On a Core Concept of Arhangel'skiĭ

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This paper is dedicated to Tsugunori Nogura in honour of his 60th birthday. Omedetō gozaimasu!

Abstract

Arhangel'skiĭ [3] has introduced a weakening of σ-compactness: *having a countable core*, for locally compact spaces, and asked when it is equivalent to σ-compactness. We settle several problems related to that paper.

The concept of *countable core* in [3] is a little hard to understand at first; Arhangel'skiĭ, however, provides equivalents which are easier to understand, and so we will take one of them as our definition, referring the reader to [3] for the original definition.

Definition. *A subset Y of a space X is* ***compact from inside*** *if every subspace F of Y which is closed in X is compact. A locally compact space X has a* ***countable core*** *if it has a countable open cover by sets compact from inside X.*

The motivation for considering this concept lies in considering the implications of the point at infinity in the one-point compactification of a locally compact space having various local countability properties — see the following Definition, Proposition, and Lemma. Let a be the point at infinity in the one-point compactification aX of a locally compact space X (we shall assume all spaces are Hausdorff).

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Definition. *Let y be a point of a space Y. A family $\mathcal{G}$ of subsets of Y is a* ***weak base at*** *y if a U included in X and containing y is open if and only if $U - \{y\}$ is open and some $G \in \mathcal{G}$ is included in U.*

Warning. It is not known — although it is conjectured — that a compact space is *weakly first countable* (defined in [1]) if each point has a countable weak base [5]. For compact spaces, weak first countability is equivalent to sequentiality plus each point having a countable weak base [5, 1.1].

Lemma 1 [3]**.** *A locally compact space has a countable core if and only if the point at infinity a has a countable weak base in aX.*

Compare Lemma 1 with 3.17 of [3]:

Proposition 2. *For any locally compact space X having a countable core, the following are equivalent:*

1. *aX is Fréchet-Urysohn at a,*
2. *aX is first countable at a,*
3. *X is σ-compact.*

Recall that a space Y is *Fréchet-Urysohn at* $y \in Y$ if whenever $y \in \overline{Z} \subseteq Y$, there is a sequence from Z converging to y. Arhangel'skiĭ [3, 1.14] gives a long list of weak conditions that imply a locally compact space with a countable core is σ-compact. Among them is countable paracompactness. In fact, the indicated proof works for countable metacompactness:

Lemma 3. *A countably metacompact, locally compact space with a countable core is σ-compact.*

Proof. Shrink the given open cover by compact-from-inside sets to a closed cover. These closed sets are then compact. □

One not mentioned in Arhangel'skiĭ's list but worth pointing out here is property wD:

Definition. *A space satisfies* ***wD*** *if for each closed discrete subspace $\{d_n\}_{n\in\omega}$, there is an infinite $A \subseteq \omega$ and a discrete collection of open sets $\{U_n : n \in A\}$ such that for $n, m \in A$, $d_n \in U_n$ but $d_n \notin U_m$, $m \neq n$.*

Theorem 4. *A locally compact space with a countable core is σ-compact if it satisfies wD.*

Proof. In a space with wD, the closure of a compact-from-inside set is countably compact. A space which is the union of countably many countably compact closed sets is countably metacompact [12, after 3.2]. □

Problem 3.4 of [3] asks *"Is there, in ZFC, a compact space such that the core of every open subspace of X is countable, but X is not perfectly normal?"* The answer is negative:

Theorem 5. *Assume MA plus $\sim CH$. Then a compact space in which the core of every open subspace is countable is perfectly normal.*

Proof. As Arhangel'skiĭ notes [3, 1.6], if the core of X is countable, $e(X)$ is countable (i.e., there are no uncountable closed discrete subspaces), and hence if the core of every open subspace is countable, X has no uncountable discrete subspace. Since X is compact, it follows that X is countably tight [2]. Hence closed subspaces of X are separable by MA plus $\sim CH$ [21], and so X is hereditarily separable. Then X is hereditarily Lindelöf [22] and thus perfectly normal. □

Similarly, we see that:

Theorem 6. *MA plus $\sim CH$ implies every open subspace of a locally compact hereditarily separable space has a countable core.*

Proof. The one-point compactification of each such subspace is hereditarily Lindelöf, so the subspace is σ-compact. □

On the other hand,

Theorem 7. *CH implies there is a locally compact, hereditarily separable space that does not have a countable core.*

