

A HAMILTONIAN APPROACH TO NONLINEAR MODULATION OF SURFACE WATER WAVES

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ABSTRACT. Many wave phenomena in physics are described by weakly nonlinear nearly monochromatic solutions in the form of modulated wave packets. The examples include ocean waves as well as waves in optics and in plasmas. There are a number of approaches to deriving the envelope equations for these theories of amplitude modulation. In this paper, we give a unified approach, based on the principles of a Hamiltonian formulation of the equations of motion. Our principal example is the system of equations of free surface water waves, for which we give a derivation of the classical nonlinear Schrödinger and Davey–Stewartson equations, as well as the higher-order Dysthe system. One consequence of our analysis from this point of view is that the Dysthe equation can be posed as a Hamiltonian PDE.

1. INTRODUCTION

Modulation theory is a well-established method in applied mathematics to study the long time evolution and stability of oscillatory solutions to partial differential equations (PDEs). Typical PDEs to which the theory is applied are nonlinear dispersive evolution equations describing wave phenomena that arise in physical applications. The usual modulational Ansatz is to anticipate a weakly nonlinear monochromatic form for solutions, and to derive equations describing the evolution of its envelope. Typically one finds the nonlinear Schrödinger equation (NLS) as a canonical equation for the first nontrivial term (or possibly the Davey–Stewartson system in the case of higher space dimensions). Well known cases include descriptions of problems in optics (Ablowitz & Segur [1], Newell & Moloney [2]), in plasmas physics (Zakharov, Musher & Rubenchik [3]), and in the problem of water waves (Zakharov [4], Hasimoto & Ono [5], Ablowitz & Segur [6]). See also Sulem & Sulem [7] for a review. For the next-order correction to this description of water waves, one finds the Dysthe equation (Dysthe [8], Stiassnie [9], Brinch-Nielsen & Jonsson [10] and more recently Agafontsev, Dias & Kuznetsov [11]). Higher-order corrections have been understood to play

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an important rôle in the modulational approximation; indeed it has been observed that the addition of the higher-order terms provides improvements for stability properties of finite amplitude waves (Dysthe [8], Janssen [12]), as compared to the NLS description. Further numerical study of the Dysthe equation and comparison to experiments are presented in Lo & Mei [13] and Ablowitz et al. [14].

One of the usual approaches to modulation theory is a direct perturbation method involving multiple spatial and temporal scales. This approach is however known for often giving rise to long calculations. An alternative approach was codified in Whitham [15], who developed an elegant method of averaged Lagrangians and a transformation theory. Another approach to such derivations was given by Zakharov and his co-workers, based on a Fourier mode coupling formalism and an expansion of the dispersion relation with respect to a small parameter [3].

In this paper we present a systematic approach to the derivation of the equations of modulation theory, based on averaged Hamiltonians. It gives a simple and straightforward method for Hamiltonian PDEs. This is in the spirit of the paper of Craig & Groves [17] which gives a uniform approach to the water waves problem in the long wave scaling regime. The method involves a formal expansion in small parameter, and elements which enter our analysis include an expansion of Fourier multiplier operators, and in the case of water waves, a precise description of the Dirichlet–Neumann operator. Our approach is closest to that of Zakharov, however what differentiates it is that throughout the approach we are careful to retain a certain point of view. Namely, that our PDEs are considered to be Hamiltonian systems, that scaling transformations and Ansätze are considered to be canonical transformations under change of symplectic form, and the formal expansion is performed in the expression of the Hamiltonian. Furthermore, several aspects of the formal expansion method are able to be placed on mathematically rigorous grounds, including two analytic results which concern the form of a Fourier multiplier operator expansion and a scale separation lemma for multiple scale functions.

We first illustrate the method in Section 2 on the nonlinear Klein–Gordon equation as a model problem. Subsequently we address the problem of the water wave equations, both in two and higher space dimensions. The approach uses only a few basic features of these equations, namely that they are dispersive Hamiltonian PDEs and that momentum and wave action are conserved quantities of motion. We distinguish two cases; in Section 3 we study the case of finite depth, and in Section 4 the case of infinite depth. In all cases we give our derivation of the NLS equation at the leading order, and our version of the Dysthe system at higher order, through which we exhibit the simplicity and straightforward nature of the approach. As a byproduct, we exhibit a Hamiltonian for this version of the Dysthe equation, for which, as far as we know, the fact that it can be written in the form of a Hamiltonian PDE has not previously been noticed.

2. A MODEL EQUATION

This section illustrates the approach to modulation theory through the example of the nonlinear Klein–Gordon equation, which is the following dispersive equation

$$(2.1) \quad \partial_t^2 v = \Delta_x v - m^2 v - v^3, \quad x \in \mathbb{R}^d, t \in \mathbb{R}$$

for real valued functions v and p . It can be written as a Hamiltonian PDE with the Hamiltonian

$$(2.2) \quad H = \int \left(\frac{1}{2} p^2 + \frac{1}{2} |\nabla v|^2 + \frac{m^2}{2} v^2 + \frac{1}{4} v^4 \right) dx$$

so that (2.1) is rewritten in first-order form

$$\begin{aligned} \partial_t v &= p \\ \partial_t p &= \Delta_x v - m^2 v - v^3. \end{aligned}$$

This exhibits the Hamiltonian character of the problem, namely

$$(2.3) \quad \partial_t \begin{pmatrix} v \\ p \end{pmatrix} = J \begin{pmatrix} \delta_v H \\ \delta_p H \end{pmatrix},$$

with the symplectic structure expressed through

$$J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix},$$

and with \mathbb{I} denoting the identity operator. The linear dispersion relation is $\omega^2(k) = |k|^2 + m^2$ and it is thus natural to define the Fourier multiplier operator $\omega(D_x) = (|D_x|^2 + m^2)^{1/2}$ where $D_x = -i\partial_x$ is the usual choice of self-adjoint form for partial derivatives.

