l. Let $\mathcal{N}'_q = \{N_a\}_{a \leq m-1}$ ordered by inclusion, with $N_0 = M \cap H_\kappa$.

$\hat{F}_q$ is a chain with possible repetitions; let us write it as:

$$\langle \hat{\vec{x}}_1, \ldots, \hat{\vec{x}}_{m-1}, t \rangle$$

where $\hat{\vec{x}}_a = \langle \hat{x}_{a,1}, \ldots, \hat{x}_{a,d_a} \rangle$ enumerates in increasing order $\hat{F}_q \cap (N_a - N_{a-1})$, $a \geq 1$. Thus the length of the vector $\vec{x}_a$ is equal to the size of $F_q \cap (N_a - N_{a-1})$. Since $\mathcal{F} \in N_{m-1}$,

$$\mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-1}) = \left\{ x \in S : \langle \hat{\vec{x}}_1, \ldots, \hat{\vec{x}}_{m-1}, x \rangle \in \mathcal{F} \right\}$$

$\in N_{m-1}$. By Lemma 27, there is a $y_m \in N_{m-1} \cap S$, $y_m \in [\max \hat{\vec{x}}_{m-1}, t)$, such that $\mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-1})$ is dense above y_m.

Next, consider:

$$\mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-2}) = \big\{\langle \vec{x}, y\rangle \in S^{d_{m-1}+1} : \langle \vec{x}, y\rangle \text{ is a chain with possible repetitions and } \mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-2}, \vec{x}) \text{ is dense above } y\big\}.$$

Then $\mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-2}) \in N_{m-2}$ and $\langle \hat{\vec{x}}_{m-1}, y_m\rangle \in \mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-2})$. As before, this time by Corollary 30, with $\mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-2}), \langle \hat{\vec{x}}_{m-1}, y_m\rangle, N_{m-2}$ playing the roles of $\mathcal{A}, A_0, N$ respectively, we can find a $y_{m-1} \in N_{m-2} \cap S$, $y_{m-1} \in [\max \hat{\vec{x}}_{m-2}, \min \hat{\vec{x}}_{m-1})$, such that $\mathcal{F}(\vec{x}_1, \ldots, \vec{x}_{m-2})$ is dense above y_{m-1}. Continuing, in m steps we find a $y_1 \in N_0$, $y_1 \in [u_1, v_1)$, such that:

$$\mathcal{F}(\emptyset) = \big\{\langle \vec{x}, y\rangle \in S^{d_1+1} : \langle \vec{x}, y\rangle \text{ is a chain with possible repetitions and } \mathcal{F}(\vec{x}) \text{ is dense above } y\big\}$$

is $\in N_0$ and dense above y_1.

One of the virtues of Souslin-tree forcing is that, roughly speaking, for each countable piece of information about the final model, there is a level such that all nodes of the tree at that level decide that information. Thus, club often, all nodes of the tree decide what's going on below, e.g. whether or not the point α is in the open set $\dot{U}_\beta$ about β for α, β's below that level. It is therefore convenient to use such a club in the proofs, thereby getting a locally countable subspace is σ-discrete on a club, or a closed discrete subspace of size $\aleph_1$ is separated on a club. By working harder, one can avoid this annoyance – see [FTT]. We'll speak more about the club later. C° will denote the set of successor elements of the club.

Let $\dot{\mathcal{X}}_1$ be a name for:

$$\Big\{\langle \alpha_1, \ldots, \alpha_{d_1}\rangle \in (C^\circ)^{d_1} : \text{for some } \langle \vec{z}, w\rangle \in \mathcal{F}(\emptyset),$$
$$\{\vec{z}, w\} \subseteq B \text{ (the generic branch) and for each } i, 1 \le i \le d_1,$$
$$ht(z_i)^- = \alpha_i \text{ (where } \gamma^- \text{ is the immediate predecessor of } \gamma \in C^o)\Big\}.$$

Then $\dot{\mathcal{X}}_1 \in M$. Let $\dot{\mathcal{X}}_1'$ be a name for

$$\{\vec{\xi} \in \mathcal{X}_1 : \min \vec{\xi} > \delta_M\}.$$

Claim: $y_1 \Vdash \dot{\mathcal{X}}_1' \neq 0$.

