NOTE ON BMM

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APRIL, 2006

Abstract. We present an ad hoc strengthening of BMM which we denote BMM^s. BMM^s is an absoluteness principle for $H(\omega_2)$ which follows from MM and which implies the saturation of NS in an inner model containing $P(\omega_1)$ with a measurable cardinal. It follows that BMM^s implies Δ^1_2-determinacy, bounding by canonical functions, and $u_2 = \omega_2$. We also discuss a limitation to pushing beyond Δ^1_2-determinacy. Substantial theorems of Bagaria, Schindler, Steel, Todorcevic, and Woodin are used.

1. Introduction

A poset $\mathbb{P}$ is stationary set preserving if $NS^{V[G]} \cap V = NS^V$ whenever $G \subset \mathbb{P}$ is V-generic. BMM^{++} asserts that

$$H(\omega_2), \in, NS)^V \prec_{\Sigma_1} (H(\omega_2), \in, NS)^{V[G]}$$

whenever G is V-generic for such a poset, and BMM is the analogous assertion without a predicate for NS. When we speak of Σ_1 definability, the structure will be clear from context. Schindler has the best lower bound for the strength of BMM, showing in [4] that BMM gives an inner model with a strong cardinal. It is believed that BMM or BMM^{++} ought to be stronger, and it is plausible that NS will play a role in the next breakthrough on this problem. We make some observations in this connection. First, observe that the following are easily seen to be consequences of BMM^{++}. The first is a bounded version of $WRP_{(2)}(\omega_2)$, and seems to require BMM^{++}.

> Suppose S, T are Σ_1-definable subsets of $[\omega_2]^\omega$ which are both stationary. Then there is $\delta < \omega_2$ such that both $S \cap [\delta]^\omega$ and $T \cap [\delta]^\omega$ are stationary in $[\delta]^\omega$.

A bounded version of $SRP(\omega_2)$ is true under BMM or BMM^{++} as well, again with the definability condition applying to the relevant structure.

2000 *Mathematics Subject Classification.* Primary 03E45; Secondary 03E35.
Key words and phrases. Bounded Martin's Maximum, Stationary Reflection.

> Suppose $S \subset [\omega_2]^\omega$ is Σ_1-definable and projective stationary. Then there is $\delta < \omega_2$ such that $S \cap [\delta]^\omega$ contains a club in $[\delta]^\omega$.

We do not see how to deduce the first principle from the second. We will consider a slightly stronger version of BMM which we call BMM^s. It is easy to see that the argument of [2] adapts to show that this principle follows from Martin's Maximum. It seems unlikely that it is a consequence of BMM^{++}.

> Suppose $a \in H(\omega_2)$ and $\varphi(x, a)$ is a Σ_1 formula with x free. Suppose $\mathbb{P}$ is stationary set preserving, $G \subset \mathbb{P}$ is V-generic and
>
> $$(H(\omega_2), \in)^{V[G]} \models \varphi(\omega_2^V, a).$$
>
> Then
>
> $$(H(\omega_2), \in)^V \models \phi(\delta, a)$$
>
> for a stationary set of $\delta < \omega_2$.

By adding a predicate for NS we obtain a principle BMM^{++s}.

Lemma 1.1. *MM implies BMM^s*

Proof. As in [2] suppose...

□

Todorcevic has shown in [5] that under BMM there exists a well-ordering of $P(\omega_1)$ which is Σ_1-definable over $H(\omega_2)$ in some parameter. Thus there is $B \subset \omega_2 \times \omega_1$ which is similarly Σ_1-definable such that $P(\omega_1) = \{X_\alpha \mid \alpha < \omega_2\}$ where $X_\alpha = \{\beta \mid (\alpha, \beta) \in B\}$. Schindler shows in [3] that BMM implies the closure of V under the sharp operation, and in [4] that V is closed under stronger operations. In particular $B^\dagger$ exists. We write $L[\mu, a]$ to mean the minimal class sized a-mouse with a measurable cardinal.

