

ON A FRAGMENT OF THE UNIVERSAL BAIRE PROPERTY FOR Σ_2^1 SETS

STUART ZOBLE
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ABSTRACT. There is a well-known global equivalence between Σ_2^1 sets having the Universal Baire property, two-step Σ_3^1 generic absoluteness, and the closure of the universe under the sharp operation. In this note, we determine the exact consistency strength of Σ_2^1 sets being $(2^\omega)^+$ -cc Universally Baire, which is below $0^\#$. In a model obtained, there is a Σ_2^1 set which is weakly ω_2 -Universally Baire but not ω_2 -Universally Baire.

1. INTRODUCTION

Consider the following two properties of a set of reals $A \subset \omega^\omega$ at an infinite cardinal κ .

- (1) For every continuous $f : \kappa^\omega \rightarrow \omega^\omega$ there is a dense set of $p \in \kappa^{<\omega}$ such that either $f^{-1}(A)$ is meager or comeager below p .
- (2) For every continuous $f : \kappa^\omega \rightarrow \omega^\omega$ there is a dense set of $p \in \kappa^{<\omega}$ such that either $f^{-1}(A) \cap \sigma^\omega$ is meager below p in σ^ω for a club of $\sigma \in [\kappa]^\omega$ or comeager below p in σ^ω for a club of $\sigma \in [\kappa]^\omega$.

The first property asserts that A is κ -Universally Baire or fully captured at κ , and the second that A is weakly κ -Universally Baire (the author's coinage) or weakly captured at κ (see [6] and Lemma 4.1 below). The implication (1) to (2) is immediate as a club of $\sigma \in [\kappa]^\omega$ are closed under a Banach-Mazur strategy in the space κ^ω . Regarding the reverse implication, any set of reals of size ω_1 is a counterexample at $\kappa = \omega_2$ assuming Martin's Maximum (see Thm. 2.6 of [6] and Thm. 3.1 of [4]). The place to look for a definable counterexample is the pointclass Σ_2^1 with κ either ω_1 or ω_2 . This is because (1) and (2) are equivalent for $\kappa = \omega$ and for Δ_2^1 sets as a whole. As the particular scenario for a counterexample suggested by [6] involves the question of the consistency strength of Σ_2^1 sets being ω_1 or ω_2 -Universally Baire, in particular whether this is possible without sharps, this paper was motivated by the following question. What fragment of the Universal Baire Property can Σ_2^1 sets have below $0^\#$, as measured by

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the weight or cellularity of the preimage space? In Theorem 3.4 of [1] a global equivalence is established between Σ_2^1 sets being Universally Baire, two-step Σ_3^1 Generic Absoluteness, and the closure of the universe under the sharp operation. This equivalence, however, is not true level-by-level. In particular, the relevant part of their argument (originating in [4]) would require that Σ_2^1 sets be $\omega_{\omega+1}$ -Universally Baire to prove the existence of $0^\#$. Using the full strength of Covering for L this can be reduced to ω_3 . On the other hand, Woodin has shown that Π_2^1 sets can be ω_2 -cc-Universally Baire in a forcing extension of L .

Definition 1.1. $A \subset \mathbb{R}$ is κ -cc-Universally Baire if $f^{-1}(A)$ has the Baire property in X for every completely regular space X of cellularity less than κ and continuous map $f : X \rightarrow \mathbb{R}$.

Theorem 1.2. (Woodin) Assume $\lambda_0 < \lambda_1 < \lambda_2$ are cardinals of L and there is an elementary embedding $\pi : L_{\lambda_1} \rightarrow L_{\lambda_2}$ with critical point λ_0 such that $\pi(\lambda_0) = \lambda_1$. Then Σ_2^1 sets are ω_2 -cc-Universally Baire in a forcing extension of L in which CH holds.

