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Resonant tunneling of fast solitons through large potential barriers

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We rigorously study the resonant tunneling of fast solitons through large potential barriers for the nonlinear Schrödinger equation in one dimension.

1 Introduction

1.1 Earlier results and heuristic discussion

In the last few years, there has been substantial progress in rigorously understanding the effective dynamics of solitons for the nonlinear Schrödinger equation (NLS) in the presence of external potentials and perturbations, both in the *"classical"* and *"quantum"* regimes. If the soliton moves in a slowly varying external potential, or in the presence of a small (even nonlinear) perturbation, the long-time dynamics of the soliton center of mass is approximately that of a classical particle in an external potential corresponding to the restriction of the perturbation to the soliton manifold, plus error terms due to radiation damping,[1]-[5], [7]-[9],[11]-[13]. On the other hand, for solitons with high velocity, we expect that the dynamics will be dominated by quantum effects, such as soliton splitting due to scattering from a delta-potential for the cubic NLS in one dimension, [13], or the *blind* collision of two fast solitons for the generalized nonlinear Schrödinger equation in the presence of an external potential, [4].

In this letter, we study the phenomenon of resonant tunneling of fast solitons through large potential barriers for NLS equations in one dimension. Since the soliton is exponentially localized in space and the potential barrier is compactly supported, the effective time of interaction between the fast traveling soliton and the potential is *short*. One expects that linear effects dominate in the interaction regime. Resonant tunneling is well-known in linear scattering theory, and refers to a situation where the reflection coefficient vanishes at certain energies of incoming waves. One expects to have an analogous phenomenon for nonlinear systems in special limiting regimes. We make this intuition mathematically precise and prove that this effect indeed occurs.

For the sake of concreteness, we consider resonant tunneling for solitons of NLS with Hartree (nonlocal) nonlinearity and in the presence of a repulsive double delta-potential. We also discuss general assumptions for nonlinearities and potentials for which our results hold.

The main ingredients of our analysis are the symplectic structure of the soliton manifold for the study of the effective dynamics of solitons in the pre- and post- collision regimes, as developed in [4], see also [8, 11, 1], and the linear scattering theory together with Strichartz estimates in one dimension to study the interacting regime, [19].

The organization of this paper is as follows. In Section 1.2, we discuss a concrete example where our analysis is applicable, namely the Hartree NLS equation with a double-delta potential. For the sake of completeness, we recall in Section 2 basic results on scattering theory and NLS equations. In Section 3, we present the effective soliton dynamics in different regimes, both close and far away from the potential, which sets the stage to prove the main result, Theorem 1.1. We finally discuss general conditions on the nonlinearity and on the external potential for which our analysis holds in Section 4.

1.2 The model and statement of the main result

In this section, we discuss the main result for the case of Hartree nonlinearity and double delta-potential. Our result holds for more general nonlinearities and other potentials, such as a box potential.

Consider the nonlinear Schrödinger equation given by

$$i\partial_t\psi = H\psi + f(\psi), \tag{1}$$

where $H = H_0 + V$ is the interacting linear Hamiltonian, $H_0 = -\frac{1}{2}\partial_x^2$, V is a (time-independent) external potential, and f is a focusing Hartree nonlinearity of the form

$$f(\psi) = (W \star |\psi|^2)\psi. \tag{2}$$

We assume that W is negative, spherically symmetric, and decaying at spatial infinity with

$$W \in L^p, 2 \le p < \infty.$$

Under this assumption, the NLS equation admits solitary wave solutions when $V = 0$. In particular, there exists an interval $I \subset \mathbb{R}$ such that, for all $\mu \in I$, the traveling wave

$$u_\sigma(t) = e^{i\mu t + i\gamma + iv(x-a-vt)}\eta_\mu(x - a - vt), \tag{3}$$

where

$$\sigma = (a, v, \gamma, \mu) \in \mathbb{R} \times \mathbb{R} \times [0, 2\pi) \times I.$$

Here, η_μ is a positive, spherically symmetric function solving the nonlinear eigenvalue problem

$$(-\frac{1}{2}\partial_x^2 + \mu)\eta_\mu + f(\eta_\mu) = 0.$$

Furthermore, $\eta_\mu \in L^2(\mathbb{R}) \cap C^2(\mathbb{R})$, and $\eta_\mu \propto e^{-\sqrt{\mu}|x|}$, as $|x| \to \infty$, see for example [6, 18]. For the sake of concreteness, we consider first the case of scattering from a repulsive double delta-potential V, which is given by

