

ON THE EXTENDER ALGEBRA BEING COMPLETE

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ABSTRACT. We show that a Woodin cardinal is necessary for the Extender Algebra to be complete. Our proof is relatively simple and does not use fine structure.

1. INTRODUCTION

Let $B_{\delta,\omega}$ denote the algebra of equivalence classes of boolean strings in V_δ formed from a countable set of atoms $\{a_n \mid n < \omega\}$. Two strings ϕ and ψ are equivalent in $B_{\delta,\omega}$ if $A(\phi) = A(\psi)$ in all generic extensions, where the map $A : B_{\delta,\omega} \rightarrow P(P(\omega))$ is defined inductively from

$$A(a_n) = \{x \subseteq \omega \mid n \in x\}.$$

Let δ be an inaccessible cardinal and $\mathcal{E} \subset V_\delta$ a set of extenders. The extender algebra $W_\delta(\mathcal{E})$ is defined as a quotient of $B_{\delta,\omega}$ by an ideal $I_\mathcal{E}$ defined from \mathcal{E} . For simplicity we require that an $E \in \mathcal{E}$ is a (κ, λ) -extender for some κ and λ , that λ is inaccessible and that E is strong to λ . $I_\mathcal{E}$ consists of elements of $B_{\delta,\omega}$ represented by strings of the form

$$i_E(\bigvee_{\alpha < \kappa} \phi_\alpha) \wedge \neg(\bigvee_{\alpha < \kappa} \phi_\alpha)$$

for such an $E \in \mathcal{E}$ and sequence $\{\phi_\alpha \mid \alpha < \kappa\}$ from V_κ . The key property of $W_\delta(\mathcal{E})$ if δ is a Woodin cardinal as witnessed by \mathcal{E} is that it satisfies the δ -chain condition, equivalently that it is a complete boolean algebra. From this alone Woodin deduces the "every real generic" theorem, one of the most important tools in Inner Model Theory.

Theorem 1.1. *(Every Real Generic; Woodin) Suppose M is a countable premouse and $\mathcal{E} \subset V_\delta^M$ is a set of extenders in M .*

- (1) *If δ is Woodin in M as witnessed by \mathcal{E} then $W_\delta(\mathcal{E})$ is a complete boolean algebra in M .*
- (2) *If $W_\delta(\mathcal{E})$ is complete in M , and M is $\omega_1 + 1$ -iterable then for every real $x \subset \omega$ in V there is an iteration $i : M \rightarrow M^*$ via a tree of countable length such that x is M^* -generic for $i(W_\delta(\mathcal{E}))$.*

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Here a premouse is merely a transitive set M and an ordinal $\delta \in M$ which satisfies a certain fragment of ZFC (see [MS]). By M^* -generic we mean that $G_x = \{[\phi] \in i(W_\delta(\mathcal{E})) \mid x \in A(\phi)\}$ is an M^* -generic filter. This will obtain exactly when

$$x \notin \bigcup_{[\phi] \in i(I_\mathcal{E})} A(\phi)$$

in which case $M^*[x] = M^*[G_x]$. The reader is referred to Theorem 7.14 of [St] for more details. The proof given there is easily modified to prove the version above, that is, it is not necessary for M to be a fine structural model, a fact well-known to inner model theorists. The purpose of this note is to prove a converse to Theorem 1.1.

Theorem 1.2. *Assume δ is an inaccessible cardinal and $V_\delta^\#$ exists. Assume $\mathcal{E} \subset V_\delta$ is a set of extenders and $W_\delta(\mathcal{E})$ is complete. Then some ordinal $\alpha \leq \delta$ is a Woodin cardinal in $L(V_\alpha)$.*

As a corollary we have that if δ is least such that some $W_\delta(\mathcal{E})$ is complete in $L(V_\delta)$ then δ is Woodin in $L(V_\delta)$. Woodin had proved an identical result using the backgrounded $L[\mathcal{E}]$ construction of [MiSt]. Our proof has the virtue of only using “coarse methods”, that is, techniques and results from [MS]. Both are not optimal in that they seem to require the existence of $V_\delta^\#$. Moreover, it seems likely that there is a direct equivalence between the completeness of $W_\delta(\mathcal{E})$ and δ being Woodin in V . Here is one version of this conjecture.