For equivalent versions of Definition 1.1, the reader is referred to Theorem 2.1 of [1]. In particular, we will use that a κ -cc Universally Baire set remains ccc-Universally Baire after forcing with a κ -cc poset. When combined with the argument of Theorem 3.4 of [1], the above shows that two-step Σ_3^1 generic absoluteness for $(2^\omega)^+$ -cc forcings $\mathbb{P} * \dot{\mathbb{Q}}$ can hold in a forcing extension of L . In this note we reduce the hypothesis of Theorem 1.2 to obtain an exact equiconsistency.

Definition 1.3. An ordinal κ is L -large to λ if for every $\alpha < \lambda$ there is an elementary $j : L_\alpha \rightarrow L_\beta$ with critical point κ such that $j(\kappa) \geq \alpha$.

Note that an ordinal κ is L -large to some cardinal of L if and only if κ is weakly compact in L . We obtain stronger notions by requiring that λ be inaccessible or weakly compact in L . Neither notion implies that $0^\#$ exists (simply collapse λ^+ and use absoluteness of $L[g]$), though L cannot see such an embedding with $\alpha \geq (\kappa^+)^L$.

Theorem 1.4. *The following are equiconsistent.*

- (a) Σ_2^1 sets are $(2^\omega)^+$ -cc-Universally Baire
- (b) There is a κ which is L -large to a weakly compact of L .

In the last section of this paper, we argue that in the model of Theorem 1.3 (a) there must be a Σ_2^1 set which has a weak capturing term at ω_2 but no full capturing term at ω_2 .

Theorem 1.5. *It is consistent that Σ_2^1 sets are weakly ω_2 -Universally Baire but not ω_2 -Universally Baire.*

In what follows, any space X^ω will carry the product topology, with the set X given the discrete topology. All pointclasses are boldface, and every statement below involving Σ_2^1 sets applies equally well to Π_2^1 sets. We would like to thank Hugh Woodin for sharing his proof of Theorem 1.2 and allowing us to include elements of it here, and Stevo Todorcevic for several helpful comments regarding earlier drafts of this paper.

2. PRELIMINARIES

Call a term \dot{A} a $Col(\omega, \kappa)$ capturing term for a set of reals A if there is a club of countable elementary submodels $X \prec H(\theta)$ with transitivity H and collapse map π such that

$$\pi(\dot{A})_g = A \cap H[g]$$

for every H -generic $g \subset Col(\omega, \pi(\kappa))$. A set of reals A has a $Col(\omega, \kappa)$ capturing term if and only if A is κ -Universally Baire (see Lemma 1.6 of [6]). The proof below uses this observation and an argument from [1].

Theorem 2.1. *The following are equivalent for a cardinal κ .*

- (1) Σ_2^1 sets are κ -Universally Baire
- (2) For all sufficiently large θ , there is a club of countable $X \prec H(\theta)$ such that $X[g]$ is Σ_2^1 elementary in V for every X -generic $g \subset Col(\omega, \kappa \cap X)$.

Proof. Let A be Σ_2^1 defined by a formula ϕ (we suppress any parameter). Assuming (2) let \dot{A} be the set of pairs (p, τ) such that $p \Vdash \phi(\tau)$. Thus \dot{A} is a capturing term for A and it follows that A is κ -Universally Baire. For the other direction assume S and T are trees for a Σ_2^1 set and its complement. Suppose the Σ_2^1 set is defined by a formula $\phi(x)$ (again suppressing parameters). The argument of Theorem 3.4 of [1] shows that

$$p[S]^{V[G]} = \{x \mid \phi(x)\}^{V[G]},$$

where $G \subset Col(\omega, \kappa)$ is V -generic. This uses Π_1^1 -Uniformization. Let ϕ be a Σ_2^1 formula defining the universal Σ_2^1 set A and let $X \prec H(\theta)$ contain S and T . Then $p[S]^{X[g]} = A \cap X[g]$ and $X[g]$ thinks $p[S]^{X[g]}$ is the universal Σ_2^1 set. Hence $X[g]$ is Σ_2^1 elementary in V . \square

Using (2) and the full strength of Covering for L we may now argue that Σ_2^1 sets being ω_3 -Universally Baire implies that $0^\#$ exists.