$$V = q(\delta(x + l) + \delta(x - l)). \tag{4}$$

Here, $q > 0$ is the *"height"* of the potential barrier, and $2l$ is its *"width"*. For such a potential, we know that the resolvent $R_V(\lambda) := (H - \lambda^2)^{-1}$ is meromorphic in $\lambda \in \mathbb{C}$ with no poles for $\mathrm{Im}\lambda > 0$, see, for example, [16]. Moreover, for $\lambda \in \mathbb{R}\backslash\{0\}$,

$$(H - \lambda^2)u = 0$$

has unique solutions $e_\pm(x, \lambda)$ satisfying

$$e_\pm(x,\lambda) = \begin{cases} e^{\pm i\lambda x} + R(\lambda)e^{\mp i\lambda x}, & \pm x < -l \\ T(\lambda)e^{\pm i\lambda x}, & \pm x > l \end{cases}. \tag{5}$$

Here

$$T(\lambda) = \frac{\lambda^2}{\lambda^2 + 2i\lambda q - q^2 + e^{4il\lambda}q^2}$$

is the transmission coefficient, and

$$R(\lambda) = \frac{-ie^{-2il\lambda}(\lambda(1 + e^{4il\lambda}) + iq(1 - e^{4il\lambda}))}{\lambda^2 + 2i\lambda q - q^2 + e^{4il\lambda}q^2}$$

is the reflection coefficient. Note that

$$|T(\lambda)|^2 + |R(\lambda)|^2 = 1, \quad \lambda \in \mathbb{R}, \tag{6}$$

which follows from the unitary character of the linear evolution. Furthermore, when

$$q^2(1 - \cos(4l\lambda)) + \lambda^2(1 + \cos(4l\lambda)) - 2q\lambda\sin(4l\lambda) = 0, \tag{7}$$

the reflection coefficient $R(\lambda) = 0$, while the transmission coefficient satisfies $|T(\lambda)| = 1$, i.e., we have *resonant tunneling.**

We are interested in the scattering of a fast soliton from the potential barrier. We make the following assumption on the initial condition, which corresponds to a state that is close to a traveling soliton,

$$\psi(0) = u_{\sigma_0} + w_0, \tag{8}$$

where $\sigma_0 = (a_0, v_0, \gamma_0, \mu_0) \in \mathbb{R} \times \mathbb{R} \times [0, 2\pi) \times I$ and w_0 is a fluctuation that we will take to be small in L^2 norm. Without loss of generality, we consider the scattering setting where $a_0 < 0$ and $v_0 > 0$.

The following result holds for the above potential and nonlinearity. The same result holds for a more general potentials and nonlinearities, see Section 4.

*The resonant tunneling condition is solved implicitly. One solution, for example, is $l = \frac{1.453}{\lambda}$ and $q = 4.129\lambda$.

Theorem 1.1. Consider the nonlinear Schrödinger equation (1), with nonlinearity (2), potential (4), and initial condition (8). Suppose that $\|w_0\|_{L^2} = O(v_0^{-1})$ and that the resonant condition (7) where λ is replaced by v_0 is satisfied with $l = O(v_0^{-1})$, $q^{-1} = O(v_0^{-1})$. Then, for v_0 sufficiently large, there exists positive constants C, α, β and δ that are independent of v_0, such that

$$\|\psi(t) - u_{\sigma_0}(t)\|_{L^2} \le C(v_0^{-\beta} + e^{-\delta(|a_0+v_0t-l|+|a_0+v_0t+l|)}),$$

uniformly in the time interval $t \in [0, \frac{|a_0|}{v_0} + \alpha \log v_0]$. □

In other words, for special values of v_0, q and l, the fast soliton tunnels through the potential barrier. For different values with $v_0 \gg 1$, $q \gg 1$, the soliton typically splits into two wave packets, see Lemma 3.5 below, and also [13] for an elegant discussion of soliton splitting from a delta potential for the cubic NLS equation in one dimension. The use of the L^2 norm (as opposed to norms in the energy space H^1) is crucial to have estimates that are uniform in v_0, [4].

2 Mathematical Setting

2.1 Linear dynamics

2.1.1 *Scattering*

We recall in this subsection some basic results of linear scattering theory, see [16]. For $\lambda \in \mathbb{R}\backslash\{0\}$, the generalized eigenfunctions $e_\pm$ appearing in (5) are given by

$$e_\pm(x,\lambda) = e^{\pm i\lambda x} - R_V(\lambda)(Ve^{\pm i\lambda x}).$$

The resolvent can be expressed in terms of the generalized eigenfunctions,

$$R_V(\lambda)(x,y) = \frac{1}{2i\lambda T(\lambda)}[e_+(x,\lambda)e_-(y,\lambda)\Theta(x-y) + e_+(y,\lambda)e_-(x,\lambda)\Theta(y-x)],$$

where Θ is the Heaviside step function. Furthermore, we have the spectral decomposition

$$\delta(x-y) = \frac{1}{2\pi}\int_0^\infty d\lambda\ (e_+(x,\lambda)\overline{e_+(y,\lambda)} + e_-(x,\lambda)\overline{e_-(y,\lambda)})$$

and

$$H(x,y) = \frac{1}{2\pi}\int_0^\infty d\lambda\ \lambda^2(e_+(x,\lambda)\overline{e_+(y,\lambda)} + e_-(x,\lambda)\overline{e_-(y,\lambda)}). \tag{9}$$

