

ON THE HEREDITARY PARACOMPACTNESS OF LOCALLY COMPACT, HEREDITARILY NORMAL SPACES

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ABSTRACT. We establish that if it is consistent that there is a supercompact cardinal, then it is consistent that every locally compact, hereditarily normal space which does not include a perfect pre-image of ω_1 is hereditarily paracompact.

This is the fifth in a series of papers ([LTo], [To], [LT], [T₁] being the previous ones) that establish powerful topological consequences in models of set theory obtained by starting with a particular kind of Souslin tree S , iterating partial orders that don't destroy S , and then forcing with S . The particular case of the theorem stated in the abstract when X is perfectly normal (and hence has no perfect pre-image of ω_1) was proved in [LT], using essentially that locally compact perfectly normal spaces are locally hereditarily Lindelöf and first countable. Here we avoid these two last properties by combining the methods of [B₂] and [T₁]. To apply [B₂], we establish the new set-theoretic result that $\text{PFA}^{++}(S)[S]$ implies Fleissner's "Axiom R". This notation is explained below; the model is a strengthening of those used in the previous four papers.

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The results established here were actually proved around 2004, modulo results of Todorćević announced in 2002 and of the second author [T₁]. We have delayed submission until a correct version of [T₁] existed in preprint form.

It is easy to find locally compact, hereditarily normal spaces which are not paracompact – ω_1 is one such. Non-trivial perfect pre-images of ω_1 may also be hereditarily normal, but are not paracompact. Our result says that consistently, any example must in fact include such a canonical example.

Theorem 1. *If it is consistent that there is a supercompact cardinal, it's consistent that every locally compact, hereditarily normal space that does not include a perfect pre-image of ω_1 is (hereditarily) paracompact.*

This is not a ZFC result, since there are many consistent examples of locally compact, perfectly normal spaces which are not paracompact.

Let us state some axioms we will be using.

PFA⁺⁺: *Suppose P is a proper partial order, $\{D_\alpha\}_{\alpha < \omega_1}$ is a collection of dense subsets of P , and $\{\dot{S}_\alpha : \alpha < \omega_1\}$ is a sequence of terms such that $(\forall \alpha < \omega_1)_P \Vdash \dot{S}_\alpha$ is stationary in ω_1 . Then there is a filter $G \subseteq P$ such that*

- (i) $(\forall \alpha < \omega_1) G \cap D_\alpha \neq \emptyset$,
and (ii) $(\forall \alpha < \omega_1) S_\alpha(G) = \{\xi < \omega_1 : (\exists p \in G) p \Vdash \xi \in \dot{S}_\alpha\}$ is stationary in ω_1 .

Baumgartner [Ba] introduced this axiom and called it “PFA⁺”. Since then, others have called this “PFA⁺⁺”, using “PFA⁺” for the weaker one-term version. As Baumgartner observed, the usual consistency proof for PFA, which uses a supercompact cardinal, yields a model for what we are calling PFA⁺⁺.

Definition. $\Gamma \subseteq [X]^{<\kappa}$ is **tight** if whenever $\{C_\alpha : \alpha < \delta\}$ is an increasing sequence from Γ , and $\omega < cf\delta < \kappa$, then $\bigcup\{C_\alpha : \alpha < \delta\} \in \Gamma$. **Axiom R:** if $\Sigma \subseteq [X]^{<\omega_1}$ is stationary and $\Gamma \subseteq [X]^{<\omega_2}$ is tight and cofinal, then there is a $Y \in \Gamma$ such that $\mathcal{P}(Y) \cap \Sigma$ is stationary in $[Y]^{<\omega_1}$. **Axiom R⁺⁺:** if $\Sigma_\alpha (\alpha < \omega_1)$ are stationary subsets of $[X]^{<\omega_1}$ and $\Gamma \subseteq [X]^{<\omega_2}$ is tight and

cofinal, then there is a $Y \in \Gamma$ such that $\mathcal{P}(Y) \cap \Sigma_\alpha$ is stationary in $[Y]^{<\omega_1}$ for each $\alpha < \omega_1$.

Fleissner introduced Axiom R in [F] and showed it held in the usual model for PFA.

Σ^+ : Suppose X is a countably tight compact space, $\mathcal{L} = \{L_\alpha\}_{\alpha < \omega_1}$ a collection of disjoint compact sets such that each L_α has a neighborhood that meets only countably many L_β 's, and \mathcal{V} is a family of $\leq \aleph_1$ open subsets of X such that:

- a) $\bigcup \mathcal{L} \subseteq \bigcup \mathcal{V}$
- b) For every $V \in \mathcal{V}$ there is an open U_V such that $\bar{V} \subseteq U_V$ and U_V meets only countably many members of \mathcal{L} .