Theorem 2.2. *Assume Σ_2^1 sets are ω_3 -Universally Baire. Then $0^\#$ exists.*

Proof. Let $\kappa = \omega_3$. Let $\kappa = \omega_3$. We first argue that there are club many $\sigma \in [\kappa]^\omega$ such that $otp(\sigma)$ is a regular cardinal of L . Let $\kappa \in X \prec H(\theta)$ be countable with transitive collapse $\pi : X \rightarrow \bar{X}$. Note that $\pi(\kappa) = otp(X \cap \kappa)$

and that there are club many such $X \cap \kappa$. Thus \bar{X} thinks that $\pi(\kappa)$ is cardinal of L . If $\pi(\kappa)$ were not a regular cardinal of L then there would be a countable L_γ which sees this. Let $g \subset Col(\omega, \pi(\kappa))$ be \bar{X} -generic, and let $z \in X[g]$ be a real coding a well-ordering of length $\pi(\kappa)$. Then by Theorem 0.4 $\bar{X}[g]$ thinks there is a level of L which sees that the ordinal coded by z is not regular. This is a contradiction as $L^{\bar{X}} = L^{\bar{X}[g]}$. We now argue that the set of $\alpha < \kappa$ such that α is a regular cardinal of L contains a club in V . Let $f : \kappa^{<\omega} \rightarrow \kappa$ be such that any $\sigma \in [\kappa]^\omega$ which is closed under f has the property that $otp(\sigma)$ is a regular cardinal of L . Let $\alpha < \kappa$ such that $f[\alpha^{<\omega}] \subset \alpha$. Since there are club many such α it suffices to show that α is a regular cardinal of L . Suppose not. Then there is a countable $X \prec H(\kappa)$ with $\alpha \in X$ such that $X \cap \alpha = \sigma$ is closed under f . Let \bar{X} be the transitivization of X with collapse map π . Then \bar{X} thinks that $\pi(\alpha)$ is not a regular cardinal of L hence $\pi(\alpha)$ is not a regular cardinal of L by absoluteness. This contradicts $\pi(\alpha) = otp(\sigma)$. It follows that there is an α with $cf(\alpha) < \omega_2 < \alpha < \omega_3$ which is a regular cardinal of L . Let $\sigma \subset \alpha$ be unbounded in α and have size $cf(\alpha)$. Then σ cannot be covered by a set in L of size ω_1 . \square

We conjecture that ω_3 -cc Universally Baire suffices for the argument above. Under this assumption ω_3 is weakly compact in L by Lemma 4 of [5] and a theorem in [2]. We close this section with an equivalence between Σ_2^1 sets being ω_1 -UB and the existence of a club of suitably closed submodels. We say that ω_2 is inaccessible to $P(\omega_1)$ if ω_2 is an inaccessible cardinal in $L[X]$ for every $X \subseteq \omega_1$.

Lemma 2.3. *The following are equivalent.*

- (1) ω_2 is inaccessible to $P(\omega_1)$ and Σ_2^1 sets are ω_1 -Universally Baire
- (2) ω_2 is inaccessible to $P(\omega_1)$ and for sufficiently large θ there is a club of $X \prec H(\theta)$ such that for every $\tau \in P(\omega_1) \cap X$ and every $L[\tau]$ -cardinal $\gamma \in X \cap \omega_2$ the order type of $\gamma \cap X$ is itself an $L[\tau \cap X]$ -cardinal.
- (3) For sufficiently large θ there is a club of $X \prec H(\theta)$ such that for every $\tau \in P(\omega_1) \cap X$ the order type of $X \cap \omega_2$ is an $L[\tau \cap X]$ -cardinal.