Here, $\bar{\cdot}$ stands for complex conjugation.

2.1.2 *Decay and Strichartz estimates*

We now discuss decay estimates involving $U(t,s) := e^{-i(t-s)H}$, the propagator corresponding to the linear dynamics generated by the interacting Hamiltonian H. For V given by (4),

$$\int dx\ |V|(1+|x|)^\gamma < \infty, \quad \gamma > \frac{5}{2}.$$

It follows from the work of [19] that

$$\|U(t,s)\|_{\mathcal{B}(L^p,L^{p'})} \le \frac{C}{(t-s)^{\frac{1}{p}-\frac{1}{2}}},$$

for $1 \le p \le 2$. Here p' is the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Earlier results in higher dimensions appeared in [14].

The above decay estimate implies that Strichartz estimates hold, see [17, 15]. We say that a pair (q,r) is admissible if $\frac{2}{q} = \frac{1}{2} - \frac{1}{r}$ with $r \in [2,\infty]$. For every $\phi \in L^2(\mathbb{R})$ and every admissible pair (q,r), the function $t \to U(t,0)\phi$ belongs to $L^q(\mathbb{R}, L^r(\mathbb{R})) \cap C(\mathbb{R}, L^2(\mathbb{R}))$, and the estimate

$$\|U(\cdot,0)\phi\|_{L^q(\mathbb{R},L^r)} \le C\|\phi\|_{L^2}, \quad \forall \phi \in L^2(\mathbb{R}),$$

where C is a constant depending on q holds. Consider $I \subset \mathbb{R}$ containing the origin. Let $J \subset \overline{I}$ such that $0 \in J$, where $\bar{\cdot}$ denotes the closure, and let (γ, ρ) be an admissible pair, with $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}))$. Then, for all admissible pairs (q, r), the function

$$t \to \Phi_f(t) = \int_0^t dsU(t,s)f(s) \in L^q(I, L^r(\mathbb{R})) \cap C(J, L^2(\mathbb{R})),$$

such that

$$\|\Phi_f\|_{L^q(I,L^r)} \leq C\|f\|_{L^{\gamma'}(I,L^{\rho'})}, \tag{10}$$

where C is a constant independent of I and depends on q and γ only.

2.2 NLS equation and the soliton manifold

We recall some basic properties of the NLS equation (1), see for example [18, 8].

The phase space for the NLS equation (1) is chosen to be $H^1(\mathbb{R}, \mathbb{C})$. The space H^1 has a real inner product (Riemannian metric)

$$\langle u, v \rangle := \mathrm{Re} \int dx \ u\overline{v}$$

for $u, v \in H^1(\mathbb{R}, \mathbb{C})$. † On H^1, one can define a symplectic 2-form

$$\omega(u,v) := \mathrm{Im} \int dx \ u\overline{v} = \langle u, iv \rangle.$$

The Hamiltonian functional corresponding to the nonlinear Schrödinger equation (1) is

$$H_V(\psi) := \frac{1}{4}\int |\partial_x\psi|^2 dx + \frac{1}{2}\int V|\psi|^2 + F(\psi), \tag{11}$$

where $F(\psi) = \int W * |\psi|^2|\psi|^2$. Using the correspondence

$$\begin{aligned} H^1 &\longleftrightarrow H^1(\mathbb{R};\mathbb{R}) \oplus H^1(\mathbb{R};\mathbb{R}) \\ \psi &\longleftrightarrow (\mathrm{Re}\psi \ \mathrm{Im}\psi) \\ i^{-1} &\longleftrightarrow J, \end{aligned}$$

where $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the complex structure on $H^1(\mathbb{R};\mathbb{R}^2)$, the nonlinear Schrödinger equation can be written as

$$\partial_t\psi = JH_V'(\psi).$$

Furthermore,

$$\begin{aligned} \langle u, v \rangle &= \int dx \ (\mathrm{Re}u \ \mathrm{Im}u) \begin{pmatrix} \mathrm{Re}v \\ \mathrm{Im}v \end{pmatrix}, \\ \omega(u,v) &= \int dx \ (\mathrm{Re}u \ \mathrm{Im}u) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Re}v \\ \mathrm{Im}v \end{pmatrix}. \end{aligned}$$