Changing variables to complex symplectic coordinates, we define $a(D_x) = \omega(D_x)^{1/2}$, and then using it to define complex symplectic coordinates¹,

$$(2.4) \quad z = \frac{1}{\sqrt{2}} \left(a(D_x)v + ia(D_x)^{-1}p \right)$$

so that

$$(2.5) \quad v = \frac{1}{\sqrt{2}} a(D_x)^{-1} (z + \bar{z}), \quad p = \frac{1}{\sqrt{2}i} a(D_x) (z - \bar{z}),$$

where the symbol $\bar{\cdot}$ denotes complex conjugation. This transformation changes the expression of the symplectic structure to standard complex form

$$J_1 = \begin{pmatrix} 0 & -i\mathbb{I} \\ i\mathbb{I} & 0 \end{pmatrix},$$

¹On the usual Sobolev space H^s the operator $a(D_x) : H^s \rightarrow H^{s-1/2}$ is bounded and invertible.

and the Hamiltonian is rewritten as

$$\begin{aligned}
 H_1(z, \bar{z}) &= \int \left(|a(D_x)z|^2 + \frac{1}{16} |a(D_x)^{-1}(z + \bar{z})|^4 \right) dx \\
 (2.6) \quad &= \int \bar{z} \omega(D_x)z dx + \frac{1}{16} \int |\omega^{-1/2}(D_x)(z + \bar{z})|^4 dx .
 \end{aligned}$$

We now introduce the modulational Ansatz

$$(2.7) \quad z = \varepsilon u(X) e^{ik_0 \cdot x}, \quad X = \varepsilon x \in \mathbb{R}^d,$$

for a modulated plane wave solution with carrier wave number k_0 . That is, $z = z(x, X)$ has the form of a multiple scale function and ε is a small parameter. However, maintaining the point of view that our scaling and Ansätze are transformations, the mapping from z to u results from a translation of Fourier space variables, followed by a spatial scaling [18], and it is invertible. In order to analyse the resulting expression in the Hamiltonian, the following result describes the action of Fourier multiplier operators on multiple scale functions.

Theorem 1. *Let $m(D_x)$ be a Fourier multiplier. For sufficiently smooth functions $f(X)$, we have*

$$\begin{aligned}
 (2.8) \quad m(D_x)(e^{ik_0 \cdot x} f(X)) &= e^{ik_0 \cdot x} m(k_0 + \varepsilon D_X) f(X) \\
 &= e^{ik_0 \cdot x} \left(m(k_0) + \varepsilon \partial_{k_j} m(k_0) D_{X_j} f(X) \right. \\
 &\quad + \frac{\varepsilon^2}{2} \partial_{k_j k_\ell}^2 m(k_0) D_{X_j X_\ell}^2 f(X) \\
 &\quad + \frac{\varepsilon^3}{3!} \partial_{k_j k_\ell k_n}^3 m(k_0) D_{X_j X_\ell X_n}^3 f(X) + \dots \\
 &\quad \left. + \frac{\varepsilon^{|j|}}{j!} \partial_{k_1}^{j_1} \dots \partial_{k_d}^{j_d} m(k_0) D_{X_1}^{j_1} \dots D_{X_d}^{j_d} f(X) + O(\varepsilon^{|j|+1}) \right).
 \end{aligned}$$

In (2.8) and hereafter, the Einstein summation notation is used for repeated indices. The proof of this theorem appears as Theorem 4.1 and extended in A2.1 of [19] where it was used to give a rigorous setting to modulation theory in the context of water waves. The proof relies on the Fourier representation of Fourier multipliers and Taylor expansions. In the formulation (2.8), the operator m acts on a multiple scale function in the simple monochromatic oscillatory form $e^{ik_0 \cdot x} u(\varepsilon x)$, and the expansion (2.8) has the result of giving a Taylor expansion about the wave number k_0 . In the X -variable, the truncation of this series at any finite order $|j|$ acts as a differential operator in the X variables, of order $|j|$.

The change of variables expressed by the modulational Ansatz (2.4)(2.7) is symplectic, up to a scaling factor, and therefore changes the symplectic structure given by J_1 only up to this factor:

$$J_2 = \varepsilon^d \begin{pmatrix} 0 & -i\mathbb{I} \\ i\mathbb{I} & 0 \end{pmatrix},$$

whose non-symplectic nature can be renormalized by simply rescaling time [18]. The transformed Hamiltonian is as follows

$$(2.9) \quad \varepsilon^d H_2(u, \bar{u}) = \varepsilon^2 \int \bar{u} \omega(k_0 + \varepsilon D_X) u \, dX \\ + \frac{\varepsilon^4}{16} \int |e^{ik_0 \cdot x} \omega^{-1/2}(k_0 + \varepsilon D_X) u + e^{-ik_0 \cdot x} \omega^{-1/2}(-k_0 + \varepsilon D_X) \bar{u}|^4 \, dX ,$$

and the evolution equations are

$$(2.10) \quad \partial_t u = -i \delta_{\bar{u}} H_2 , \quad \partial_t \bar{u} = i \delta_u H_2 .$$

The second equation of (2.10) is of course the complex conjugate of the first because of the original reality conditions imposed on v and p . The quartic term in the Hamiltonian simplifies further because of the following scale separation lemma:

Theorem 2. *Let g be a function on \mathbb{R}^d which is periodic over the fundamental domain \mathbb{R}^d/Γ , a torus, and denote $\mathbf{E}(g)$ its average value. For any function $f(X)$ in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, we have*

$$(2.11) \quad \int g(X/\varepsilon) f(X) \, dX = \mathbf{E}(g) \int f(X) \, dX + O(\varepsilon^N)$$

for any N .

The proof of this result can be found in [20]. It expresses that on the LHS of (2.11), the contributions of the fast oscillations homogenize, and this approximation is true at order $O(\varepsilon^N)$ for $f(X) \in \mathcal{S}(\mathbb{R}^d)$. In general the order of approximation is related to the smoothness of the given test function f .

Using the scale separation result, the Hamiltonian is now expressed as

$$(2.12) \quad \varepsilon^{d-2} H_2(u, \bar{u}) = \frac{1}{2} \int \left(\bar{u} \omega(k_0 + \varepsilon D_X) u + u \omega(-k_0 + \varepsilon D_X) \bar{u} \right) \, dX \\ + \frac{3\varepsilon^2}{8} \int \left(\omega^{-1/2}(k_0 + \varepsilon D_X) u \right)^2 \left(\omega^{-1/2}(-k_0 + \varepsilon D_X) \bar{u} \right)^2 \, dX + O(\varepsilon^N).$$

Following Theorem 1, the Taylor expansion of the linear dispersion relation gives

$$(2.13) \quad \omega(k_0 + \varepsilon D_X) f(X) \\ = \omega(k_0) f(X) + \varepsilon \partial_{k_j} \omega(k_0) D_{X_j} f(X) + \frac{1}{2} \varepsilon^2 \partial_{k_j k_l}^2 \omega(k_0) D_{X_j X_l}^2 f(X) \\ + \frac{1}{6} \varepsilon^3 \partial_{k_j k_l k_m}^3 \omega(k_0) D_{X_j X_l X_m}^3 f(X) + \dots .$$