Proof. By the argument of Theorem 2.2, condition (3) implies that ω_2 is inaccessible to $P(\omega_1)$. Thus (2) and (3) are equivalent. Again by a boldface version of an argument from Theorem 2.2, (1) implies (2). Let $X \prec H(\theta)$ be as in (2). Let $\pi : X \rightarrow \bar{X}$ be the collapse map and let $g \subset Col(\omega, \omega_1 \cap X)$ be \bar{X} -generic. Let $y = \pi(\tau)_g$ be a real in $\bar{X}[g]$. Since g is also $L[\pi(\tau)]$ -generic we have

$$\pi(\omega_2) > (\pi(\omega_1)^+)^{L[\pi(\tau)]} \geq (\omega_1)^{L[y]}$$

so that $\bar{X}[g]$ is correct about Σ_2^1 facts in the parameter y . \square

3. EQUICONSISTENCY RESULTS

Fix a surjection $f_\gamma : \omega_1 \rightarrow \gamma$ for each γ between ω_1 and ω_2 . If γ is a cardinal of L let S_γ denote the set of $\alpha < \omega_1$ such that the order type of $f_\gamma[\alpha]$ is an L -cardinal. Let S be the set of $\sigma \in [\omega_2]^\omega$ such that

$$\sigma \cap \omega_1 \in \bigcap_{\gamma \in \sigma} S_\gamma.$$

If we assume that ω_2 is inaccessible in L and that there are stationary many $\sigma \in [\omega_2]^\omega$ such that $otp(\sigma)$ is an L -cardinal then it follows that S is stationary. Now let \mathbb{Q} be the countable support product of \mathbb{Q}_γ , ranging over ordinals $\gamma < \omega_2$ which are L -cardinals, where \mathbb{Q}_γ is the poset for shooting a club through S_γ with countable conditions. It follows that \mathbb{Q} is (ω, ∞) -distributive. If CH holds then \mathbb{Q} satisfies the ω_2 -chain condition. The following key lemma is implicit in Woodin's proof of Theorem 1.2.

Lemma 3.1. *Suppose*

- (1) *Every subset of ω_1 is L -generic for some poset $\mathbb{P} \in L$ with $|\mathbb{P}| < \omega_2$*
- (2) *There are stationary many $\sigma \in [\omega_2]^\omega$ such that $otp(\sigma)$ is an L -cardinal*
- (3) *ω_2 is inaccessible in L and CH holds.*

Then Σ_2^1 sets are ω_1 -Universally Baire in $V[G]$ where $G \subset \mathbb{Q}$ is V -generic.

Proof. We show that condition (3) of Lemma 2.3 is satisfied in $V[G]$. As discussed above, \mathbb{Q} preserves cardinals under these hypotheses and by design there is in $V[G]$ a club of $\sigma \in [\omega_2]^\omega$ such that $otp(\sigma)$ is a cardinal of L . Further, condition (1) continues to hold in $V[G]$. Suppose $X \prec H(\theta)$ is such that $otp(X \cap \omega_2)$ is an L -cardinal. Let $\tau \in P(\omega_1) \cap X$. Then there are $\mathbb{P}, H \in X$ such that X thinks that $\mathbb{P} \in L$, $|\mathbb{P}| < \omega_2$, $H \subset \mathbb{P}$ is L -generic and $\tau \in L[H]$. Let $\pi : X \rightarrow \bar{X}$ be the transitivity map. As $otp(X \cap \omega_2) = (\omega_2)^{\bar{X}}$ is a limit cardinal of L , it follows that $\pi(H) \subset \pi(\mathbb{P})$ is L -generic and $\tau \cap X \in L[\pi(H)]$. Thus $otp(X \cap \omega_2)$ remains a cardinal in $L[\tau \cap X]$ as desired. \square

Theorem 3.2. *The following are equiconsistent.*

- (a) *ω_2 is inaccessible in L and Σ_2^1 are ω_1 -Universally Baire.*
- (b) *There are club many $\sigma \in [\omega_2]^\omega$ such that $otp(\sigma)$ is an L -cardinal.*
- (c) *ω_2 is inaccessible in L and there are stationary many $\sigma \in [\omega_2]^\omega$ such that $otp(\sigma)$ is a cardinal of L .*
- (d) *There is a κ which is L -large to an L -inaccessible.*