The Hamiltonian H_V is conserved, and it is invariant under global gauge transformations, $H_V(e^{i\gamma}\psi) = H_V(\psi)$. The associated conserved quantity is the mass $m(\psi) := \frac{1}{2}\int dx\ |\psi|^2$. The assumptions on the nonlinearity imply that $\partial_\mu m(\mu) > 0$. It follows that η_μ is a local minimizer of $H_{V=0}(\psi)$ restricted to the balls $\mathcal{B}_m := \{\psi \in H^1 : N(\psi) = m\}$, for $m > 0$; [10]. They are critical points of the functional

$$\mathcal{E}_\mu(\psi) := \frac{1}{4}\int dx \ (|\partial_x\psi|^2 + 2\mu|\psi|^2) + F(\psi), \tag{12}$$

where $\mu = \mu(m)$ is a Lagrange multiplier.

†The tangent space at an element $\psi \in H^1$ is $\mathcal{T}_\psi H^1 = H^1$.

2.2.1 *Nonlinearity*

The Hartree nonlinearity given in (2) is in $C^2(H^1, H^{-1})$. From a direct application of Hölder's and Young's inequality, we have that for $W \in L^p(\mathbb{R}),\quad p \geq 2,\, r = \frac{2p}{p-2} \in [0, \infty]$,

$$\|f(u_\sigma) - f(u_\sigma + v)\|_{L^{r'}} \leq C\|v\|_{L^2},$$

and

$$\|f(u_\sigma + v) - f(u_\sigma) - f'(u_\sigma)v\|_{L^{\frac{2p}{p+2}}} \leq C\|v\|_{L^2}^2$$

where C depends on $\|W\|_{L^p}$ and $\|u_\sigma\|_{L^2}$.

2.2.2 *Soliton Manifold*

We define the transformation $T_{av\gamma}$ by

$$\psi_{av\gamma} := T_{av\gamma}\psi = e^{i(v(x-a)+\gamma)}\psi(x-a),$$

where $v, a \in \mathbb{R}$ and $\gamma \in [0, 2\pi)$. The soliton manifold is

$$\mathcal{M}_s := \{\eta_\sigma := T_{av\gamma}\eta_\mu,\ \sigma = (a, v, \gamma, \mu) \in \mathbb{R} \times \mathbb{R} \times [0, 2\pi) \times I\}.$$

The tangent space to the soliton manifold $\mathcal{M}_s$ at $\eta_\mu \in \mathcal{M}_s$ is given by

$$\mathcal{T}_{\eta_\mu}\mathcal{M}_s = span\{E_t, E_g, E_b, E_s\},$$

where

$$E_t := \partial_a T_{a00}\eta_\mu|_{a=0} = -\partial_x \eta_\mu; \qquad E_g := \partial_\gamma T_{00\gamma}\eta_\mu|_{\gamma=0} = i\eta_\mu$$
$$E_b := \partial_v T_{0v0}\eta_\mu|_{v=0} = ix\eta_\mu; \qquad E_s := \partial_\mu \eta_\mu.$$

In the following, we denote by

$$e_1 := -\partial_x, \quad e_2 := ix, \quad e_3 := i; \quad e_4 := \partial_\mu, \tag{13}$$

which, when acting on $\eta_\sigma \in \mathcal{M}_s$, generate the basis vectors $\{e_\alpha\eta_\sigma\}_{\alpha=1,\cdots,4}$ of $\mathcal{T}_{\eta_\sigma}\mathcal{M}_s$. The soliton manifold $\mathcal{M}_s$ inherits the symplectic structure from (H^1, ω). For $\sigma = (a, v, \gamma, \mu) \in \mathbb{R} \times \mathbb{R} \times [0, 2\pi) \times I$, the matrices

$$\Omega_\sigma := P_\sigma J^{-1} P_\sigma \in \mathcal{T}^*_{\eta_\sigma}\mathcal{M}_s \times \mathcal{T}^*_{\eta_\sigma}\mathcal{M}_s,$$

where $\mathcal{T}^*_{\eta_\sigma}\mathcal{M}_s$ is the cotangent space at η_σ, and P_σ is the L^2-orthogonal projection onto $\mathcal{T}_{\eta_\sigma}\mathcal{M}_s$, define the induced symplectic structure on $\mathcal{M}_s$. Explicitly,

$$\begin{aligned}\Omega_\sigma|_{\mathcal{T}_{\eta_\sigma}\mathcal{M}_s} &:= \{\omega(e_\alpha\eta_\sigma, e_\beta\eta_\sigma)\}_{1\leq\alpha,\beta\leq 4} \\ &= \begin{pmatrix} 0 & -m(\mu) & 0 & -vm'(\mu) \\ m(\mu) & 0 & 0 & am'(\mu) \\ 0 & 0 & 0 & m'(\mu) \\ vm'(\mu) & -am'(\mu) & -m'(\mu) & 0 \end{pmatrix},\end{aligned} \tag{14}$$

which is invertible if $m'(\mu) > 0$.