Form a subtree of height ω of the cone above y_1, such that each element of each level of the subtree decides $\dot{\vec{\xi}} \in \dot{\mathcal{X}}_1$ and a corresponding $\langle \vec{z}, w \rangle(\vec{\xi})$, and such that the $\vec{\xi}$'s of one level all have minimal components greater than the maximal components of the $\vec{\xi}$'s of the previous level.
By elementarity, there is such a subtree in M. Therefore the sup ζ of the heights in S of the elements of the subtree is less than δ.
We can thus take $\{y_{1,j} : j < \omega\}$ strictly ascending below v_1, all of height less than ζ, and associated strictly increasing $\vec{\xi}^1_j$'s and their corresponding $\langle \vec{z}^1_j, w^1_j \rangle$'s, with $\langle \vec{z}^1_j, w^1_j \rangle \in \mathcal{F}(\emptyset) \cap M$ and $y_{1,j} \Vdash \langle \vec{z}^1_j, w^1_j \rangle \subseteq \dot{B}$, and $\pi_d(\vec{\xi}^1_j) = ht(\pi_d(\vec{z}^1_j))^-$, where $\pi_d(\vec{\xi}^1_j)$ (respectively, $\pi_d(\vec{z}^1_j)$) are d-th components.

Now, finally, we want to look at some specific partial orders. The first one will show that:

Theorem 32

PFA(S)[S] implies that if $\{x_\alpha : \alpha < \omega_1\}$ *is a right-separated subspace of a compact space Z with finite products Fréchet-Urysohn, then there is a club* $C \subseteq \omega_1$ *such that* $\{x_\alpha : \alpha \in C\}$ *is* σ*-discrete.*

Refinements of the argument which we won't get into will enable us to

a) replace "C" by "ω_1",
b) replace "right-separated" by "locally countable",
c) replace "finite products Fréchet-Urysohn" by "sequential", and finally,
d) replace "sequential" by "countably tight".

In order to get the σ-closed-discrete version of $\sum$, we seem to need a somewhat different partial order [FTT]. (This paper is currently being revised; wait a few weeks before trying to go through it.)

To prove the Theorem, we have S-names $\dot{Z}$, $\dot{U}_\alpha$, $\alpha < \omega_1$, such that S forces $\dot{Z}$ is such a space and:

i) $\alpha \in \dot{U}_\alpha$, which is open,

ii) $\beta < \alpha$ implies $\alpha \notin \bigcup \left\{ \overline{\dot{U}}_\beta : \beta < \alpha \right\}$.

Let C be a closed unbounded subset of ω_1 such that for each $\delta \in C$, every node of the δth level of S decides all statements of the form $\alpha \in \overline{\dot{U}}_\beta$, $\alpha < \beta < \delta$.

Let $C^\circ = \{\delta \in C : \sup(C \cap \delta) < \delta\}$. For $\delta \in C^\circ$, let $\delta^- = \sup(C \cap \delta)$. Note that every member of C is a δ^- for some $\delta \in C^\circ$. For $\delta \in C$, let δ^+ be the least element of C greater than δ.

Let $\mathcal{P}$ be the collection of all pairs $p = \langle f_p, \mathcal{N}_p \rangle$ where:

1) f_p is a finite partial function from $S \mid C^\circ$ to ω. Let $\mathrm{dom}_l\, f_p = \{s : f_p(s) = l\}$. We require that each non-empty $\mathrm{dom}_l\, f_p$ consists of nodes of different heights.
2) $\mathfrak{N}_p$ is a finite $\in$-chain of countable elementary submodels of H_κ where κ is regular and sufficiently large, containing all relevant objects, such that $\mathfrak{N}_p$ separates each $\mathrm{dom}_l\, f_p$ in the sense that if $s, s' \in \mathrm{dom}_l\, f_p$ with $s \neq s'$, then there is an $N \in \mathfrak{N}_p$ such that $s \in N$ and $s' \notin N$.
3) If $s, s' \in \mathrm{dom}_l\, f_p$ and s' strictly extends s and $\mathrm{ht}(s') = \tau$ and $\mathrm{ht}(s) = \sigma$, then $s' \Vdash \sigma^- \notin \dot{\bar{U}}_{\tau^-}$.

The ordering on $\mathcal{P}$ is given by:

$$p \leq q \text{ if } f_p \text{ extends } f_q \text{ and } \mathfrak{N}_p \supseteq \mathfrak{N}_q$$

The rationale for the "if s extends s'" clause is that we are coding the σ-discrete subspace by a generic branch, and don't care what happens off that branch. The superscript minuses are there because we only expect conditions of height α to know about things of smaller index.