Proof. (a) implies (b) outright by Lemma 2.3. The argument for (b) implies (c) is implicit in the proof of Theorem 2.2. Assume (c). Let $g \subset Col(\omega, < \omega_1)$ be V -generic. Let $\kappa = \omega_2^V$. Then in $L[g]$ there is a stationary set

of $\sigma \in [\kappa]^\omega$ such that the order type of σ is an L -cardinal. Thus if $h \subset \text{Col}(\omega_1, < \omega_2^V)$ is $L[g]$ -generic then in $L[g][h]$ then the hypotheses of Theorem 3.1 are satisfied so that (a) holds in the forcing extension described there. Thus (a), (b) and (c) are equiconsistent. Assume (a). We will show that ω_1^V is L -large to ω_2^V in $V[g]$ where $g \subset \text{Col}(\omega, \omega_1)$ is V -generic. Let $X \prec H(\theta)$ be countable. Let $\pi : X \rightarrow H$ be the transitive collapse. Let $Y \prec H(\theta)$ with $\pi, H \in Y$ and let $j : Y \rightarrow M$ be its transitivization. Note that $j \circ \pi^{-1} = j(\pi)$. Call this map k . We have

$$k \upharpoonright L_{\omega_2^H} : L_{\omega_2^H} \rightarrow L_\gamma$$

is fully elementary with critical point ω_1^H and this map is an element of M . Because Y sees that H is countable we have

$$k(\omega_1) = Y \cap \omega_1 > \omega_2^H.$$

Let $g \subset \text{Col}(\omega, \omega_1^H)$ be H -generic. Let $\alpha < \omega_2^H$ be arbitrary and let $x \in H[g]$ be a real coding a well-ordering of length α . The sentence asserting the existence of a transitive model of a sufficient fragment of set theory containing y which sees an embedding $k : L_\alpha \rightarrow L_\beta$ with critical point ω_1^H such that $j(\omega_1^H) > \alpha$ is Σ_2^1 in the parameter x . Hence $H[g]$ sees such an embedding. As α is arbitrary we conclude that $H[g]$ thinks that ω_1^H is L -large to ω_2^H . Now apply π . To connect (d) back to (a) assume that κ is L -large to some L -inaccessible λ . Let $g_\lambda \subset \text{Col}(\omega, < \lambda)$ be V -generic. Then g_λ is also L -generic for the same forcing. By folding the embeddings witnessing our hypothesis (d) into countable submodels and collapsing, we see that κ is L -large to $\lambda = \omega_1^{L[g_\lambda]}$ in $L[g_\lambda]$. For ordinals $\gamma < \lambda$ let g_γ denote $g \cap \text{Col}(\omega, < \gamma)$. We claim that in $L[g_\kappa]$ there are stationary many $\sigma \in [\lambda]^\omega$ such that $\text{otp}(\sigma)$ is an L -cardinal. It will then follow that (c) holds after forcing with $\text{Col}(\omega_1, < \lambda)$. Let $f : \lambda^{<\omega} \rightarrow \lambda$ belong to $L[g_\kappa]$ be arbitrary. Let $\delta < \lambda$ be a cardinal of $L[g_\kappa]$ such that $f[\delta^{<\omega}] \subseteq \delta$. Let α be a regular cardinal of $L[g_\kappa]$ below λ which is greater than $(\delta^+)^L$ so that $f \upharpoonright \delta^{<\omega} \in L_\alpha[g_\kappa]$. In $L[g_\lambda]$ there is an elementary embedding

$$j : L_\alpha \rightarrow L_\beta$$

with critical point κ such that $j(\kappa) > \alpha$. By standard arguments (using the fact that $\text{Col}(\omega, < \kappa)$ is κ -cc in L) this embedding extends to a fully elementary