2.2.3 *Group structure*

The anti-selfadjoint operators $\{e_\alpha\}_{\alpha=1,\cdots,4}$ defined in (13) form the generators of the Lie algebra g corresponding to the Heisenberg group H^3, where the latter is given by

$$(a, v, \gamma) \cdot (a', v', \gamma') = (a'', v'', \gamma''),$$

with $a'' = a + a'$, $v'' = v + v'$, and $\gamma'' = \gamma' + \gamma + \frac{1}{2}va'$. Elements of g satisfy the commutation relations

$$[e_1, e_2] = -e_3, \tag{15}$$

and the rest of the commutators are zero. This group structure has been noted in [11].

2.2.4 *Zero modes*

The solitary wave solutions transform covariantly under translations and gauge transformations, i.e.,

$$\mathcal{E}'_{\mu}(T_{a00}T_{00\gamma}\eta_{\mu}) = 0$$

for all $a \in \mathbb{R}$ and $\gamma \in [0, 2\pi)$. Here, the prime stands for the Fréchet derivative. There are associated zero modes of the *Hessian*,

$$\mathcal{L}_{\mu} := -\frac{1}{2}\partial_x^2 + \mu + f'(\eta_{\mu}), \tag{16}$$

associated to these symmetries. In particular,

$$i\mathcal{L}_{\mu} : \mathcal{T}_{\eta_{\mu}}\mathcal{M}_s \to \mathcal{T}_{\eta_{\mu}}\mathcal{M}_s \tag{17}$$

with $(i\mathcal{L}_{\mu})^2 X = 0$, for any vector $X \in \mathcal{T}_{\eta_{\mu}}\mathcal{M}_s$.

2.2.5 *Skew-orthogonal decomposition*

We define

$$\Sigma := \{\sigma = (a, v, \gamma, \mu) \in \mathbb{R} \times \mathbb{R} \times [0, 2\pi) \times I\},$$

and let

$$\Sigma^0 := \{\sigma = (a, v, \gamma, \mu) \in \mathbb{R} \times \mathbb{R} \times [0, 2\pi) \times I_0, \text{ with } \overline{I}_0 \subset I\backslash\partial I \text{ bounded}\}.$$

We consider the neighbourhood $U_{\delta} \subset H^1$ defined by

$$U_{\tilde{\delta}} := \{\psi \in H^1, \sup_{\sigma\in\Sigma^0} \|\psi - \eta_{\sigma}\|_{L^2} < \tilde{\delta}\}.$$

We have the following proposition, whose proof follows from an application of the Implicit Function Theorem, [4].

Proposition 2.1. For $\tilde{\delta} \ll 1$, there exists a unique mapping

$$\sigma(\cdot) : U_{\tilde{\delta}} \to \Sigma$$

such that

$$\psi = \eta_{\sigma(\psi)} + w, \tag{18}$$

and $\omega(w, X) = 0$, for all $X \in \mathcal{T}_{\eta_{\sigma}(\psi)}\mathcal{M}_s$. □

The group element $T_{av\gamma} \in \mathsf{H}^3$ is given by

$$T_{av\gamma} = e^{-a\partial_x} e^{ivx} e^{i\gamma}.$$

It follows from (15) that $T^{-1}_{av\gamma} Y T_{av\gamma} \in \mathsf{g}$ if $Y \in \mathsf{g}$. Furthermore, from the translational invariance, we have that $\omega(T_{av\gamma}u, T_{av\gamma}v) = \omega(u, v)$, for $u, v \in L^2$. Therefore, Proposition 2.1 implies that

$$\omega(w, Y\eta_{\sigma}) = \omega(w', Y'\eta_{\sigma'}) = 0,$$

for all $Y \in \mathsf{g}$, where $Y' = T^{-1}_{av\gamma} Y T_{av\gamma} \in \mathsf{g}$, $w' = T^{-1}_{av\gamma} w$, and $\eta_{\sigma'} = T^{-1}_{av\gamma}\eta_{\sigma}$.