Similarly,

$$\omega^{-1/2}(\pm k_0 + \varepsilon D_X) = \omega^{-1/2}(\pm k_0) \mp \frac{\varepsilon}{2} \omega^{-3/2}(\pm k_0) \partial_{k_j} \omega(\pm k_0) D_{X_j} + \dots .$$

These expressions in the Hamiltonian provide the expansion of H_2 in powers of ε that we seek:

$$\begin{aligned}
(2.14) \quad \varepsilon^{d-2} H_2(u, \bar{u}) &= \int \omega(k_0) |u|^2 dX \\
&+ \frac{\varepsilon}{2} \int \partial_{k_j} \omega(k_0) (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) dX \\
&+ \varepsilon^2 \int \left(\frac{1}{4} \partial_{k_j k_\ell}^2 \omega(k_0) (\bar{u} D_{X_j X_\ell}^2 u + u \overline{D_{X_j X_\ell}^2 u}) + \frac{3}{8} \omega^{-2}(k_0) |u|^4 \right) dX \\
&+ \varepsilon^3 \int \left(\frac{1}{12} \partial_{k_j k_\ell k_m}^3 \omega(k_0) (\bar{u} D_{X_j X_\ell X_m}^3 u + u \overline{D_{X_j X_\ell X_m}^3 u}) \right. \\
&\quad \left. - \frac{3}{8} \omega^{-3}(k_0) \partial_{k_j} \omega(k_0) |u|^2 (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) \right) dX \\
&+ O(\varepsilon^4),
\end{aligned}$$

where we have used that $\omega(k_0)$ is an even function.

Of the four orders represented in perturbation theory in this Hamiltonian, the first two can be eliminated through elementary considerations. Firstly, one transforms the system into a reference coordinate frame moving with the group velocity $\partial_k \omega(k_0)$. This is accomplished by subtracting from (2.14) a multiple of the impulse (or momentum),

$$\begin{aligned}
(2.15) \quad I &= \int v \partial_x p dx = \frac{1}{2} \int (\bar{z} D_x z - z D_x \bar{z}) dx, \\
\varepsilon^{d-2} I &= k_0 \int |u|^2 dX + \frac{\varepsilon}{2} \int (\bar{u} D_X u + u \overline{D_X u}) dX,
\end{aligned}$$

which is a conserved integral of motion [16][17][18], yielding

$$\begin{aligned}
(2.16) \quad \varepsilon^{d-2} (H_2 - \partial_k \omega(k_0) \cdot I) &= (\omega(k_0) - k_0 \cdot \partial_k \omega(k_0)) \int |u|^2 dX \\
&+ \varepsilon^2 \int \left(\frac{1}{4} \partial_{k_j k_\ell}^2 \omega(k_0) (\bar{u} D_{X_j X_\ell}^2 u + u \overline{D_{X_j X_\ell}^2 u}) + \frac{3}{8} \omega^{-2}(k_0) |u|^4 \right) dX \\
&+ \varepsilon^3 \int \left(\frac{1}{12} \partial_{k_j k_\ell k_m}^3 \omega(k_0) (\bar{u} D_{X_j X_\ell X_m}^3 u + u \overline{D_{X_j X_\ell X_m}^3 u}) \right. \\
&\quad \left. - \frac{3}{8} \omega^{-3}(k_0) \partial_{k_j} \omega(k_0) |u|^2 (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) \right) dX \\
&+ O(\varepsilon^4).
\end{aligned}$$

Secondly one adjusts the phase of solutions, subtracting a multiple of the L^2 norm

$$(2.17) \quad \varepsilon^{d-2} M = \int |u|^2 dX,$$

to obtain the reduced Hamiltonian

$$\begin{aligned}
(2.18) \quad \varepsilon^{d-2} \hat{H}_2 &:= \varepsilon^{d-2} \left(H_2 - \partial_k \omega(k_0) \cdot I - (\omega(k_0) - k_0 \cdot \partial_k \omega(k_0)) M \right) \\
&= \varepsilon^2 \int \left(\frac{1}{4} \partial_{k_j k_\ell}^2 \omega(k_0) (\bar{u} D_{X_j X_\ell}^2 u + u \overline{D_{X_j X_\ell}^2 u}) + \frac{3}{8} \omega^{-2}(k_0) |u|^4 \right) dX \\
&+ \varepsilon^3 \int \left(\frac{1}{12} \partial_{k_j k_\ell k_m}^3 \omega(k_0) (\bar{u} D_{X_j X_\ell X_m}^3 u + u \overline{D_{X_j X_\ell X_m}^3 u}) \right. \\
&\quad \left. - \frac{3}{8} \omega^{-3}(k_0) \partial_{k_j} \omega(k_0) |u|^2 (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) \right) dX \\
&+ O(\varepsilon^4) .
\end{aligned}$$

Posed as a system of equations for u , (2.3) becomes

$$(2.19) \quad \partial_t u = -i \delta_{\bar{u}} \hat{H}_2.$$

The subtraction of (2.17) from H_2 is justified by the fact that, although it is not by itself a constant of motion for the original equation (2.1), it is however a conserved quantity for the approximate system (2.19). Indeed, it is a straightforward calculation to show that $dM/dt = 0$ using (2.19) or, equivalently, that \hat{H}_2 Poisson commutes with M , meaning that

$$\{\hat{H}_2, M\} = \int (\delta_{\bar{u}} \hat{H}_2 \delta_u M - \delta_u \hat{H}_2 \delta_{\bar{u}} M) dX = 0 .$$

The conservation of (2.17) is directly related to the phase invariance of solutions of (2.19).