$$j : L_\alpha[g_\kappa] \rightarrow L_\beta[g_{j(\kappa)}].$$

The embedding is defined by $j(\text{val}(\tau, g_\kappa)) = \text{val}(j(\tau), g_{j(\kappa)})$ and since it extends j we will also denote it by j . Now, let σ denote the set $j[\delta]$. Let z be a real in $L_\beta[g_{j(\kappa)}]$ coding a well-ordering of length δ . The structure $L_\beta[\hat{g}]$ has a tree T consisting of pairs (s, t) with s a finite approximation to

a set of ordinals σ closed under f , and t a finite approximation to an order isomorphism between δ and σ . $L_\beta[g_{j(\kappa)}]$ must see a branch through this tree and the result follows by reflection using j . \square

Lemma 3.3. *Suppose κ is a weakly compact cardinal and \mathbb{P} is κ -cc. Suppose Π_2^1 sets are $< \kappa$ -Universally Baire in $V[G]$ where $G \subset \mathbb{P}$ is V -generic. Then Π_2^1 sets are κ -cc-UB in $V[G]$.*

Proof. We will use the fact that a set $A \subset \omega^\omega$ is κ -cc Universally Baire iff there are trees S, T on some $\omega \times \lambda$ which project to A and its complement and continue to project to complements after forcing with any κ -cc poset. So let $\dot{\mathbb{Q}}$ be a \mathbb{P} -name for a poset which is forced by \mathbb{P} to have the κ -cc. Thus $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ -cc in V . Now suppose \dot{x} is a $\mathbb{P} * \dot{\mathbb{Q}}$ -name for a real. Since $\mathbb{P} * \dot{\mathbb{Q}}$ is κ -cc and κ is weakly compact there is an elementary suborder $\mathbb{A} \prec \mathbb{P} * \dot{\mathbb{Q}}$ which has size strictly less than κ , decides \dot{x} , and has the property that maximal \mathbb{A} antichains are maximal antichains in $\mathbb{P} * \dot{\mathbb{Q}}$. The upshot of this is that over $V[G]$ where $G \subset \mathbb{P}$ is V -generic, every real which is generic for a κ -cc forcing is generic for a forcing of size $< \kappa$. Let A be a Σ_2^1 set and for each forcing \mathbb{Q} of size $< \kappa$ (whose underlying set is some ordinal below κ say) let $S_{\mathbb{Q}}, T_{\mathbb{Q}}$ be \mathbb{Q} -UB representation of A . These trees may be joined to produce the desired κ -cc Universal Baire representation of A . \square

Theorem 3.4. *The following are equiconsistent.*

- (1) Σ_2^1 sets are $(2^\omega)^+$ -cc-Universally Baire
- (2) ω_2 is weakly compact in L and there are stationary many $\sigma \in [\omega_2]^\omega$ such that $otp(\sigma)$ is a cardinal of L
- (3) There is a κ which is L -large to an weakly compact cardinal of L . compact.

Proof. This is identical to the proof of Theorem 3.2, using lemma 3.4 in the argument from (2) to (1) to get the stronger conclusion. Of course we are using that CH holds in all models under consideration. We need to show that (1) implies that ω_2 is weakly compact in L . Assume (a) and let \mathbb{Q} be the poset for forcing Martin's Axiom. Let $\mathbb{P} = Col(\omega, \omega_1) * \mathbb{Q}$ and note that \mathbb{P} is ω_2 -cc. In the extension $V[G]$ by \mathbb{P} we will have Σ_2^1 sets ccc-Universally Baire. Thus Σ_2^1 sets are Lebesgue measurable and have the property of Baire which implies that ω_1 is inaccessible to reals (see [4]). Thus in $\omega_1 = \omega_2^V$ is weakly compact in L in $V[G]$ by a result of Harrington and Shelah (see [2] or Lemma 7 of [3]). \square

4. WEAK CAPTURING DOES NOT IMPLY CAPTURING

A weakening of the Universal Baire property is presented in [6]. A set of reals A is weakly captured at κ if there is a $Col(\omega, \kappa)$ -term \dot{A} such that for

sufficiently large θ , for a club of countable $H \prec H(\theta)$, and for a comeager set of $g : \omega \rightarrow otp(H \cap \kappa)$,

$$\pi_H(\dot{A})_g = A \cap H[g],$$

where $otp(H \cap \kappa)$ is the order type of $H \cap \kappa$ and π_H is the transitivization map. A less metamathematical characterization is the following.