It is routine to show that this partial order does what it is intended to do. To show that the partial order is proper and preserves S, note that by compactness, $\mathcal{X}_1$, as a subspace of a finite power of Z, has a complete accumulation point x; by right-separation, x does not project to any of the x_α's. By Fréchet-Urysohn, there is a sequence $\{x_{\alpha_n}\}_{n<\omega}$ from $\mathcal{X}_1$ which converges to x. Since the projections of x are not in any of the $\overline{U}_\alpha$'s for s's of height α in the condition we are trying to get away from, for n sufficiently large, x_{α_n} will not be in them either.

Since for $x \in \operatorname{dom} f_q - M$, $ht(x) \geq \delta$, such x decides whether or not $\pi_d(\vec{\xi}^1_j)^- \in \dot{\overline{U}}_{ht(x)^-}$. Since $\overline{U}_{ht(x)^-}$ is compact and for fixed d the $\pi_d(\vec{\xi}^1_j)^-$'s are distinct, there is a $j_x \in \omega$ such that for each $d \leq d_1$, x forces:

$$\left\{\pi_d(\vec{\xi}^1_j)^- : j \geq j_x\right\} \cap \dot{\overline{U}}_{ht(x)^-} = 0. \qquad (\dagger)$$

To see this, note that x certainly forces that there is such a j_x. Then for some j_x, some extension of x forces $(\dagger)$. But then x must have already forced this, since it had decided whether $\pi_d(\vec{\xi}^1_j)^- \in \dot{\overline{U}}_{ht(x)^-}$.

Let $j_1 = \max\{j_x : x \in \operatorname{dom} f_q - M\}$. Let $\mathbf{z}_{1,d}$ be the element of height $\pi_d(\vec{\xi}^1_{j_1})^+$ below x_d, for $x_d \in F_q \cap (N_1 - N_0)$. Let $\vec{\mathbf{z}}_1 = \langle \mathbf{z}_{1,1}, \ldots, \mathbf{z}_{1,d_1} \rangle$. Let $\mathbf{w}_1 = w^1_{j_1}$. Then $\langle \hat{\vec{\mathbf{z}}}_1, \mathbf{w}_1 \rangle = \langle \vec{z}^1_{j_1}, w^1_{j_1} \rangle \in \mathcal{F}(\emptyset)$ and for all $x \in \operatorname{dom} f_q - M$, x forces $\pi_d(\vec{\xi}^1_j)^- \in \dot{\overline{U}}_{ht(x)^-}$, for all $d \le d_1$. Notice that $\langle \vec{\mathbf{z}}_1, \mathbf{w}_1 \rangle \in M$.

We now need to iteratively peel off the remaining "layers" of F_q. Let $\dot{\mathcal{X}}_2$ be a name for:

$$\Big\{\langle\alpha_1,\ldots,\alpha_{d_2}\rangle \in (C^\circ)^{d_2} : \text{ for some } \langle\vec{z},w\rangle, \langle\hat{\vec{z}},w\rangle \in \mathcal{F}(\vec{\mathbf{z}}_1), \{\vec{z},w\} \subseteq B \\ \text{and for each } i, 1 \le i \le d_2, ht(z_i)^- = \alpha_i\Big\}.$$

We now carry out the same argument as before, with an infinite strictly ascending sequence of $y_{2,j}$'s below v_1 extending y_{1,j_1} and deciding $\vec{\xi} \in \mathcal{X}_2$, where $\min\vec{\xi} > \max\vec{\xi}^1_{j_1}$. As before, we obtain a $\vec{\mathbf{z}}_2 \in M$, each $\mathbf{z}_{2,d}$ below $x_d \in F_q \cap (N_2 - N_1)$, and with each $ht(\mathbf{z}_{2,d}) > ht(\mathbf{z}_{1,d_1})$, such that for each $x \in \operatorname{dom} f_q - M$, x forces $\pi_d(\vec{\xi}^1_j)^- \in \overline{U}_{ht(x)^-}$, for all $d \le d_2$.

