THE EQUATION $\delta^1_2 = u_2$

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Abstract. We give a proof of the well-known result that $\delta^1_2 = u_2$ assuming sharps for reals. We do not know the origin of this result but appropriate references will be added as soon as we find them.

Introduction

This note contains nothing new, except we avoid the EM blueprint definition of sharps, preferring instead to use their representation as mice, which should always be possible. It is hoped that this will be useful to anyone who has puzzled over the equation in the title, perhaps in the course of reading Section 3.1 of [1].

Definition 0.1. δ^1_2 is the supremum of all prewellorderings of $\mathbb{R}$ which are Δ^1_2 definable from a real.

Definition 0.2. For a real x let I_x be the club class of indiscernibles for $L[x]$. The second smallest ordinal in

$$I = \bigcap_{x \in \mathbb{R}} I_x$$

is denoted u_2. The first is $u_1 = \omega_1$.

Definition 0.3. $x^{\#}$ is the minimal amenable structure $N^x_0 = (L_\alpha[x], U)$ such that N^x_0 has a largest cardinal κ, thinks that U is a normal measure on κ, and crucially, all iterated ultrapowers of N^x_0 are well-founded.

Minimality is included for uniqueness. Amenability means that $U \cap L_\beta[x] \in N^x_0$ for every $\beta < \alpha$. If N^x_0 is iterated, the sequence of critical points form the $L[x]$-indiscernibles I_x. We let N^x_β denote the iterates with maps $\pi^x_{\beta,\gamma}$ for $\beta \leq \gamma$. The critical point of $\pi^x_{\alpha,\alpha+1}$ will be denoted κ^x_α. Thus $\kappa^x_{\omega_1} = \omega_1$ and letting $\pi^x = \pi^x_{\omega_1,\omega_1+1}$ we have

$$u_2 = \sup\{\pi^x(\omega_1) \mid x \in \mathbb{R}\},$$

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or equivalently

$$u_2 = \sup\{\pi^x(f)(\omega_1) \mid x \in \mathbb{R} \text{ and } f \in {\omega_1}^{\omega_1} \cap L[x]\}.$$

The following lemmas will show that the lengths of Δ^1_2-definable prewellorderings are ordinals of the form $\pi^x(f)(\omega_1)$. Note that if x is a real and ϕ a formula then the sentence asserting the existence of a transitive model M containing x such that $M \models \phi(x)$ is $\Sigma^1_2(x)$. More precisely the statement is: there exists a real z which codes a well-founded set M, $x \in M$ and $M \models \phi(x)$. We can append any finite fragment of ZFC to ϕ as well. From now on we drop superscripts.

Lemma 0.4. $\delta^1_2 \geq u_2$.

Proof. Let $F = {\omega_1}^{\omega_1} \cap L[x]$. The prewellordering induced on F by π has length $\pi(\omega_1)$. Since every $f \in F$ is of the form $\pi_{\alpha,\omega_1}(\bar{f})$ for some $\bar{f} \in N_\alpha$ we can transfer this ordering to reals as follows. Let F_α be the set of functions from κ_α to κ_α in N_α. For reals p and q say $p \preceq q$ if p codes a function $\bar{f} \in F_\alpha$, q codes a function $\bar{g} \in F_\beta$, and letting $\gamma = \max(\alpha, \beta)$ we have

$$\pi_{\alpha,\gamma+1}(\bar{f})(\kappa_\gamma) \leq \pi_{\beta,\gamma+1}(\bar{g})(\kappa_\gamma).$$

Let $\phi(p, q, N_0)$ be the sentence asserting that p, q code relevant functions and there is a countable ordinal γ bigger than α and β such that N_γ exists. Let $\psi(p, q, N_0)$ say that in addition the functions coded by p, q map to the same function. We now argue that our prewellordering is Δ^1_2 in any real coding N_0. We have $p \preceq q$ if and only if there is a transitive set M containing N_0, p, q such that $M \models \phi(p, q, N_0) \wedge \psi(p, q, N_0)$ if and only if for every transitive set M, if M contains p, q, N_x and models $M \models \phi(p, q, N_0)$ then $M \models \psi(p, q, N_0)$. □

The following lemma does not require sharps but makes crucial use of the Boundedness Lemma asserting that Σ^1_1-definable prewellorderings have countable length. The reader is referred to the proof of Theorem 100b of [0] for more details.

Lemma 0.5. *Suppose A is a prewellordering of $\mathbb{R}$ which is Δ^1_2 definable from a real x. Let $\kappa = \omega_1$. Then the length of A is less than $(\kappa^+)^{L[x]}$.*

Proof. Let A be the projection of $T \in L[x]$ on $\omega \times \omega \times \kappa$. Let $g \subset Col(\omega, \omega_1)$ be V-generic. Thus g is $L[x]$-generic and $\omega_1^{L[x][g]} = (\kappa^+)^{L[x]}$. Let $z \in L[x][g]$ be a real coding a well-ordering of length κ. We following need to be checked.

(1) $p[T]^{V[g]}$ is a prewellordering extending $p[T]^V$
(2) $p[T]^{V[g]}$ is Σ^1_1-definable from z
(3) The above are true in $L[x][g]$

By the Boundedness lemma it follows that the length of $p[T]$ is countable in $V[g]$. Similarly $L[x][g]$ determines that the length of $p[T]$ is a countable ordinal. By (1) we will be done if we show that these lengths are the same. Let γ denote the ordinal computed in $L[x][g]$, and y a real there coding it. If there is a real in $V[g]$ whose rank in $p[T]$ is $\gamma + 1$ then there is a transitive model M containing x, y which witnessed this. Such a model would then belong to $L[x][g]$ by Schonfield absoluteness, a contradition. □

Since $(\kappa^+)^{L[x]} < u_2$ this lemma shows that $\delta^1_2 \leq u_2$.

References

[0] Jech, T., *Set Theory*, Academic Press, 1978.

[1] Woodin, W.H., *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*, de Gruyter Series in Logic and its Applications 1, 1999

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