Lemma 4.1. *The following are equivalent.*

- (1) A is weakly captured at κ .
- (2) For every continuous $f : \kappa^\omega \rightarrow \omega^\omega$ there is a dense set of $p \in \kappa^{<\omega}$ such that either $f^{-1}(A) \cap \sigma^\omega$ is meager below p in σ^ω for a club of $\sigma \in [\kappa]^\omega$ or comeager below p in σ^ω for a club of $\sigma \in [\kappa]^\omega$.

Proof. (1) implies (2) is immediate as any condition $p \in \kappa^{<\omega}$ has a refinement $p \subseteq q$ such that $q \Vdash_{Col(\omega, \kappa)} f(\dot{g}) \in \dot{A}$ or $q \Vdash_{Col(\omega, \kappa)} f(\dot{g}) \notin \dot{A}$. For the other direction, if τ is a standard $Col(\omega, \kappa)$ term for a real then τ gives rise to a function $f_\tau : \kappa^\omega \rightarrow \omega^\omega$ defined by $f_\tau(g) = \tau_g$ which is continuous on a comeager set. Define \dot{A} to be the set of (p, τ) such that $f^{-1}(A) \cap \sigma^\omega$ is comeager below p in σ^ω for a club of $\sigma \in [\kappa]^\omega$. A straightforward argument shows that \dot{A} is a weak capturing term for A . \square

Theorem 4.2. *It is consistent that Σ_2^1 is weakly captured at ω_2 but not fully captured at ω_2 .*

Proof. Suppose κ is L -large to an L -weakly compact, and let $L[g][h]$ be the model of Theorem 2.2 in which $\kappa = \omega_1$ and $\lambda = \omega_2$. We have shown that there is a stationary set $S \subset [\omega_2]^\omega$ such that whenever $X \prec H(\theta)$ is such that $X \cap \omega_2 \in S$ then $X[g]$ is Σ_2^1 elementary in V for every X -generic $g \subset Col(\omega, \omega_1 \cap X)$. The forcing \mathbb{Q} of Lemma 3.1 puts a club through S and so in the extension all Σ_2^1 sets are ω_1 -Universally Baire. We first argue that $WRP_{(2)}(\omega_2)$ holds in $L[g][h][G]$. For $a \subseteq \lambda$ let \mathbb{Q}_a denote the countable support product of \mathbb{P}_γ taken over L -cardinals $\gamma \in a$, with S the underlying stationary set. Returning to $L[g]$ where λ is still weakly compact, let p be a condition and \dot{T} a term such that

$$p \Vdash_{Col(\omega_1, < \lambda) * \mathbb{Q}_\lambda}^{L[g]} \dot{T} \text{ is stationary in } [\omega_2]^\omega.$$

By the usual reflection argument we have an inaccessible $\delta < \omega_2$ such that

$$p \Vdash_{Col(\omega_1, < \delta) * \mathbb{Q}_\delta}^{L[g]} \dot{T}_\delta \text{ is stationary in } [\delta]^\omega,$$

where \dot{T}_δ denotes $\dot{T} \cap V_\delta$. Now let $h \subset Col(\omega_1, < \lambda)$ be $L[g]$ -generic and $G \subset \mathbb{Q}$ be $L[g][h]$ generic below the condition p . Let h_δ and G_δ be the restrictions to $Col(\omega_1, < \delta)$ and \mathbb{Q}_δ respectively. These are $L[g]$ -generic as well and