Question 1.3. Suppose δ is a cardinal and $\mathcal{E} \subset V_\delta$ is a set of extenders each of which is strong to its length. Suppose $W_\delta(\mathcal{E})$ is complete and has size δ . Must δ be a Woodin cardinal?¹

Our argument uses two theorems from [MS]. The first, a slight variation on their Corollary 5.11, concerns iterability of countable submodels of the universe under a smallness assumption, and the second shows that complicated iteration trees actually give rise to Woodin cardinals.

Theorem 1.4. *(Iterability; Martin, Steel) Suppose no $\alpha \leq \delta$ is Woodin in $L(V_\alpha)$. Suppose $\pi : N \rightarrow V_\theta$ is elementary with N countable. Let M be the preimage of $L(V_\delta)$. Then M is ω_1 -iterable via the strategy of picking the unique cofinal well-founded branch at limit stages.*

For an iteration tree T of limit length λ on a premouse M define

$$\delta(T) = \sup_{\alpha < \lambda} \inf_{\alpha \leq \gamma < \lambda} \text{str}(E_\gamma^T)$$

¹One could also consider the δ -generator version of the extender algebra; see [St]

as in [MS]. If b and c are cofinal branches of T then $V_{\delta(T)}^{M_b} = V_{\delta(T)}^{M_c}$ is well-founded and we call this the “common part” of T and denote it $M(T)$. The following is Corollary 2.3 of [MS].

Theorem 1.5. (*Distinct Branches; Martin, Steel*) *Suppose T is an iteration tree of limit length λ on a premouse M and b and c are distinct cofinal branches. Suppose $\alpha \geq \delta(T)$ belongs to $wfp(M_b) \cap wfp(M_c)$. Then $L_\alpha(M(T))$ thinks that $\delta(T)$ is a Woodin cardinal.*

2. PROOF OF THEOREM 1.2

Suppose $V_\delta^\#$ exists and $W_\delta(\mathcal{E})$ is complete. Then $W_\delta(\mathcal{E})$ is complete in $L(V_\delta)$.² We assume that no $\alpha \leq \delta$ is Woodin in $L(V_\alpha)$. Let θ be sufficiently large and

$$\pi : N \rightarrow V_\theta$$

be elementary with N countable and transitive and $V_\delta^\#, \mathcal{E}$ in the range of π . Let M denote the collapse of $L(V_\delta)$, that is

$$M = L^N(\pi^{-1}(V_\delta)),$$

and let $\bar{\delta}$ denote the preimage of δ . Then M is ω_1 -iterable by Theorem 1.4. The (unique) iteration strategy is to pick the unique cofinal well-founded branch at every limit stage. We will work inside of $M[g]$ where $g \subset Col(\omega, V_{\bar{\delta}}^M)$ is V -generic. It is easy to see that $Col(\omega, V_{\bar{\delta}}^M)$ is $\bar{\delta}$ -cc in M and that $M[g] = L_\eta[g]$ where $\eta = M \cap OR$. It follows that $M[g]$ satisfies the Axiom of Choice (M may not), a fact which we shall use later. Our use of the existence of $V_\delta^\#$ is the following.

Claim 2.1. For any $\gamma < \eta$ and $M[g]$ -generic $h \subset Col(\omega, \gamma)$ the model $M[g][h]$ is Σ_2^1 correct in V .

Proof. M is of the form $L_\eta(V_{\bar{\delta}}^M)$. By standard facts about sharps $\pi^{-1}(V_\delta^\#) = \pi^{-1}(V_\delta)^\#$. Thus η , which is the ordinal height of N as well as M , is a limit cardinal of $L(M)$, in fact an inaccessible cardinal of $L(M)$ (we use here that θ is a suitable reflection point). Thus g is generic over $L(M)$, h is generic over $L(M)[g]$, and η is a limit cardinal of $L(M)[g][h]$. It follows from Schonfield absoluteness that $M[g][h]$ is Σ_2^1 -correct in V . \square

We will work inside $M[g]$ where $g \subset Col(\omega, V_{\bar{\delta}})$ is a fixed M -generic in V . Let $x \in M[g]$ be a real coding the generic g in the sense that $L[x] = L[g]$. In $M[g]$ we construct an iteration tree on M to make x

² $L(V_\delta)$ may not see an arbitrary \mathcal{E} but we may assume without generality that \mathcal{E} is the set of all extenders in V_δ .

generic over the final model. Note that $(\omega_1)^{M[g]} = (\bar{\delta}^+)^M < \eta$. Call this ordinal κ . We claim that either

- (1) At some limit stage $\alpha \leq \kappa$, $M[g]$ does not see a cofinal well-founded branch of $T \upharpoonright \alpha$, or
- (2) The construction succeeds in producing an iterate M^* over which x is generic via a tree which is countable in $M[g]$.