3 Effective dynamics

We decompose the dynamics into three regimes. For $\beta, \delta \in (0, 1)$, we let

$$t_0 = 0, \qquad t_1 = \frac{|a_0| - v_0^{\delta}}{v_0},$$

$$t_2 = \frac{|a_0| + v_0^{\delta}}{v_0}, \qquad t_3 = t_2 + \beta \log v_0.$$

The *pre-collision* regime corresponds to $t \in [t_0, t_1]$, the *collision* regime to $t \in [t_1, t_2]$, and the *post-collision* regime to $t \in [t_2, t_3]$. Below, we will show that nonlinear effects dominate in the pre- and post- collision regimes, such that one can neglect the external potential since the potential is compactly supported and the traveling wave is exponentially localized in space, while the dynamics is almost linear in the collision regime due to the high velocity of the soliton, or equivalently, short time of interaction. The same decomposition was used in [13], except that in the latter reference, Strichartz estimates are used to control the dynamics in all three regimes and the inverse scattering method is applied for the post-collision regime. Here, we only use Strichartz estimates in the collision regime, and we apply the skew-orthogonal projection onto the soliton manifold in the pre- and post- collision regime.

3.1 Nonlinear regime

The analysis in this subsection applies to the pre- and post- collision regimes. We discuss first the post-collision regime. A similar result holds for the pre-collision regime, which is simpler.

3.1.1 *Reparametrized equations of motion in the post-collision regime*

Suppose that the solution at time $t = t_2$ has the form

$$\psi(t_2) = \eta_{\sigma_2} + w_2 \tag{19}$$

where $\sigma_2 = (a_0 + v_0 t_2, v_0, \gamma_0, \mu_0)$, and that $\|w_2\|_{L^2} = O(|v_0|^{-\alpha})$, for some $\alpha > 0$.

Lemma 3.1. Consider the NLS equation (1) with initial condition (19), and assume that, for $t \in [t_2, t_3]$, $\psi(t) \in U_{\tilde{\delta}}$, $\tilde{\delta} \ll 1$. Then, for $v_0 \gg 1$, the parameters $\sigma = (a, v, \gamma, \mu)$ satisfy the equations

$$\partial_t a = v_0 + O(\|w\|_{L^2}^2 + qe^{-\xi|a|}), \tag{20}$$

$$\partial_t v = O(\|w\|_{L^2}^2 + qe^{-\xi|a|}) \tag{21}$$

$$\partial_t \gamma = \mu_0 + \frac{v_0^2}{2} + O(\|w\|_{L^2}^2 + qe^{-\xi|a|}), \tag{22}$$

$$\partial_t \mu = O(\|w\|_{L^2}^2 + qe^{-\xi|a|}), \tag{23}$$

for some constant $\xi > 0$ independent of v_0. □

Proof. We first find the equation of motion in the center of mass reference frame, i.e., for

$$u = T_{av\gamma}^{-1} \psi = e^{-ivx - i\gamma} \psi(x + a).$$

Using the skew-orthogonal property 2.1, we have

$$u = \eta_\mu + w', \quad w' = T_{av\gamma}^{-1} w, \tag{24}$$

where

$$\omega(w', X) = 0, \tag{25}$$

for all $X \in \mathcal{T}_{\eta_\mu} \mathcal{M}_s$, and

$$|v(t_2) - v_0|, \ |a(t_2) - a_0 - v_0 t_2|, \ |\mu(t_2) - \mu_0|, \ |\gamma(t_2) - \gamma_0| = O(v_0^{-\alpha}).$$

We define the coefficients

$$\begin{aligned} c_1 &:= \partial_t a - v, \\ c_2 &:= -\frac{1}{2} \partial_t v, \\ c_3 &:= \mu - \frac{1}{2} v^2 + \frac{1}{2} \partial_t a v - \partial_t \gamma, \\ c_4 &:= -\partial_t \mu. \end{aligned} \tag{26, 27}$$

Note that

$$e^{-i(vx+\gamma)} \partial_x^2 \psi(x + a) = \partial_x^2 u + 2iv \partial_x u - v^2 \tag{28}$$

$$e^{-i(vx+\gamma)} f(\psi(x + a)) = f(u). \tag{29}$$

Differentiating u with respect to t and using (1), (27)-(29), we get

$$\partial_t u = -i((-\frac{1}{2} \partial_x^2 + \mu) u - f(u)) + \sum_{\alpha=1}^{3} c_\alpha e_\alpha u - i V_a u, \tag{30}$$

where $V_a(x) = V(x + a)$. In other words,

$$\partial_t u = -i \mathcal{E}'_\mu(u) + \sum_{\alpha=1}^{3} c_\alpha e_\alpha u - i V_a u, \tag{31}$$