Lemma 1.2. *Assume BMM^s. Then NS is saturated in $L[\mu, B]$ and hence bounding holds in V as well as Δ^1_2-determinacy and $u_2 = \omega_2$.*

Proof. The point is that BMM^s is strong enough to seal a putative least antichain in $L[\mu, B]$. For $\sigma \in [\omega_2]^\omega$ let $\pi_\sigma : \sigma \to otp(\sigma)$ be the collapse of σ and let B_σ denote the image $\pi_\sigma[B \cap \sigma]$. We may think of the $L[\mu, B]$-least antichain in $P(\omega_1)/NS$ as a subset $A \subset \omega_2$. Let $A_\sigma = \pi_\sigma[A \cap \sigma]$. For a club of σ it will be true that the least antichain of $L[\mu, B_\sigma]$ is A_σ. In every case, let us use A_σ to denote the least antichain in $L[\mu, B_\sigma]$. Define the set $S \subset [\omega_2]^\omega$ to consist of all σ satisfying

(1) $L[B_\sigma]$ thinks that B_σ enumerates $P(\omega_1)$ in length ω_2
(2) $L[\mu, B_\sigma]$ thinks that the least NS antichain exists and is coded by some $A_\sigma \subset otp(\sigma)$
(3) There exists $\alpha \in A_\sigma$ such that $\sigma \cap \omega_1 \in X_{\pi_\sigma(\alpha)}$

This set is Σ_1 definable. If N is a transitive model containing $sup(\sigma)$ which sees π_σ and $B_\sigma^\dagger$, and sees that $B \cap sup(\sigma)$ exists, then N will be correct about the membership of σ in S. It follows that there exists a $\delta < \omega_2$ such that

(a) $(B \cap \delta)^\dagger$ embeds elementarily into $B^\dagger$, and
(b) S contains a club in $[\delta]^\omega$.

Thus the diagonal union of the members of $A \cap \delta$ contains a club which is a contradiction. Thus $L[\mu, B]$ is a model of NS saturated hence $u_2 = \omega_2$ by a theorem of Woodin. Further, bouding holds in $L[\mu, B]$ by standard arguments, and Δ^1_2-determinacy by a theorem of Steel. These last three assertions are absolute to V. □

Since BMM^{++} implies a bounded form of $WRP_{(2)}(\omega_2)$ one might think that more strength could be obtained using an argument of [1].

Lemma 1.3. *ZFC plus BMM^{++} plus NS saturated does not prove that the $M_1^\#$ operation is Σ_1-definable over $(H(\omega_2), \in, NS)$.*

Proof. Woodin has shown that it is consistent relative to large cardinals that $N = L^{M_1^\#}(P(\omega_1))$ satisfies NS saturated plus BMM^{++}. First note that $H(\omega_2)$ is closed under this operation under these assumptions. Work inside N and assume toward a contradiction that $M_1^\#(a)$ is uniformly Σ_1-definable for $a \in H(\omega_2)$. Using NS saturated it follows that $H(\omega_2)$ is closed under the M_1^* operation, and that this operation is also Σ_1 definable. $M_1^*(a)$ is the minimal a-mouse with a top sharp whose critical point is closed under the $M_1^\#$-operation. Using the bounded form of $WRP_{(2)}(\omega_2)$ it follows that $M_1^*(B)$ exists where B is a Σ_1 well-ordering of $P(\omega_1)$. This is a contradiction. □

It can be shown using arguments from Woodin's proof of PD from $MM(c)$ (elements of which are used in this note) that $BMM^{++} + NS$ saturated implies the closure of $H(\omega_2)$ under the $M_1^\#$ operation. But it is not at all clear how to get $M_2^\#$. This open question as well as a few others are listed below.

(1) Does ZFC + MM prove that the $M_1^\#$ operation is not uniformly Σ_1 definable over $(H(\omega_2, \in, NS)$? Can it be under CH?
(2) Does BMM imply BMM^s?
(3) Does BMM^{++} imply bounding or $u_2 = \omega_2$?

(4) Does BMM^{++} imply that $M_1^{\#}$ exists? BMM^{++} plus bounding?
(5) Does $BMM^{++} + NS$ saturated imply that $M_2^{\#}$ exists?

References

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