Then $\mathcal{L} = \bigcup_{n < \omega} \mathcal{L}_n$, where each \mathcal{L}_n is a discrete collection in $\bigcup \mathcal{V}$.

Balogh [B₁] proved that MA_{ω_1} implies the restricted version of Σ^+ in which we take the L_α 's to be points. We will call that " Σ' ".

Definition. A space is (strongly) κ -collectionwise Hausdorff if for each closed discrete subspace $\{x_d\}_{d \in D}$, $|D| \leq \kappa$, there is a disjoint (discrete) family of open sets $\{U_d\}_{d \in D}$ with $d \in U_d$. A space is (strongly) collectionwise Hausdorff if it is (strongly) κ -collectionwise Hausdorff for all κ .

It is easy to see that normal (κ -) collectionwise Hausdorff spaces are strongly (κ -) collectionwise Hausdorff.

Balogh [B₂] proved:

Lemma 2. $MA_{\omega_1} + \text{Axiom R}$ implies locally compact hereditarily strongly \aleph_1 -collectionwise Hausdorff spaces which do not include a perfect pre-image of ω_1 are paracompact.

The consequences of MA_{ω_1} he used are Σ' and Szentmiklóssy's result [S] that compact spaces with no uncountable discrete subspaces are hereditarily Lindelöf. Our plan is to find a model in which these two consequences and Axiom R hold, as well as normality implying (strongly) \aleph_1 -collectionwise

Hausdorffness for the spaces under consideration. The model we will consider is of the same genre as those in [LTo], [To], [LT], and [T₁]. One starts off with a particular kind of Souslin tree S , a *coherent* one, which is obtainable from \diamond or by adding a Cohen real. One then iterates in standard fashion as in establishing MA_{ω_1} or PFA, but omitting partial orders that adjoin uncountable antichains to S . In the PFA case for example, this will establish $\text{PFA}(S)$, which is like PFA except restricted to partial orders that don't kill S . In fact it will also establish $\text{PFA}^{++}(S)$, which is the corresponding modification of PFA^{++} . We then force with S . For more information on such models, see [Mi] and [L₁]. We use $\text{PFA}^{++}(S)[S]$ *implies* φ to mean that whenever we force over a model of $\text{PFA}^{++}(S)$ with S , φ holds. Similarly for $\text{PFA}(S)[S]$, etc.

In [T₁] it is established that:

Lemma 3. *$\text{PFA}(S)[S]$ implies that locally compact normal spaces are \aleph_1 -collectionwise Hausdorff.*

By doing some preliminary forcing (as in [LT]), one can actually get full collectionwise Hausdorffness, but we won't need that here.

We will assume all spaces are Hausdorff, and use " X^* " to refer to the one-point compactification of a locally compact space X .

There is a bit of a gap in Balogh's proof of Lemma 2. Balogh asserted that:

Lemma 4. *If X is locally compact and does not include a perfect pre-image of ω_1 , then X^* is countably tight.*

and referred to [B₁] for the proof. However in [B₁], he only proved this for the case in which X is countably tight. It is not obvious that that hypothesis can be omitted, but in fact it can. We need a definition and lemma.

Definition. *A space Y is ω -bounded if each separable subspace of Y has compact closure.*

Lemma 5. [G], [Bu]. *If Y is ω -bounded and does not include a perfect pre-image of ω_1 , then Y is compact.*

We then can establish Lemma 4 as follows.

Proof. By Lemma 5, every ω -bounded subspace of X is compact. By [B₁], it suffices to show X is countably tight. Suppose, on the contrary, that there is a $Y \subseteq X$ which is not closed, but is such that for all countable $Z \subseteq Y$, $\overline{Z} \subseteq Y$. Since X is a k -space, there is a compact K such that $K \cap Y$ is not closed. Then $K \cap Y$ is not ω -bounded, so there is a countable $Z \subseteq K \cap Y$ such that $\overline{Z} \cap K \cap Y$ is not compact. But $\overline{Z} \subseteq Y$, so $\overline{Z} \cap K \cap Y = \overline{Z} \cap K$, which is compact, contradiction.