Proof. The Kunen line [17] is locally compact, hereditarily separable, normal, but not Lindelöf, so not σ-compact. Arhangel'skiĭ [3, 1.14] proved that locally compact normal spaces with a countable core are σ-compact. We can see that because normal spaces are wD. □

Arhangel'skiĭ in Problem 3.14 also asks whether there is a consistent example of a compact space which is not perfectly normal, but is such that every open subspace has a countable core. This remains open. Compact S-spaces are natural candidates, but Ostaszewski's locally compact, first countable S-space [20] does not have a countable core since it is normal, but is not σ-compact. It is of course an open subspace of its one-point compactification, which is a compact S-space.

Arhangel'skiĭ [3, 3.13] proves that if every open subspace of a locally compact space has a countable core, the space has cardinality $\leq 2^{\aleph_0}$. Fedorčuk's compact S-space from $\diamondsuit$ [13] has cardinality $> 2^{\aleph_0}$, so some open subspace of it does not have a countable core.

Problem 1.16 of [3] asks whether a locally compact space with a G_δ-diagonal and a countable core is σ-compact. We shall provide a partial answer:

Lemma 8 [5, 1.2]**.** *Let X be compact. If every point of X has a countable weak base, then X is countably tight.*

Theorem 9. *MA plus $\sim CH$ implies every locally compact space with a countable core, a G_δ-diagonal, and with Lindelöf number $\leq \aleph_1$ is σ-compact and (hence) metrizable.*

Proof. Locally compact spaces with a G_δ-diagonal are locally metrizable and hence first countable. Paracompact locally metrizable spaces are metrizable. By Lemmas 1 and 8 (or by Lemma 22 below), the one-point compactification of X is countably tight. If X were not σ-compact, it would have a locally countable subspace S of size $\aleph_1$. There would then be a collection $\mathcal{V}$ of $\leq \aleph_1$ open sets covering X such that for each $V \in \mathcal{V}$, there is an open U_V with $\overline{V} \subseteq U_V$, and U_V containing only countably many members of S. By the following lemma of Balogh, that implies S is σ-closed-discrete in X. But $e(X) = \aleph_0$, giving a contradiction. □

Lemma 10 [7, 1.1]**.** *Assume MA. Let X be a compact, countably tight space, Y a locally countable subspace of X of size $< 2^{\aleph_0}$, and $\mathcal{V}$ a family of $< 2^{\aleph_0}$ open subsets of X such that:*

a) $Y \subseteq \bigcup \mathcal{V}$,

b) for every $V \in \mathcal{V}$, there is an open $U_V \subseteq X$ such that $\overline{V} \subseteq U_V$ and $U_V \cap Y$ is countable.

Then Y is σ-closed-discrete in $\bigcup \mathcal{V}$.

We can get a weaker conclusion from a weaker axiom:

Theorem 11. $\mathfrak{b} = 2^{\aleph_0}$ *implies every locally compact, first countable space with a countable core and size* $< 2^{\aleph_0}$ *is σ-compact.*

Proof. In [11, 12.2], it is shown that first countable spaces of size $< \mathfrak{b}$ are wD. $\square$

For future reference, let us denote by $\sum$ the assertion obtained from the statement of Lemma 10 by replacing "$< 2^{\aleph_0}$" by "$\aleph_1$" and omitting "assume MA".

Here is another application of MA and small Lindelöf number. It is interesting since MA plus $\sim CH$ is not enough to prove that locally compact, countably tight spaces are sequential [19]; that requires PFA (for compact spaces [6], which easily implies the conclusion for locally compact ones).

Theorem 12. *MA implies locally compact, countably tight spaces with hereditary Lindelöf number* $< 2^{\aleph_0}$ *are sequential.*

Proof. The CH case is routine, since points are G_δ, so the space is first countable. Using MA plus $\sim CH$, it suffices to show countably compact subspaces are closed [15, 6.5]. By [23, 1.24], under MA plus $\sim CH$, a countably compact space with Lindelöf number $< 2^{\aleph_0}$ is **ω-bounded**, i.e., countable sets have compact closures. By [19], in a countably tight space, ω-bounded subspaces are closed. $\square$

The obvious try after seeing Theorem 9 would be to use reflection in order to prove it consistent that if there is a locally compact space with a G_δ-diagonal and a countable core, then there is one with Lindelöf number $\aleph_1$. Indeed PFA implies MA plus Fleissner's *Axiom R* [14]; the latter axiom is used by Balogh [9] to prove a number of promising results, e.g.,

Lemma 13 [9, 1.4, 1.6]**.** *Assume Axiom R. Then:*

a) If X is locally Lindelöf, countably tight, regular, and not paracompact, then X has a non-paracompact open subspace with Lindelöf number $\aleph_1$.

b) If, in addition, closures of Lindelöf subspaces have Lindelöf number $\leq \aleph_1$, then that open subspace may also be taken to be closed.