This uses the proof of Theorem 1.1. There are a couple subtle points here so we elaborate. First, we have shown that $M[g]$ is a model of ZFC . In fact, $M[g] = L_\eta[g]$ has a definable well-ordering. This ordering is used to construct the tree T on M . At a stage β find the least extender E which belongs to $i_{0,\beta}(\mathcal{E})$ (which is a subset of $\subset M_\beta$ and hence $M[g]$) and which generates an element $[\phi]$ of $i_{0,\beta}(I_\mathcal{E})$ which is satisfied by x in the sense that $x \in A(\phi)$. Apply this extender to the appropriate model on the tree (according to the requirements in the definition of iteration tree). At limit stages pick the unique cofinal well-founded branch if it exists. Suppose case (1) does not obtain and the construction lasts κ stages producing a tree T of length κ with cofinal well-founded branch b and an embedding $i_b : M \rightarrow M_b$. For a model M_β on the tree let M_β^* denote $i_{0,\beta}(V_{\bar{\delta}}^M)$. Thus T may be viewed as a tree on these smaller structures. Inside $M[g]$ let $j : H \rightarrow H(\xi)$ where ξ is large enough, H is countable and transitive and j is elementary with $x, b, T \in \text{ran}(j)$. Let $\alpha = H \cap \kappa$. It is easy to see that $\alpha \in b$, $j^{-1}(M_b^*) = M_\alpha^*$ and

$$j \upharpoonright M_\alpha^* = i_{\alpha,b} \upharpoonright M_\alpha^*$$

giving the usual contradiction. We now show that both cases contradict our original smallness assumption. The second case leads to a contradiction as follows. Let $i^* : M \rightarrow M^*$ which we have assumed is definable in $M[g]$. By assumption, x is generic for $i^*(W_{\bar{\delta}}(\mathcal{E}))$. Clearly $M^*[x] = M[g]$. By the chain condition $i^*(\bar{\delta})$ is a regular cardinal of $M^*[x]$ and hence is equal to κ . This is a contradiction because $M[g]$ sees that $i^*(\bar{\delta})$ is a countable ordinal. So we may assume that the first case obtains. Let λ be the limit length of the tree T for which $M[g]$ does not see a well-founded cofinal branch. In the outside world, there is a unique cofinal well-founded branch b_λ with final well-founded model M_λ .

Claim 2.2. For cofinally many $\gamma < \eta$ there is $h \subset \text{Col}(\omega, \gamma)$ which is $M[g]$ -generic such that $M[g][h]$ sees distinct cofinal branches of T with γ in the well-founded part of both final models.

Proof. In any such $M[g][h]$ there is a cofinal branch b of T with γ in the well-founded part of M_b by Claim 2.1 as the sentence asserting the

existence of such a branch is Σ_2^1 in any code for a well-ordering of length γ . If this branch is unique then it must belong to $M[g]$ by homogeneity of the forcing $Col(\omega, \gamma)$. If this were true for a tail of $\gamma < \eta$ then $M[g]$ would see M_b contrary to our assumption. \square

Thus in any such $M[g][h]$ the model $L_\eta(M(T))$ thinks that $\delta(T)$ is Woodin by Theorem 1.5 and hence $M[g]$ sees that $L_\eta(M(T))$ thinks that $\delta(T)$ is Woodin. Let $i_\lambda : M \rightarrow M_{b_\lambda}$ be the branch embedding, which exists outside of M . Thus M_{b_λ} thinks that some ordinal $\alpha \leq i_\lambda(\bar{\delta})$ is Woodin in $L(V_\alpha)$ so M thinks the same of some ordinal $\alpha \leq \bar{\delta}$ contradicting our assumption.

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