Lemma 3 takes care of the hereditary strong \aleph_1 -collectionwise Hausdorffness we need, since if open subspaces are \aleph_1 -collectionwise Hausdorff, all subspaces are, and open subspaces of locally compact spaces are locally compact. The proposition that

Σ : *in a compact countably tight space, locally countable subspaces of size \aleph_1 are σ -discrete.*

is implied by $\text{PFA}(S)[S]$ was announced by Todorćević in the Toronto Set Theory Seminar in 2002.

From Σ it is standard to get the result of Szentmiklóssy quoted earlier: since the compact space has no uncountable discrete subspace, it has countable tightness. If it were not hereditarily Lindelöf, it would have a right-separated subspace of size \aleph_1 . But Σ implies it has an uncountable discrete subspace, contradiction.

Σ' is established by a minor variation of the forcing for Σ . A proof exists in the union of [L₂] and [FT]. Σ^+ , however, is not so clear, and has not yet been proved from $\text{PFA}(S)[S]$. Thus, instead of using it to get \aleph_1 -collectionwise Hausdorffness in locally compact normal spaces with no perfect pre-image of ω_1 , as we did in an earlier version of this paper, we are instead quoting Lemma 3, which is a new result of the second author, based on methods of [To].

Thus all we have to do is prove that $\text{PFA}^{++}(S)[S]$ implies Axiom R. In order to prove that $\text{PFA}^{++}(S)[S]$ implies Axiom R, we first note that a

straightforward argument using the forcing $\text{Coll}(\omega_1, X)$ (whose conditions are countable partial functions from ω_1 to X , ordered by inclusion) shows that $\text{PFA}^{++}(\text{S})$ implies Axiom R^{++} .

It then suffices to prove:

Lemma 6. *If Axiom R^{++} holds and S is a Souslin tree, then Axiom R^{++} still holds after forcing with S .*

Proof. First note that if X is a set, P is a c.c.c. forcing and τ is a P -name for a tight cofinal subset of $[X]^{<\omega_2}$, then the set of $a \in [X]^{<\omega_2}$ such that every condition in P forces that a is in the realization of τ is itself tight and cofinal. The tightness of this set is immediate. To see that it is cofinal, let b_0 be any set in $[X]^{<\omega_2}$. Define sets b_α ($\alpha \leq \omega_1$) and σ_α ($\alpha < \omega_1$) recursively by letting σ_α be a P -name for a member of the realization of τ containing b_α and letting $b_{\alpha+1}$ be the set of members of X which are forced by some condition in P to be in σ_α . For limit ordinals $\alpha \leq \omega_1$, let b_α be the union of the b_β ($\beta < \alpha$). Then b_{ω_1} is forced by every condition in P to be in τ .

Since we are assuming that the Axiom of Choice holds, Axiom R^{++} does not change if we require X to be an ordinal. Fix an ordinal γ and let ρ_α ($\alpha < \omega_1$) be S -names for stationary subsets of $[\gamma]^{<\omega_1}$. Let T be a tight cofinal subset of $[\gamma]^{<\omega_2}$. For each countable ordinal α and each node $s \in S$, let $\tau_{s,\alpha}$ be the set of countable subsets a of γ such that some condition in S extending s forces that a is in the realization of ρ_α . Applying Axiom R^{++} , we have a set $Y \in [\gamma]^{<\omega_2}$ such that each $\mathcal{P}(Y) \cap \tau_{s,\alpha}$ is stationary in $[Y]^{<\omega_1}$.

Since S is c.c.c., every club subset of $[Y]^{<\omega_1}$ that exists after forcing with S includes a club subset of $[Y]^{<\omega_1}$ existing in the ground model. Letting $(\rho_\alpha)_G$ (for each $\alpha < \omega_1$) be the realization of ρ_α , we have by genericity then that after forcing with S , each $\mathcal{P}(Y) \cap (\rho_\alpha)_G$ will be stationary in $[Y]^{<\omega_1}$.

This completes the proof of Theorem 1.

We do not know the answer to the following question; a positive answer would likely enable us to dispense with Axiom R , and possibly with the supercompact cardinal.

Problem. Does MA_{ω_1} imply every locally compact, hereditarily strongly collectionwise Hausdorff space which does not include a perfect pre-image of ω_1 is paracompact?

We also do not know whether in our main result, we can replace “perfect pre-image of ω_1 ” by “copy of ω_1 ”.

Remark. That $PFA(S)[S]$ does not imply Axiom R is proved in [T₃].

The problem of finding in models of $PFA(S)[S]$ necessary and sufficient conditions for locally compact normal spaces to be paracompact is studied in [T₂] by extending the methods of [B₂] and this note.

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