Expressed in coordinates, the system (2.19) is

$$\begin{aligned}
\partial_t u &= -i \varepsilon^2 \left[\frac{1}{2} \partial_{k_j k_l}^2 \omega(k_0) D_{X_j X_l}^2 u + \frac{3}{4} \omega^{-2}(k_0) |u|^2 u \right. \\
&\left. + \varepsilon \left(\frac{1}{6} \partial_{k_j k_l k_m}^3 \omega(k_0) D_{X_j X_l X_m}^3 u - \frac{3}{2} \omega^{-3}(k_0) \partial_{k_j} \omega(k_0) |u|^2 D_{X_j} u \right) \right].
\end{aligned}$$

Introducing the slow time scale $\tau = \varepsilon^2 t$ gives directly the cubic NLS equation at the lowest order of approximation, and an equation reminiscent of the Dysthe equation when the next order of approximation is taken into account. Explicitly, this is

$$\begin{aligned}
(2.20) \quad i \partial_\tau u &= -\frac{1}{2} \partial_{k_j k_l}^2 \omega(k_0) \partial_{X_j X_l}^2 u + \frac{3}{4 \omega^2(k_0)} |u|^2 u \\
&+ \frac{i \varepsilon}{6} \partial_{k_j k_l k_m}^3 \omega(k_0) \partial_{X_j X_l X_m}^3 u + \frac{3 i \varepsilon}{2} \omega^{-3}(k_0) \partial_{k_j} \omega(k_0) |u|^2 \partial_{X_j} u .
\end{aligned}$$

We note that the NLS equation takes essentially a universal form, as seen by this derivation. However the nonlinear term of higher order does not have in general a universal form, as we will find in the next sections devoted to the water wave problem, since it depends on the detailed structure of the nonlinear terms.

3. THE WATER WAVE PROBLEM WITH FINITE DEPTH

We now apply this analysis to the water wave problem that describes the motion of a free surface on top of an incompressible, ideal and irrotational fluid in a channel in two or three dimensions. We consider the Euler equations in Hamiltonian formulation where the free surface $\eta(x)$ and the boundary values $\xi(x)$ of the velocity potential constitute the canonical variables [4]

$$(3.1) \quad \partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \end{pmatrix} = J \nabla H.$$

The Hamiltonian functional is the total energy

$$(3.2) \quad H = \frac{1}{2} \int \xi(x) G(\eta) \xi(x) + g\eta^2(x) dx .$$

In the above Hamiltonian, g is the acceleration of gravity, and $G(\eta)$ is the free-surface Dirichlet–Neumann operator associated with the Laplace equation for the velocity potential φ :

$$(3.3) \quad G(\eta)\xi(x) = \nabla\varphi(x, \eta(x)) \cdot N(\eta)(1 + |\partial_x\eta|^2)^{1/2},$$

where $N(\eta)$ is the exterior unit normal on the free surface.

The time-dependent fluid domain consists of the region

$$S(\eta) = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : -h < y < \eta(x, t)\},$$

where h is the (constant) fluid depth, and $n = 2$ or 3 is the space dimension. For a given function $f(x)$, let $\hat{f}(k) = \sqrt{2\pi}^{-(n-1)} \int_{\mathbb{R}^{n-1}} f(x) e^{-ik \cdot x} dx$ denote its Fourier transform, and P_0 is the projection that associates to f its zero frequency mode $\hat{f}(0)$.

3.1. Canonical transformations and Ansatz. Use the Dirichlet–Neumann operator $G_0(D_x)$ for a flat surface elevation and a finite-depth channel given by [21],

$$G_0(D_x) = |D_x| \tanh(h|D_x|)$$

to define the Fourier multiplier

$$a(D_x) = \sqrt[4]{(g/G_0(D_x))} ,$$

and use this in turn for the canonical transformation $(\eta, \xi) \rightarrow (z, \bar{z}, \tilde{\eta}, \tilde{\xi})$ given by

$$(3.4) \quad \eta = \frac{1}{\sqrt{2}} a^{-1}(D_x)(z + \bar{z}) + \tilde{\eta} , \quad \tilde{\eta} = P_0 \eta$$

$$(3.5) \quad \xi = \frac{1}{\sqrt{2}i} a(D_x)(z - \bar{z}) + \tilde{\xi} , \quad \tilde{\xi} = P_0 \xi .$$

As above, the notation is that $D_x = -i\partial_x$. This operator reduces to $G_0(D_x) = |D_x|$ if the depth is infinite. The presence of the mean fields $\tilde{\eta}$ and $\tilde{\xi}$ is due to the fact that $a^{-1}(0) = 0$, so that the change of variables

$(\eta, \xi) \rightarrow (z, \bar{z})$ is not invertible². The new variables $(z, \bar{z}, \tilde{\eta}, \tilde{\xi})^\top$ are expressed in terms of $(\eta, \xi)^\top$ as follows;

$$(z, \bar{z}, \tilde{\eta}, \tilde{\xi})^\top = A_1 (\eta, \xi)^\top$$

where the 4×2 matrix A_1 is given by

$$(3.6) \quad A_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} a(D)(\mathbb{I} - P_0) & ia^{-1}(D)(\mathbb{I} - P_0) \\ a(D)(\mathbb{I} - P_0) & -ia^{-1}(D)(\mathbb{I} - P_0) \\ \sqrt{2}P_0 & 0 \\ 0 & \sqrt{2}P_0 \end{pmatrix}.$$

The equations of motion become

$$(3.7) \quad \partial_t \begin{pmatrix} z \\ \bar{z} \\ \tilde{\eta} \\ \tilde{\xi} \end{pmatrix} = J_1 \begin{pmatrix} \delta_z H \\ \delta_{\bar{z}} H \\ \delta_{\tilde{\eta}} H \\ \delta_{\tilde{\xi}} H \end{pmatrix}$$

with the symplectic form given by the matrix

$$(3.8) \quad J_1 = A_1 J A_1^\top = \begin{pmatrix} 0 & -i(\mathbb{I} - P_0) & 0 & 0 \\ i(\mathbb{I} - P_0) & 0 & 0 & 0 \\ 0 & 0 & 0 & P_0 \\ 0 & 0 & -P_0 & 0 \end{pmatrix},$$

where we have used that $P_0^2 = P_0$.

We look for solutions in the form of monochromatic waves with a modulated complex amplitude depending upon the second variables $X = \varepsilon x$,

$$(3.9) \quad z = \varepsilon u(X, t) e^{ik_0 \cdot x}, \quad \bar{z} = \varepsilon \bar{u}(X, t) e^{-ik_0 \cdot x}$$

$$(3.10) \quad \tilde{\eta} = \varepsilon^\alpha \tilde{\eta}_1(X, t), \quad \tilde{\xi} = \varepsilon^\beta \tilde{\xi}_1(X, t),$$

where the exponents $\alpha \geq 1$ and $\beta \geq 1$ are to be chosen.