$$val(\dot{T}_\delta, h_\delta * G_\delta) = val(\dot{T}, h * G) \cap [\delta]^\omega$$

in $L[g][h][G]$, where. The key point is that the stationarity of $T_\delta = \text{int}(\dot{T}_\delta, h_\delta * G_\delta)$ is preserved. It suffices to show that the stationarity of T is preserved by $\mathbb{Q}_{\lambda \setminus \delta}$ over $L[h][G_\delta]$. The key point is that $\{\sigma \cap \delta \mid \sigma \in S\}$ contains a club in $[\delta]^\omega$. Thus if \dot{C} is a name for a club subset of $[\delta]^\omega$, we can find a dense set of conditions $t \in \mathbb{Q}_{\lambda \setminus \delta}$ with a corresponding $\sigma \in S$ such that $\sigma \cap \delta \in T$ and t forces that $\sigma \in \dot{C}$. Now let G be $L[g][h]$ -generic for the forcing \mathbb{Q} . This forcing does not add countable sets of ordinals. Let A be a Σ_2^1 set in $L[g][h][G]$. Then $A = A^{L[g][h]}$. Fix such an A . We claim that A is weakly captured in $L[g][h][G]$. Otherwise there is a condition $t \in \mathbb{Q}$, terms \dot{T}_m and \dot{T}_c , and $p \in \omega_2^{<\omega}$ such that t forces

- (1) \dot{T}_m and \dot{T}_c are both stationary subset of S
- (2) $\sigma \in \dot{T}_m$ implies $\dot{f}^{-1}(A) \cap \sigma^\omega$ is meager below p
- (3) $\sigma \in \dot{T}_c$ implies $\dot{f}^{-1}(A) \cap \sigma^\omega$ is comeager below p .

We may assume that there is $\delta < \omega_2$ such that t forces that both \dot{T}_m and \dot{T}_c reflect to δ . Let $\bar{t} \leq t$ and $\bar{p} \leq p$ such that

$$\bar{t} \Vdash_{\mathbb{Q}} \dot{f}^{-1}(A) \cap \delta^\omega \text{ is comeager below } \bar{p}.$$

It follows that \bar{t} forces that $\dot{f}^{-1}(A) \cap \sigma^\omega$ is comeager below \bar{p} for a club of $\sigma \in [\delta]^\omega$, a contradiction. To finish the proof of the theorem we must show that Σ_2^1 sets are not ω_2 -Universally Baire in $L[g][h][G]$. Let D be the set of $\alpha < \lambda$ such that $cf(\alpha) = \omega$ in L . As $L[g][h][G]$ is a λ -cc extension of L we know that D remains stationary in $L[g][h][G]$. Thus the set of regular cardinals of L below ω_2 cannot be club, as they would be if Σ_2^1 sets were ω_2 -Universally Baire by the argument of Theorem 2.2 \square

REFERENCES

- [1] Feng, Q., Magidor, M., Woodin, H., *Universally Baire sets of reals*, MSRI Pub. 26, 1992, pp. 203-242
- [2] Harrington, L., Shelah, S., *Some exact equiconsistency results in set theory*, Notre Dame Journal of Formal Logic 26, 1985, pp. 178-188
- [3] Schindler, R., *Forcing axioms and projective sets of reals*, Proceedings of FotFs III, in: "Classical and New Paradigms of computation and their complexity hierarchies" (Löwe et al., eds., Kluwer, 2004), pp. 207-222
- [4] Todorćević, S., Zoble, S., *Baire reflection*, preprint, 2006
- [5] Woodin, H., *On the strength of projective uniformization*, Logic Colloquium '81, J. Stern (ed.), 1982, pp. 365-383
- [6] Zoble, S., *Stationary Reflection and the Universal Baire Property*, forthcoming in Fundamenta Mathematicae, 2005

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO
 E-mail address: szoble@math.toronto.edu