where $\mathcal{E}_\mu$ is defined in (12). Recall that

$$\mathcal{E}'_\mu(\eta_\mu) = 0,$$

which implies

$$\mathcal{E}'_\mu(u) = \mathcal{L}_\mu(w') + N_\mu(w'), \tag{32}$$

where

$$\mathcal{L}_\mu = (-\frac{1}{2}\partial_x^2 + \mu + f'(\eta_\mu)) \equiv \mathcal{E}''_\mu(\eta_\mu)$$

and

$$N_\mu(w') = f(\eta_\mu + w') - f(\eta_\mu) + f'(\eta_\mu)(w').$$

Substituting (24) and (32) into (31), we obtain

$$\partial_t w' = (-i\mathcal{L}_\mu + \sum_{\alpha=1}^{3} c_\alpha e_\alpha - iV_a)w' + N_\mu(w') + \sum_{\alpha=1}^{4} c_\alpha e_\alpha \eta_\mu - iV_a\eta_\mu. \tag{33}$$

To obtain the equations of motion for a, v, γ and μ, We project Eq. 33 onto $\mathcal{T}_{\eta_\mu}\mathcal{M}_s$ using the skew-orthogonal property.

We have

$$\partial_t\langle iw', X\rangle = \partial_t\mu\langle iw', \partial_\mu X\rangle + \langle i\partial_t w', X\rangle = 0. \tag{34}$$

Substituting the expression for $\partial_t w'$ given by (33) in (34), and the fact that

$$e^*_\alpha = -e_\alpha,\ \alpha = 1, \cdots, 4, \tag{35}$$

we have

$$\langle \mathcal{L}_\mu w', X\rangle + \langle i\sum_{\alpha=1}^{4} c_\alpha e_\alpha w', X\rangle + \langle V_a w', X\rangle + \langle iN_\mu(w'), X\rangle + \langle i\sum_{\alpha=1}^{4} c_\alpha e_\alpha \eta_\mu, X\rangle + \\ + \langle V_a\eta_\mu, X\rangle = 0. \tag{36}$$

Some of the terms in the above equation cancel due to the zero modes of the Hessian. It follows from (17), that

$$X' = i\mathcal{L}_\mu X \in \mathcal{T}_{\eta_\mu\mathcal{M}_s} \text{ if } X \in \mathcal{T}_{\eta_\mu\mathcal{M}_s},$$

and hence

$$\langle \mathcal{L}_\mu w', X\rangle = \langle w', \mathcal{L}_\mu X\rangle = -\omega(w, X') = 0.$$

Together with (35) and (36), this yields

$$\sum_{\alpha=1}^{4} c_\alpha\,\omega(e_\alpha\eta_\mu, X) = \langle V_a\eta_\mu, X\rangle + \sum_{\alpha=1}^{4} c_\alpha\,\langle ie_\alpha w', X\rangle + \langle V_a w', X\rangle + \langle iN_\mu(w'), X\rangle. \tag{37}$$

We now estimate each term appearing in the right-hand-side of (37) with $X = e_\beta\eta_\mu$, $\beta = 1, \cdots, 4$. Due to the exponential localization in space of the soliton profile and the fact that the potential is compactly supported, we have that

$$\|V_a e_\beta \eta_\mu\|_{L^2} = O(qe^{-\xi|a|}),\ \beta = 1, \cdots, 4,$$

and

$$\|X\|_{L^2} = \|e_\beta\eta_\mu\|_{L^2} = O(1),\ \beta = 1, \cdots, 4$$
$$\|e_\alpha X\|_{L^2} = \|e_\alpha e_\beta \eta_\mu\|_{L^2} = O(1),\ \alpha, \beta = 1, \cdots, 4.$$

Hence, Hölder's inequality, and the fact that V is real yield the estimates

$$|\langle V_a\eta_\mu, X\rangle| = |\langle \eta_{\mu_1}, V_a X\rangle| = O(qe^{-\xi|a|})$$
$$|\langle V_a w', X\rangle| = |\langle w', V_a X\rangle| \leq \|V_a e_\beta \eta_{\mu_1}\|_{L^2}\|w'\|_{L^2} = O(qe^{-\xi|a|}\|w'\|_{L^2}).$$

We also have from Hölder's inequality that

$$\begin{aligned}|\sum_{\alpha=1}^{4} c_\alpha \langle ie_\alpha w', X\rangle| &= |\sum_{\alpha=1}^{4} c_\alpha \langle iw', e_\alpha X\rangle| \\ &\leq C|c|\|w'\|_{L^2}\end{aligned}$$

where $|c| := \max_{\alpha=1,\cdots,4}|c_\alpha|$. Now, it follows from Section 2.2.1 and Hölder's inequality that

$$|\langle iN_\mu(w'), X\rangle| = O(\|w'\|_{L^2}^2).$$

We hence obtain the estimate

$$|c| \leq C(\|w'\|_{L^2}^2 + qe^{-\xi|a|})$$

for some C and $\xi > 0$. Note that

$$\|w'\|_{L^2} = \|w\|_{L^2},$$

and hence the claim of the lemma. ■

3.1.2 *Control of the fluctuation*

We now control the L^2-norm of the fluctuation w using conservation of charge, the skew-orthogonal property, and the reparametrized equations of motion.