Equivalently, the system has the form

$$(3.11) \quad \begin{pmatrix} u \\ \bar{u} \\ \tilde{\eta}_1 \\ \tilde{\xi}_1 \end{pmatrix} = A_2 \begin{pmatrix} z \\ \bar{z} \\ \tilde{\eta} \\ \tilde{\xi} \end{pmatrix}$$

where A_2 is the 4×4 diagonal matrix with the entries on the diagonal being $(\varepsilon^{-1} e^{-ik_0 \cdot x}, \varepsilon^{-1} e^{ik_0 \cdot x}, \varepsilon^{-\alpha}, \varepsilon^{-\beta})$. The equations of motion are

$$(3.12) \quad \partial_t v = J_2 \nabla_v H$$

with $v = (u, \bar{u}, \tilde{\eta}_1, \tilde{\xi}_1)^\top$ and

$$J_2 = \varepsilon^{n-1} A_2 J_1 A_2^\top$$

²Actually the projection P_0 is not well defined on the classical Sobolev spaces H^s , but it is so on weighted Sobolev spaces.

$$\begin{aligned}
&= \begin{pmatrix} 0 & -i\varepsilon^{n-3}e^{-ik_0 \cdot x}(\mathbb{I} - P_0)(e^{ik_0 \cdot x}) & 0 & 0 \\ i\varepsilon^{n-3}e^{ik_0 \cdot x}(\mathbb{I} - P_0)(e^{-ik_0 \cdot x}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^{n-1-\alpha-\beta}P_0 \\ 0 & 0 & -\varepsilon^{n-1-\alpha-\beta}P_0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -i\varepsilon^{n-3}\mathbb{I}' & 0 & 0 \\ i\varepsilon^{n-3}\mathbb{I}' & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^{n-1-\alpha-\beta} \\ 0 & 0 & -\varepsilon^{n-1-\alpha-\beta} & 0 \end{pmatrix},
\end{aligned}$$

where \mathbb{I}' is the identity on the class of functions $\{u(X)\}$, and the final 2×2 block retains essentially the standard symplectic form on the two-dimensional space of constants $(\tilde{\eta}_1, \tilde{\xi}_1)$.

3.2. Expansion of Hamiltonian. We expand the Hamiltonian in powers of ε

$$(3.13) \quad H = H^{(2)} + H^{(3)} + H^{(4)} + \dots$$

using the Taylor series expansion of the Dirichlet–Neumann operator [21]

$$(3.14) \quad G = G_0 + G_1 + G_2 + \dots$$

with the result that

$$(3.15) \quad H^{(2)} = \frac{1}{2} \int (\xi G_0 \xi + g \eta^2) dx,$$

$$(3.16) \quad H^{(3)} = \frac{1}{2} \int \xi G_1(\eta) \xi dx,$$

$$(3.17) \quad H^{(4)} = \frac{1}{2} \int \xi G_2(\eta) \xi dx,$$

where

$$G_1(\eta) = D_x \eta \cdot D_x - G_0 \eta G_0,$$

$$G_2(\eta) = -\frac{1}{2} (|D_x|^2 \eta^2 G_0 + G_0 \eta^2 |D_x|^2 - 2G_0 \eta G_0 \eta G_0).$$

Retaining terms of up to second order in η is sufficient for the purposes of the present paper, as this will include all of the contributions to the NLS and Dysthe systems. Denoting by $\omega(D_x) = (gG_0)^{1/2}$ the linear dispersion relation, we get that

$$(3.18) \quad H^{(2)} = \int \left(\bar{z} \omega(D_x) z + \frac{i}{\sqrt{2}} a(D_x) (z - \bar{z}) G_0 \tilde{\xi} + \frac{1}{2} \tilde{\xi} G_0 \tilde{\xi} + \frac{1}{2} g \tilde{\eta}^2 + \frac{g}{\sqrt{2}} \tilde{\eta} a^{-1}(D_x) (z + \bar{z}) \right) dx.$$

In (3.18), the first term is calculated as before

$$(3.19) \quad \int \bar{z} \omega(D_x) z dx = \varepsilon^{3-n} \int \bar{u}(X) \omega(k_0 + \varepsilon D_X) u(X) dX.$$

To calculate the second and last term of (3.18), we apply Theorem 2 and find that all terms are nonresonant, and therefore they do not contribute to the Hamiltonian. The remaining terms in $H^{(2)}$ are

$$(3.20) \quad \begin{aligned} \varepsilon^{n-3} H^{(2)} &= \int \bar{u}(X) \omega(k_0 + \varepsilon D_X) u(X) dX \\ &\quad + \frac{1}{2} \varepsilon^{2\beta-2} \int \tilde{\xi}_1 G_0 \tilde{\xi}_1 dX + \frac{g}{2} \varepsilon^{2\alpha-2} \int \tilde{\eta}_1^2 dX . \end{aligned}$$

In the above expression, we have

$$(3.21) \quad \int \tilde{\xi}_1 G_0 \tilde{\xi}_1 dX = \varepsilon^2 h \int \tilde{\xi}_1 |D_X|^2 \tilde{\xi}_1 dX + O(\varepsilon^4) .$$

We now compute $H^{(3)}$

$$(3.22) \quad H^{(3)} = \frac{1}{2} \int \eta (|D_x \xi|^2 - (G_0 \xi)^2) dx .$$

From the terms that come out of the expansion, it is natural to choose the exponents α and β such that $\alpha = \beta + 1$. We find

$$\begin{aligned} \varepsilon^{n-3} H^{(3)} &= \varepsilon^{\beta+1} \int \left(\alpha_1(k_0) |u|^2 \tilde{\eta}_1 - i |u|^2 k_0 \cdot D_X \tilde{\xi}_1 \right) dX \\ &\quad + \varepsilon^{\beta+2} \alpha_2^j(k_0) \int (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) \tilde{\eta}_1 dX \\ &\quad - \varepsilon^{\beta+2} \frac{i}{2} \int (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) D_{X_j} \tilde{\xi}_1 dX , \end{aligned}$$

where the coefficients are

$$(3.23) \quad \begin{aligned} \alpha_1(k_0) &= \frac{a^2(k_0)}{2} \left(|k_0|^2 - G_0^2(k_0) \right) \\ \alpha_2^j(k_0) &= \frac{a^2(k_0)}{2} \left(k_{0j} + |k_0|^2 \frac{\partial_{k_j} a}{a(k_0)} \right. \\ &\quad \left. - G_0(k_0) \partial_{k_j} G_0(k_0) - G_0^2(k_0) \frac{\partial_{k_j} a}{a(k_0)} \right) . \end{aligned}$$