Continuing, after m steps we will find $\langle \hat{\vec{\mathbf{z}}}_1, \ldots, \hat{\vec{\mathbf{z}}}_m, \mathbf{w}_1 \rangle \in \mathcal{F}$, each component of each $\vec{\mathbf{z}}_a$ below some v_i, and hence in M. Since $\langle \hat{\vec{\mathbf{z}}}_1, \ldots, \hat{\vec{\mathbf{z}}}_m, \mathbf{w}_1 \rangle \in \mathcal{F}$, there is an $\langle r, t_r \rangle \in \mathcal{D}_0 \cap M$ such that $\hat{F}_r = \langle \hat{\vec{\mathbf{z}}}_1, \ldots, \hat{\vec{\mathbf{z}}}_m, \mathbf{w}_1 \rangle$. Then $\mathbf{w}_1 = t_r$. Now $\mathbf{w}_1 = w^1_{j_1}$ is below y_{1,j_1}, since otherwise y_{1,j_1}, could not force it to be in B. Therefore it is below v_1 and so $t_r \leq t$. We claim that $\langle r, t_r \rangle$ is compatible with $\langle q, t \rangle$, which will finish the proof.

The PFA(S)[S] proofs are mainly more difficult versions of the PFA proofs; one has to show that a "natural" partial order $\mathcal{P}$ is both proper and preserves S, and that forcing with S then creates the desired object from the generic filter forced by $\mathcal{P}$. For example, $\mathcal{P}$ might create a tree of discrete subspaces indexed and ordered by the Souslin tree, so that forcing a generic branch through the Souslin tree will make uncountably many of these discrete subspaces cohere, so that there will be an uncountable discrete subspace.

How about the proof that *there is a model of* PFA(S)[S] *in which every locally compact normal space is CWH*? How does that fit in with the general scenario, given that the conclusion contradicts PFA? The answer is interesting:

1. By starting with a particular ground model, we need only prove that PFA(S)[S] *implies locally compact normal spaces are* $\aleph_1$-*CWH* (which also contradicts PFA). (We actually prove $\aleph_1$-CWH on a club $C \subseteq \omega_1$. One can then quote a theorem of Taylor to get full $\aleph_1$-CWH, or else complicate the proof to eliminate the club. For the first alternative, see [Talb]; for the second, see [FTT].)
2. Both PFA and PFA(S)[S] imply each closed discrete subspace of size $\aleph_1$ in a locally compact normal space can be expanded to a discrete collection of compact sets of countable character.
3. After forcing with a Souslin tree, normal spaces are collectionwise normal with respect to collections of $\aleph_1$ sets of countable character.

Thus the essential core of the proof is a (new) PFA consequence, added to that is a Souslin tree forcing version of a countably closed forcing argument. We shall see that 2) shares several features with the argument for making locally countable subspaces σ-discrete.

I won't give the proof for 3); it is the same as the argument for getting *normal first countable implies* $\aleph_1$*-CWH*, which was presented by Paul Larson at a couple of topology conferences around 2004, and appears in our recent paper in Fundamenta [LT10]. The idea for that proof is a mashup of Fleissner's $V = L$ proof and my countably closed forcing proof for getting *normal* + *character* $\leq \aleph_1$ *implies* $\aleph_1$*-CWH*. We inductively define a name for a partition of the closed discrete subset in the extension, such that any assignment of open sets to members of the discrete set that witnesses normality for that partition is actually a separation. (We are sliding over some details involving a club.)

2) begins to look similar to the idea for getting Balogh's $\sum$ when we throw in a couple more easy topological ideas. The first uses an old idea of Steve Watson [Wat82]. I'll leave the easy proof to you:

Lemma 33

Let $D = \{x_\alpha : \alpha < \omega_1\}$ be a closed discrete subspace of a locally compact normal space. Then $\{x_\alpha : \alpha < \omega_1\}$ has a right-separated expansion by compact G_δ's.

The following lemma is also not difficult, and is also left to you:

Lemma 34

If D as above has such a right-separated expansion which is also σ-relatively discrete, then D has a discrete expansion by compact G_δ's.

Thus, as with $\sum$, we shall force a right-separated collection to be σ-discrete.

In order to prove the CWH result, it will suffice to expand the points in a club $C \subseteq \omega_1$ to compact G_δ's which are σ-left-separated by the right-separating U's. We shall do this by simultaneously both approximating a countable partition of ω_1 by finite partial functions from ω_1 into ω and approximating finitely many of the desired compact G_δ's by finite decreasing sequences of compact G_δ's.