Unfortunately, the subspace given in a) need not have a countable core, even if X does, and there is no reason to believe closures of Lindelöf subspaces have Lindelöf number $\leq \aleph_1$.

Since closed subspaces of a space with a countable core also have countable cores [3, 1.5], they have countable extent. A weak condition, which when added to countable extent yields Lindelöf number $\leq \aleph_1$ is *submeta-$\aleph_1$-Lindelöfness*:

Definition. *Let $\mathcal{U}$ be an open cover of a space X and let $x \in X$. $Ord(x,\mathcal{U}) = |\{U \in \mathcal{U} : x \in U\}|$. X is* ***submeta-$\aleph_1$-Lindelöf*** *if every open cover has a refinement $\bigcup_{n<\omega} \mathcal{U}_n$ such that each $\mathcal{U}_n$ is an open cover, and for each $x \in X$, there is an n such that $Ord(x,\mathcal{U}_n) \leq \aleph_1$.*

Following a similar proof of Balogh [8, 1.1], it is not difficult to prove:

Lemma 14. *For any space X, $L(X) \leq \aleph_1$ if and only if $e(X) \leq \aleph_1$ and X is submeta-$\aleph_1$-Lindelöf.*

Corollary 15. *MA plus $\sim CH$ implies every submeta-$\aleph_1$-Lindelöf, locally compact, countably tight space with a countable core is σ-compact.*

The condition that *countable sets have Lindelöf closures* is crucial in the investigation of locally compact spaces by Eisworth and Nyikos [12] and in unpublished work by the author. I first deduced propositions from this with the aid of *P-ideal dichotomy*, but later realized that having a countable core is such a strong requirement that this set-theoretic proposition is not needed.

Definition. *A subspace Y of a space X is* ***conditionally compact*** *if every infinite subset of Y has a limit point in X.*

Observe that *compact-from-inside subspaces of a space X are conditionally compact.* The following observation, proved but not stated in [12], is crucial:

Lemma 16 [12]**.** *Suppose K has a conditionally compact dense set D and every countable subset of K has Lindelöf closure. Then $E = \bigcup\{\overline{Q} : Q$ is a countable subset of $D\}$ is ω-bounded.*

Proof. Let S be a countable subset of E. Then $\overline{S} \subseteq E$. $\overline{S}$ is pseudocompact, since if there were an infinite discrete collection $\{U_n\}_{n<\omega}$ of non-empty open sets in $\overline{S}$, then taking $s_n \in S \cap U_n$, $\{s_n\}_{n<\omega}$ would be a closed discrete subspace of $\overline{S}$ and hence of K. But $\{s_n\}_{n<\omega}$ has a limit point in K, contradiction. Now $\overline{S}$ is also Lindelöf, hence normal. But then it is countably compact and hence compact. □

From Lemma 16 we easily obtain:

Theorem 17. *If X is a locally compact, countably tight space with a countable core, and countable subsets of X have Lindelöf closure, then X is σ-compact.*

The point is that since the space is countably tight, the E of Lemma 16 is just $\overline{D}$, so the space is the union of countably many closed countably compact sets. Alternatively, we have previously noted that an ω-bounded subspace of a countably tight space is closed.

Corollary 18. *If X is a locally compact, countably tight space with a countable core which is not σ-compact, then X has a separable closed subspace (hence locally compact with a countable core) which is not σ-compact.*

Corollary 19. *If there is a locally compact space with a G_δ-diagonal and a countable core which is not σ-compact, then there is a separable, pseudocompact one.*

Both corollaries are straightforward, except for pseudocompactness. Given a separable example X, let $\{V_n\}_{n<\omega}$ be an open cover by sets compact from the inside. Each $\overline{V_n}$ is separable, locally compact, and has countable core. If all of them were Lindelöf, so would be X, so some $\overline{V_n}$ is not σ-compact. Arhangel'skiĭ [3, proof of 1.11] points out that the closure of a compact-from-inside subspace is pseudocompact.