The contributions from $H^{(4)}$ are

$$(3.24) \quad \varepsilon^{n-3} H^{(4)} = \varepsilon^2 \alpha_3(k_0) \int |u|^4 dX + \varepsilon^3 \alpha_4^j(k_0) \int |u|^2 (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) dX ,$$

with coefficients

$$\begin{aligned} \alpha_3(k_0) &= \frac{1}{4} \left(G_0(k_0)^2 G_0(2k_0) - |k_0|^2 G_0(k_0) \right) , \\ \alpha_4^j(k_0) &= \frac{3}{8} G_0(k_0) |k_0|^{-2} \left(G_0(2k_0) G_0(k_0) - |k_0|^2 \right) k_{0j} \\ &\quad + \frac{h}{8} \left(1 - \tanh^2(h|k_0|) \right) \left(2G_0(2k_0) G_0(k_0) - |k_0|^2 \right) k_{0j} \\ &\quad + \frac{h}{2} G_0(k_0)^2 \left(1 - \tanh^2(2h|k_0|) \right) k_{0j} . \end{aligned}$$

The conserved impulse vector valued integral is given by

$$(3.25) \quad I = - \int \eta \partial_x \xi \, dx$$

which in these coordinates is given by

$$(3.26) \quad \varepsilon^{n-3} I = \int \left(k_0 |u|^2 + \varepsilon \bar{u} D_X u - i \varepsilon^{2\beta} \tilde{\eta}_1 D_X \tilde{\xi}_1 \right) dX .$$

In analogy with the Klein–Gordon equation, we also make use of the quantity

$$\varepsilon^{n-3} M = \int |u|^2 \, dX ,$$

which is sometimes known as the wave action integral. After subtracting terms proportional to I and M ,

$$(3.27) \quad \hat{H} = H - \partial_k \omega(k_0) \cdot I - (\omega(k_0) - k_0 \cdot \partial_k \omega(k_0)) M ,$$

we obtain the renormalized Hamiltonian

$$\begin{aligned} \varepsilon^{n-3} \hat{H} &= \varepsilon^2 \int \left(\frac{1}{2} \partial_{k_j k_l}^2 \omega(k_0) \bar{u} D_{X_j X_l}^2 u + \alpha_3(k_0) |u|^4 \right. \\ &\quad + \varepsilon^{2\beta-2} \left(\frac{\hbar}{2} \tilde{\xi}_1 |D_X|^2 \tilde{\xi}_1 + \frac{g}{2} \tilde{\eta}_1^2 + i \partial_{k_j} \omega \tilde{\eta}_1 D_{X_j} \tilde{\xi}_1 \right) \\ &\quad + \alpha_1(k_0) \varepsilon^{\beta-1} \tilde{\eta}_1 |u|^2 - i \varepsilon^{\beta-1} |u|^2 k_0 \cdot D_X \tilde{\xi}_1 \\ &\quad + \frac{\varepsilon}{6} \partial_{k_j k_l k_m}^3 \omega(k_0) \bar{u} D_{X_j X_l X_m}^3 u \\ &\quad + \varepsilon \left(\alpha_2^j(k_0) \varepsilon^{\beta-1} \tilde{\eta}_1 - \frac{i}{2} \varepsilon^{\beta-1} D_{X_j} \tilde{\xi}_1 + \alpha_4^j(k_0) |u|^2 \right) \left(\bar{u} D_{X_j} u + u \overline{D_{X_j} u} \right) \\ &\quad \left. + O(\varepsilon^2) \right) dX . \end{aligned}$$

Here, as in the case of the Klein–Gordon equation, it can be shown that M is a conserved quantity of this system at the order of approximation that is being considered.

3.3. Hamilton equations. It is now natural to choose $\beta = 1$ (and thus $\alpha = 2$) for the exponents of the mean fields. Introducing the slow time

variables $\tau = \varepsilon^2 t$, the corresponding equations of motion are

$$\begin{aligned}
i\partial_\tau u &= -\frac{1}{2}\partial_{k_j k_l}^2 \omega \partial_{X_j X_l}^2 u + 2\alpha_3 |u|^2 u + (\alpha_1 \tilde{\eta}_1 - ik_0 \cdot D_X \tilde{\xi}_1) u \\
&\quad + i\varepsilon \left(\frac{1}{6} \partial_{k_j k_l k_m}^3 \omega \partial_{X_j X_l X_m}^3 u - \alpha_2^j (\tilde{\eta}_1 \partial_{X_j} u + \partial_{X_j} (u \tilde{\eta}_1)) \right. \\
&\quad \left. + \frac{1}{2} (\partial_{X_j} \tilde{\xi}_1 \partial_{X_j} u + \partial_{X_j} (u \partial_{X_j} \tilde{\xi}_1)) - 4\alpha_4^j |u|^2 \partial_{X_j} u \right) + O(\varepsilon^2) , \\
\varepsilon \partial_\tau \tilde{\eta}_1 &= h |D_X|^2 \tilde{\xi}_1 + k_0 \cdot \partial_X |u|^2 - i \partial_{k_j} \omega D_{X_j} \tilde{\eta}_1 \\
&\quad + \frac{\varepsilon}{2} \partial_{X_j} (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) + O(\varepsilon^2) , \\
\varepsilon \partial_\tau \tilde{\xi}_1 &= -(g \tilde{\eta}_1 + \alpha_1 |u|^2 + i \partial_{k_j} \omega D_{X_j} \tilde{\xi}_1) \\
&\quad - \varepsilon \alpha_2^j (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) + O(\varepsilon^2) .
\end{aligned}$$

For simplicity of notation, we have dropped the explicit dependency on k_0 of the coefficients α_i and ω . This is the NLS equation in the case $n = 2$ and the Davey–Stewartson system in a certain form, in the case $n = 3$.