From now on, we assume $\mathrm{PFA}(S)$. We have an S-name $\dot{Z}$, such that S forces $\dot{Z}$ is a locally compact normal space. It is convenient to assume that $\{\alpha : \alpha < \omega_1\}$ is a closed discrete subspace of Z. We shall usually omit the "ˇ" that should be placed over elements of the ground model. Let $\dot{\mathcal{E}}$ be a name such that S forces $\dot{\mathcal{E}}$ to be the collection of non-empty compact G_δ's of $\dot{Z}$. We shall assume that for each $\alpha < \omega_1$, we have S-names $\dot{U}_\alpha$, $\dot{K}_\alpha$, $\dot{K}_{\alpha,\beta}$, $\beta < \alpha$, such that S forces:

i) $\alpha \in \dot{U}_\alpha$,

ii) $\dot{U}_\alpha$ is open, $\overline{\dot{U}_\alpha}$ is compact,

iii) $\alpha \neq \beta$ implies $\alpha \notin \overline{\dot{U}_\beta}$,

iv) $\alpha \in \dot{K}_\alpha \subseteq \dot{U}_\alpha$,

v) $\dot{K}_\alpha \in \dot{\mathcal{E}}$, $\dot{K}_{\alpha,\beta} \in \dot{\mathcal{E}}$,

vi) $\beta < \alpha$ implies $\dot{K}_\alpha \cap \overline{\dot{U}_\beta} = 0$

vii) for each α, $\{\dot{K}_{\alpha,\beta} : \beta < \alpha\} \subseteq \dot{\mathcal{E}}$ is discrete, with $\beta \in \dot{K}_{\alpha,\beta} \subseteq \dot{K}_\beta$, and if $\alpha < \gamma$, then $\dot{K}_{\gamma,\beta} \subseteq \dot{K}_{\alpha,\beta}$.

vii) is easy to accomplish: discretely separate $\{\beta : \beta < \alpha\}$, shrink the separating open sets to compact G_δ's, and then intersect with the corresponding K_β's. We then can recursively shrink the compact G_δ's to get $K_{\gamma,\beta} \subseteq K_{\alpha,\beta}$. That is, having gotten say the discrete collection $\{K'_{\gamma,\beta} : \beta < \gamma\}$, let
$K_{\gamma,\beta} = K'_{\gamma,\beta} \cap \bigcap\{K_{\alpha,\beta} : \alpha < \gamma\}$.
Let C be a closed unbounded subset of ω_1 such that for each $\delta \in C$, every node of the δth level of S decides all statements of form $\dot{K}_{\gamma,\beta} \cap \overline{\dot{U}}_\alpha = 0$ for all $\beta < \gamma \leq \alpha < \delta$.

Let $\mathcal{P}$ be the collection of all triples $p = \langle f_p, \mathcal{E}_p, \mathfrak{N}_p \rangle$ where:

1) f_p is a finite partial function from $S \mid C^\circ$ to ω. Let $\operatorname{dom}_l f_p = \{s : f_p(s) = l\}$. We require that each non-empty $\operatorname{dom}_l f_p$ consists of nodes of different heights.
2) $\mathfrak{N}_p$ is a finite $\in$-chain of countable elementary submodels of H_κ where κ is regular and sufficiently large, containing all relevant objects, such that $\mathfrak{N}_p$ separates each $\operatorname{dom}_l f_p$ in the sense that if $s, s' \in \operatorname{dom}_l f_p$ with $s \neq s'$, then there is an $N \in \mathfrak{N}_p$ such that $s \in N$ and $s' \notin N$.
3) $\mathcal{E}_p$ is a finite partial function from $\omega \times S \mid C^\circ$ to ω_1 such that, letting π_2 be the projection map from $\omega \times S \mid C^\circ$ onto $S \mid C^\circ$,
   a) $\pi_2[\operatorname{dom} \mathcal{E}_p] = \operatorname{dom} f_p$
   b) $\mathcal{E}_p(n, s) \geq ht(s)$,
   c) whenever $s \in N \in \mathfrak{N}_p$, $\mathcal{E}_p(n, s) \in N$,
   d) if $s, s' \in \operatorname{dom}_l f_p$ and s' strictly extends s and $ht(s') = \tau$, then

$$s' \Vdash \bigcap \{\dot{K}_{\mathcal{E}_p(n,s), ht(s)^-} : \langle n, s \rangle \in \operatorname{dom} \mathcal{E}_p\} \cap \dot{\bar{U}}_{\tau^-} = 0$$

For $p, q \in \mathcal{P}$, we let $p \leq q$ if and only if:

4) $f_p \mid \operatorname{dom} f_q = f_q$,
5) $\mathcal{E}_p \mid \operatorname{dom} \mathcal{E}_q = \mathcal{E}_q$.
6) $\mathfrak{N}_p \supseteq \mathfrak{N}_q$.