It follows from Corollary 18 that the CH-example of Jakovlev [16] discussed in [3] has a separable closed subspace which is locally compact, locally countable, has a countable core, and is not σ-compact.

Theorem 17 can be improved at the cost of making an additional assumption. Recall $\sum$ was defined earlier.

Theorem 20. $\sum$ *implies if X is locally compact and does not include a perfect pre-image of ω_1, then either:*

a) X is σ-compact,

or b) $e(X) > \aleph_0$,

or c) X has a countable discrete subspace D such that $\overline{D}$ is not Lindelöf.

Proof. We need three lemmas. Recall a space is **$\aleph_1$-Lindelöf** if every open cover of size $\aleph_1$ has a countable subcover; equivalently, if every subset of size $\aleph_1$ has a complete accumulation point.

Lemma 21 [4, 3.2]**.** *If X is Tychonoff, countably tight, $\aleph_1$-Lindelöf, and countable discrete subspaces have Lindelöf closures, then X is Lindelöf.*

Lemma 22 [7, 2.1]**.** *A locally compact space does not include a perfect pre-image of ω_1 if and only if the one-point compactification of the space is countably tight.*

Lemma 23. $\sum$ *implies every locally compact space of Lindelöf number $\leq \aleph_1$, not including a perfect pre-image of ω_1, but with countable extent, is σ-compact.*

Proof. See the proof of Theorem 9. □

Continuing the proof of Theorem 20, since locally compact spaces are Tychonoff, it suffices by Lemma 21 to show that every subset of X of size $\aleph_1$ has a complete accumulation point. If not, we have a locally countable and hence σ-discrete subset of size $\aleph_1$, and hence an uncountable discrete subspace Y with no complete accumulation point. But then by countable tightness and condition c) in Theorem 20, we get that the closure of Y has Lindelöf number $\leq \aleph_1$, so is Lindelöf by Lemma 23, so Y does indeed have a complete accumulation point, contradiction. □

Corollary 24. $\sum$ *implies that if X is a locally compact, countably tight space with a countable core, and countable discrete subspaces of X have Lindelöf closure, then X is σ-compact.*

Proof. It suffices to show X does not include a perfect pre-image of ω_1. Such a subspace Y would be ω-bounded and hence closed, since X is countably tight. But then Y would be σ-compact by Theorem 17, contradiction. □

We also have:

Theorem 25. *If X is a countably tight, locally compact space with a countable core, and every subspace of X of size $\aleph_1$ is metalindelöf, then X is σ-compact.*

We need:

Lemma 26 [10, 2.7]**.** *If $d(X) \leq \aleph_1$ and X is countably tight and every subspace of X of size $\aleph_1$ is metalindelöf, then X is hereditarily metalindelöf.*

Proof of Theorem 25. If X were not σ-compact, it would have a separable closed subspace which was not σ-compact. But that subspace would be locally compact and metalindelöf, so it would be Lindelöf and, in fact, σ-compact. □

Note that "metalindelöf" cannot be replaced by "weakly θ-refinable": Jakovlev's space [16], as noted by Arhangel'skiĭ [3], is σ-discrete and hence hereditarily weakly refinable. It has a countable core, but is not σ-compact.

There are not so many familiar weak topological properties that ensure separable subspaces have Lindelöf closures. One that Arhangel'skiĭ has introduced is **ω-monolithic**, i.e., separable subspaces have closures with countable networks. Another candidate is **linear Lindelöfness**, i.e. every well-ordered-by-inclusion open cover has a countable subcover.

Lemma 27 [12, proof of 3.4]**.** $2^{\aleph_0} < \aleph_\omega$ *implies every separable closed subspace of a linearly Lindelöf regular space is Lindelöf.*

Thus by Theorem 17 we have:

Theorem 28. $2^{\aleph_0} < \aleph_\omega$ *implies every countably tight, locally compact, linearly Lindelöf space with a countable core is Lindelöf.*

We can get other sufficient conditions for countable core to imply σ-compactness by using Axiom R.