When $n = 3$, in order to recovering the more usual form of the Davey–Stewartson system, we neglect all the Dysthe terms, obtaining

$$(3.28) \quad i\partial_\tau u = -\frac{1}{2}\partial_{k_j k_l}^2 \omega \partial_{X_j X_l}^2 u + 2\alpha_3 |u|^2 u + (\alpha_1 \tilde{\eta}_1 - ik_0 \cdot D_X \tilde{\xi}_1) u$$

$$(3.29) \quad h |D_X|^2 \tilde{\xi}_1 + k_0 \cdot \partial_X |u|^2 - \partial_{k_j} \omega \partial_{X_j} \tilde{\eta}_1 = 0$$

$$(3.30) \quad g \tilde{\eta}_1 + \alpha_1 |u|^2 + \partial_{k_j} \omega \partial_{X_j} \tilde{\xi}_1 = 0 .$$

From (3.30) we can solve $\tilde{\eta}_1$. After substitution in (3.28)(3.29), the system reads

$$\begin{aligned}
(3.31) \quad i\partial_\tau u &= -\frac{1}{2}\partial_{k_j k_l}^2 \omega \partial_{X_j X_l}^2 u + \alpha_5 |u|^2 u - \alpha_6^j u \partial_{X_j} \tilde{\xi}_1 \\
\mathcal{L} \tilde{\xi}_1 + \alpha_6^j \partial_{X_j} |u|^2 &= 0 ,
\end{aligned}$$

which is a standard form of the Davey–Stewartson system, where

$$\mathcal{L} = h |D_X|^2 + \frac{1}{g} \partial_{k_j} \omega \partial_{k_l} \omega \partial_{X_j X_l}^2$$

is a second-order differential operator with constant coefficients, and

$$\alpha_5 = 2\alpha_3 - \frac{\alpha_1^2}{g} , \quad \alpha_6^j = \frac{\alpha_1}{g} \partial_{k_j} \omega + k_{0j} .$$

When one retains the Dysthe terms, there is no further apparent simplification. However, by construction, the present method provides a conserved

Hamiltonian for the above Dysthe system, which takes the form

$$\begin{aligned}
H = & \int \left(\frac{1}{2} \partial_{k_j k_l}^2 \omega(k_0) \partial_{X_j} \bar{u} \partial_{X_l} u + \alpha_3(k_0) |u|^4 \right. \\
& + \frac{h}{2} \tilde{\xi}_1 |D_X|^2 \tilde{\xi}_1 + \frac{g}{2} \tilde{\eta}_1^2 + i \partial_{k_j} \omega \tilde{\eta}_1 D_{X_j} \tilde{\xi}_1 + \alpha_1(k_0) \tilde{\eta}_1 |u|^2 - |u|^2 k_{0j} \partial_{X_j} \tilde{\xi}_1 \\
& + \frac{\varepsilon}{6} \partial_{k_j k_l k_m}^3 \omega(k_0) \Im(\partial_{X_j} \bar{u} \partial_{X_l X_m}^2 u) \\
& \left. + \varepsilon \left(\alpha_2^j(k_0) \tilde{\eta}_1 - \frac{1}{2} \partial_{X_j} \tilde{\xi}_1 + \alpha_4^j(k_0) |u|^2 \right) (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) \right) dX ,
\end{aligned}$$

where \Im stands for the imaginary part.

4. THE WATER WAVE PROBLEM WITH INFINITE DEPTH

It is of interest to extend the analysis to the water wave problem with infinite depth as it gives a simplified Dysthe system. The only difference in the initial Hamiltonian lies in the form of the Dirichlet–Neumann operator $G_0(D_x)$ for the unperturbed fluid domain given by a flat surface elevation. It has now the form $G_0(D_x) = |D_x|$. As a consequence, it modifies the ordering in $H^{(2)}$ as (3.21) is now replaced by

$$(4.1) \quad \int \tilde{\xi}_1 G_0 \tilde{\xi}_1 dX = \varepsilon \int \tilde{\xi}_1 |D_X| \tilde{\xi}_1 dX .$$

This results in a modification of the values of the coefficients $\alpha_i(k_0)$; namely, the $\alpha_1(k_0)$ as defined in (3.23) vanishes, $\alpha_3(k_0) = \frac{|k_0|^3}{4}$ and $\alpha_4^j(k_0) = \frac{3}{8} |k_0| k_{0j}$. The resulting modified Hamiltonian is rewritten as

$$\begin{aligned}
(4.2) \quad \varepsilon^{n-3} \hat{H} = & \varepsilon^2 \int \left(\frac{1}{2} \partial_{k_j k_l}^2 \omega(k_0) \bar{u} D_{X_j X_l}^2 u + \alpha_3(k_0) |u|^4 \right. \\
& + \frac{1}{2} \varepsilon^{2\beta-3} \tilde{\xi}_1 |D_X| \tilde{\xi}_1 + \varepsilon^{2\beta-2} \left(\frac{g}{2} \tilde{\eta}_1^2 + i \partial_{k_j} \omega \tilde{\eta}_1 D_{X_j} \tilde{\xi}_1 \right) \\
& - i \varepsilon^{\beta-1} |u|^2 k_0 \cdot D_X \tilde{\xi}_1 + \frac{\varepsilon}{6} \partial_{k_j k_l k_m}^3 \omega(k_0) \bar{u} D_{X_j X_l X_m}^3 u \\
& + \varepsilon \left(\alpha_2^j(k_0) \varepsilon^{\beta-1} \tilde{\eta}_1 - \frac{i}{2} \varepsilon^{\beta-1} D_{X_j} \tilde{\xi}_1 + \alpha_4^j(k_0) |u|^2 \right) (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) \\
& \left. + O(\varepsilon^2) \right) dX .
\end{aligned}$$

The presence of the term $\varepsilon^{2\beta-3} \tilde{\xi}_1 |D_X| \tilde{\xi}_1$ in the second line of the RHS of (4.2) suggests that β should be equal to 2. Under this condition, (4.2) takes

the form

$$(4.3) \quad \varepsilon^{n-3} \hat{H} = \varepsilon^2 \int \left(\frac{1}{2} \partial_{k_j k_l}^2 \omega(k_0) \bar{u} D_{X_j X_l}^2 u + \frac{|k_0|^3}{4} |u|^4 \right. \\ \left. + \varepsilon \left(\frac{1}{6} \partial_{k_j k_l k_m}^3 \omega(k_0) \bar{u} D_{X_j X_l X_m}^3 u + \frac{1}{2} \tilde{\xi}_1 |D_X \tilde{\xi}_1 \right. \right. \\ \left. \left. - i |u|^2 k_0 \cdot D_X \tilde{\xi}_1 \right. \right. \\ \left. \left. + \frac{3}{8} |k_0| |k_{0j}| |u|^2 (\bar{u} D_{X_j} u + u \overline{D_{X_j} u}) \right) + O(\varepsilon^2) \right) dX ,$$

where we have written terms up to $O(\varepsilon)$ inside the integral. The conserved impulse vector valued integral I reduces to

$$(4.4) \quad I = - \int \eta \partial_x \xi \, dx = \varepsilon^{3-n} \int \left(k_0 |u|^2 + \varepsilon \bar{u} D_X u \right) dX + O(\varepsilon^{7-n}) .$$