Lemma 35

Let $D_s = \{p \in \mathcal{P} : s \in \operatorname{dom} f_p\}$*. Let* $D_{s,n} = \{p \in \mathcal{P} : \langle n, s\rangle \in \operatorname{dom} \mathcal{E}_p\}$*. Then for each* $s \in S \mid C^\circ$*, and each* $n < \omega$*,* D_s *and* $D_{s,n}$ *are dense.*

Proof.

Given any $q \in \mathcal{P}$, if $q \notin D_s$, take $m > \max\{f_q(t) : t \in \operatorname{dom} f_q\}$. Then $\langle f_q \cup \{\langle s, m\rangle\}, (\mathcal{E}_q \cup \{\langle\langle 0, s\rangle, ht(s)\rangle\}, \mathfrak{N}_q\rangle$ is the required extension of q in D_s. Given $q \in D_s - D_{s,n}$, suppose k is least such that $\langle k, s\rangle \in \operatorname{dom} \mathcal{E}_q$. Let $q' = \langle f_q, \mathcal{E}_q \cup \{\langle\langle n, s\rangle, \mathcal{E}_q(k, s)\rangle\}, \mathfrak{N}_q\rangle$. Then q' is $\leq q$ and is a member of $D_{s,n}$. □

Lemma 36

PFA(S)[S] implies that C has a σ-left-separated, right- separated expansion by compact G_δ's, and hence a discrete expansion by compact G_δ's.

Proof.

Let G be $\mathcal{P}$-generic for the D_s's and the $D_{s,n}$'s. Let $f = \bigcup\{f_p : p \in G\}$. Let $e = \bigcup\{\mathcal{E}_p : p \in G\}$. Then $e : \omega \times S \,|\, C^\circ \to \omega_1$. For $\gamma = ht(s)^-$, $s \in B \,|\, C^\circ$, where B is the generic branch of S, let $E_\gamma = \bigcap\{K_{e(n,s),\gamma} : n < \omega\}$. Then S forces that $\{E_\gamma : \gamma \in C\}^\cdot$ is the required right-separated, σ-left-separated expansion of C by compact G_δ's. □

Topological Consequences of PFA(S)[S] concerning Tightness and Sequentiality

In [Tod], Todorcevic proves PFA(S)[S] implies:

i) separable countably tight compacta have cardinality $\leq 2^{\aleph_0}$,

ii) countably tight compacta are sequentially compact,

iii) every countably tight compactum has a G_δ point,

iv) every non-Lindelöf subspace of a compact countably tight space has an uncountable discrete subspace,

v) countably tight compacta are sequential,

vi) if Y is a non-separable subspace of a regular X, then either X includes an uncountable discrete subspace or a subset Z such that the closure of Z in X has no point of countable π-character.

I expect these PFA consequences can be used to obtain results not following from PFA, besides those for (locally) compact hereditarily normal spaces we have mentioned here.

The following theorem is the key ingredient in many of these sequential proofs, and in particular for proving PFA(S)[S] implies $\sum$. See [Tod] and [Lar].

Theorem 37

The coherent Souslin tree forces that if X is a subset of a compact countably tight space K, then for any ground model ultrafilter $\mathcal{U}$ on ω, if X is $\mathcal{U}$-sequentially closed, then it is closed.

To prove this, set up a partial order $\mathcal{P}$ that forces an uncountable free sequence in K. Prove $\mathcal{P}$ is proper and preserves S by showing that if X were sequentially closed but not closed, X would be countably compact non-Lindelöf. We then have a sequence from X $\mathcal{U}$-converging to a point outside of X and can do the usual tricks. However, there are additional complications involving a filter, just as there were in the original PFA $\rightarrow$ Moore-Mrowka proof.

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