Theorem 29. *Axiom R implies that if X is locally separable, countably tight, and is locally compact with a countable core, and if every subspace of X of size $\leq \aleph_1$ is metalindelöf, then X is σ-compact.*

This follows from:

Lemma 30. *Axiom R implies a locally separable, countably tight, regular space is hereditarily paracompact if and only if every subspace of size* $\leq \aleph_1$ *is metalindelöf.*

Proof. One direction is trivial. To go the other way, we shall first obtain paracompactness via Lemma 13. Let V be an open subspace with $L(V) \leq \aleph_1$. Covering V by $\leq \aleph_1$ separable open sets, we see that $d(V) \leq \aleph_1$. Then by Lemma 25, $\overline{V}$ is hereditarily paracompact. To get the whole space hereditarily paracompact, note it is a sum of separable, hence hereditarily Lindelöf, clopen sets. □

Theorem 29 can, for example, be applied to locally compact spaces with a G_δ-diagonal and a countable core. Surprisingly, by adding an additional condition, we can obtain ZFC results:

Theorem 31. *A locally compact, locally separable, countably tight, locally connected space with a countable core is* σ*-compact if every subspace of size* $\leq \aleph_1$ *is metalindelöf.*

This follows from:

Lemma 32. *A locally compact, locally separable, countably tight, locally connected space is hereditarily paracompact if and only if every subspace of size* $\leq \aleph_1$ *is metalindelöf.*

Proof. Every Lindelöf subspace of the space X is included in a countable union of separable open sets, and hence has Lindelöf closure by Lemma 25. By 5.9 of [12], since X is locally compact, locally separable, countably tight and locally connected, X is the sum of clopen subspaces of Lindelöf number $\aleph_1$. But each of these has density $\leq \aleph_1$, and so is hereditarily paracompact by Lemma 25, since hereditarily metalindelöf, locally separable regular spaces are hereditarily paracompact. □

Thus in ZFC, we have, for example,

Corollary 33. *A locally compact, locally connected space with a countable core and a* G_δ*-diagonal is* σ*-compact if and only if every subspace of size* $\leq \aleph_1$ *is metalindelöf.*

The forward direction is because paracompact, locally metrizable spaces are metrizable.

Combining Axiom R with Lemma 26, we obtain:

Theorem 34. *Axiom R plus* $2^{\aleph_0} < \aleph_\omega$ *implies that if* X *is countably tight, linearly Lindelöf, regular, and locally separable, then* X *is Lindelöf.*

Proof. Each point has an open neighborhood, the closure of which is separable and linearly Lindelöf, so the space is locally Lindelöf. A Lindelöf subspace is included in a separable subspace, so its closure is Lindelöf. Thus, by Lemma 13, if the space were not Lindelöf and hence not paracompact, it would have a closed non-paracompact subspace with Lindelöf number $\aleph_1$. But a linearly Lindelöf space with Lindelöf number $< \aleph_\omega$ is Lindelöf. □

Note that e.g. PFA implies Axiom R plus $2^{\aleph_0} < \aleph_\omega$.

Although there is a ZFC example, due to Kunen [18] and discussed in [3] which is locally compact, has a countable core, and is not σ-compact and hence is not Lindelöf, one might wonder whether having a countable core confers some degree of Lindelöfness on a locally compact space. We already know that every set of power $\aleph_1$ has a limit point; must such a set actually have a complete accumulation point? Arhangel'skiĭ proves that Kunen's space is not $\aleph_1$-Lindelöf. He also proves that the locally compact, locally countable space constructed by Jakovlev [16] using CH has a countable core but is not $\aleph_1$-Lindelöf.

Theorem 35. CH *implies that if there is a locally compact space with a countable core and a* G_δ*-diagonal which is not* σ*-compact, then there is one which is not* $\aleph_1$*-Lindelöf.*

Proof. By Corollary 19, we may assume our space is separable. Every locally compact space with a G_δ-diagonal is first countable, so by CH, the space has cardinality $\aleph_1$. But an $\aleph_1$-Lindelöf space of size $\aleph_1$ is Lindelöf, and a locally compact Lindelöf space is σ-compact. □

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In conclusion, the problem I find most intriguing in [3] is the one concerning spaces with a G_δ-diagonal.

Conjecture. *It is undecidable whether locally compact spaces with a countable core and a* G_δ*-diagonal are* σ*-compact.*

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