Proceeding as before, Hamilton's canonical equations are

$$(4.5) \quad \partial_t u = -i \varepsilon^2 \left[\frac{1}{2} \partial_{k_j k_l}^2 \omega(k_0) D_{X_j X_l}^2 u + \frac{|k_0|^3}{2} |u|^2 u \right. \\ \left. + \varepsilon \left(\frac{1}{6} \partial_{k_j k_l k_m}^3 \omega(k_0) D_{X_j X_l X_m}^3 u - i k_0 \cdot u D_X \tilde{\xi}_1 \right. \right. \\ \left. \left. + \frac{3|k_0|}{2} |u|^2 k_0 \cdot D_X u \right) \right] + O(\varepsilon^4) \\ \partial_t \tilde{\eta}_1 = |D_X \tilde{\xi}_1 + i k_0 \cdot D_X |u|^2 + O(\varepsilon) \\ \partial_t \tilde{\xi}_1 = O(\varepsilon) .$$

Introducing the slow time $\tau = \varepsilon^2 t$ and solving for $\tilde{\xi}_1$ at lowest order in terms of u :

$$(4.6) \quad \tilde{\xi}_1 = -i |D_X|^{-1} k_0 \cdot D_X |u|^2 + O(\varepsilon)$$

lead to the nonlinear Schrödinger equation at lowest order, and to a Dysthe-type equation [8] when one considers the next-order corrections. This latter takes the form

$$(4.7) \quad 2i \partial_\tau u = -\partial_{k_j k_l}^2 \omega(k_0) \partial_{X_j X_l}^2 u + |k_0|^3 |u|^2 u \\ + \varepsilon \left(\frac{i}{3} \partial_{k_j k_l k_m}^3 \omega(k_0) \partial_{X_j X_l X_m}^3 u - 3i |k_0| |u|^2 k_0 \cdot \partial_X u \right. \\ \left. + 2u k_{0j} k_{0l} |D_X|^{-1} \partial_{X_j X_l}^2 |u|^2 \right) .$$

We note the presence of the nonlocal operator $|D_X|^{-1} = (-\Delta)^{-1/2}$ in the equation, which is a consequence of the mean flow terms, and which can be expressed using the Hilbert transform.

Again, there is a conserved Hamiltonian for the Dysthe equation that we have derived, both for dimensions $n = 2$ or 3 , given by

$$(4.8) \quad H = \frac{1}{2} \int \left[\partial_{k_j k_l}^2 \omega(k_0) \partial_{X_j} u \partial_{X_l} \bar{u} + \frac{|k_0|^3}{2} |u|^4 \right. \\ \left. + \varepsilon \left(\frac{1}{3} \partial_{k_j k_l k_m}^3 \omega(k_0) \Im(\partial_{X_j} \bar{u} \partial_{X_l X_m}^2 u) - k_{0j} k_{0l} (\partial_{X_j} |u|^2) |D_X|^{-1} \partial_{X_l} |u|^2 \right. \right. \\ \left. \left. + \frac{3}{2} |k_0| k_{0j} |u|^2 \Im(\bar{u} \partial_{X_j} u) \right) \right] dX .$$

In the above formulas,

$$\partial_{k_j k_l}^2 \omega(k_0) = |k_0|^{-1} \left(\delta_{jl} - \frac{k_{0j} k_{0l}}{|k_0|^2} \right)$$

and

$$\partial_{k_j k_l k_m}^3 \omega(k_0) = -|k_0|^{-3} \left(\delta_{jl} k_{0m} + \delta_{lm} k_{0j} + \delta_{mj} k_{0l} + \frac{k_{0j} k_{0l} k_{0m}}{|k_0|^2} \right) .$$

It can be checked that evaluating

$$\partial_\tau u = -i \delta_{\bar{u}} H$$

using the closed form (4.8) of the Hamiltonian yields (4.7).

In order to compare our equations (4.7) in Hamiltonian form with the Dysthe-type equations that have previously appeared in the literature, introduce the new variables

$$(4.9) \quad \psi = \frac{1}{\sqrt{2}} \left(\frac{|k_0|}{g} \right)^{1/4} \left(1 + \frac{\varepsilon}{4|k_0|^2} k_0 \cdot D_X \right) u,$$

which is similar to Stiassnie [9]. This variable can be thought of as a first-order approximation of η as given by (3.4), which is to say that $\psi(X)$ is directly related to the free-surface elevation. Inverting (4.9), inserting it in (4.7) and retaining only $O(\varepsilon)$ terms, leads to the following equation

$$(4.10) \quad 2i \partial_\tau \psi = -\partial_{k_j k_l}^2 \omega(k_0) \partial_{X_j X_l}^2 \psi + 2g^{1/2} |k_0|^{5/2} |\psi|^2 \psi \\ + \varepsilon \left(\frac{i}{3} \partial_{k_j k_l k_m}^3 \omega(k_0) \partial_{X_j X_l X_m}^3 \psi - 6ig^{1/2} |k_0|^{1/2} |\psi|^2 k_0 \cdot \partial_X \psi \right. \\ \left. - ig^{1/2} |k_0|^{1/2} \psi^2 k_0 \cdot \partial_X \bar{\psi} + 4g^{1/2} |k_0|^{-1/2} \psi k_{0j} k_{0l} |D_X|^{-1} \partial_{X_j X_l}^2 |\psi|^2 \right) .$$

This equation contains all the usual Dysthe terms including the high-order nonlinear term in $\psi^2 k_0 \cdot \partial_X \bar{\psi}$ as can be seen in [8][12][9][13][14]. The Hamiltonian structure is however lost in the transformation to the ψ variables. We also note that the numerical coefficients in (4.10) are very similar to those in the equation derived by Stiassnie [9]. However, in general, we should not expect to obtain precisely the same coefficients as in previous work, since the various approaches do not use precisely the same physical variables (in particular the choice of the velocity potential). Our own choice of physical variables is quite natural, consisting of the canonically conjugate variables η and ξ as related to $(z, \bar{z}, \bar{\eta}, \bar{\xi})$ through (3.4)(3.5).

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