Lemma 3.2. Consider (1) with initial condition (19). Suppose the hypotheses of Lemma 3.1 hold. Then, for $v_0 \gg 1$

$$\sup_{t\in[t_1,\beta\log v_0]} \|w'(t)\|_{L^2}^2 \leq C|v_0|^{-\alpha/2},$$

for some constants $C > 0$, $\beta \in (0,1)$. □

Proof. From the conservation of the L^2-norm of the solution,

$$\|\psi(t)\|_{L^2} = \|\phi\|_{L^2},$$

and the skew-orthogonal decomposition (Proposition 2.1), we have

$$\|\psi\|_{L^2}^2 = \|w\|_{L^2}^2 + \|\eta_{\mu_1}\|_{L^2}^2 = \|\phi\|_{L^2}^2, \tag{38}$$

where we have used

$$\langle w, \eta_\sigma\rangle = -\omega(w, i\eta_\sigma) = 0,$$

and

$$\|\eta_\sigma\|_{L^2} = \|\eta_\mu\|_{L^2}.$$

Differentiating (38) with respect to t, and recalling that $m(\mu) = \frac{1}{2}\|\eta_\mu\|_{L^2}^2$, we get

$$\partial_t \|w\|_{L^2}^2 = -2\partial_t\mu\, \partial_\mu m(\mu). \tag{39}$$

Now, (23) implies that

$$|\partial_t\mu\, \partial_\mu m(\mu)| \leq C(qe^{-\xi|a|} + \|w\|_{L^2}^2). \tag{40}$$

Together with (39) and the Duhamel formula, this yields

$$\|w\|_{L^2}^2 \leq C(e^{ct}\|w_2\|_{L^2}^2 + q\int_0^t ds\, e^{c(t-s)}e^{-\xi|a|}). \tag{41}$$

For times $t < Cv_0^\epsilon$, $\epsilon \in (0,1)$, we know from (21) that

$$v(s) \geq c_0 v_0, \tag{42}$$

for some constant $c_0 > 0$, and hence

$$a(s) \geq a_0 + v_0 t_2 + c_0 v_0 s \gg 1.$$

This yields

$$\int_0^t ds\, e^{c(t-s)}e^{-\xi a(s)} \le Ce^{ct-\xi v_0^\delta}, \quad \delta \in (0,1).$$

Together with (41), we get the estimate

$$\|w\|_{L^2}^2 \le C(qe^{ct-\xi v_0^\delta} + \frac{1}{v_0^\alpha}e^{ct}), \tag{43}$$

for some positive constants C and c that are independent of v_0. Choose $\tau < \frac{\alpha \log v_0}{2c}$. For $t_2 \le t \le t_2+\tau$, (43) implies

$$\sup_{t\in[0,\tau]} \|w\|_{L^2}^2 < Cv_0^{-\alpha/2}.$$

■

We have the following proposition, which follows from Lemmas 3.1 and 3.2.

Proposition 3.3. Consider (1) with initial condition (19). Then, for $v_0 \gg 1$,

$$\sup_{t\in[t_2,t_3]} \|\psi(t) - u_{\sigma_0}(t)\|_{L^2} \le Cv_0^{-\tilde{\alpha}},$$

for some C and $\tilde{\alpha} > 0$. □

The same result holds for $t \in [t_0, t_1]$ (pre-collision regime).

3.2 Almost linear regime

This regime corresponds to $t \in [t_1, t_2]$.

Proposition 3.4. At resonance, and for $v_0 \gg 1$, there exist constants $C > 0$ and $\tilde{\beta}, \tilde{\delta} \in (0,1)$ such that

$$\|\psi(t_1+s) - u_{\sigma_0}(t_1+s)\|_{L^2} \le C(l^{1/4} + l + q^{-1} + v_0^{-\tilde{\beta}} + e^{-\tilde{\delta}(|a_0+v_0t-l|+|a_0+v_0t+l|)})$$

uniformly in $t \in [t_1, t_2]$. □

Proof. We divide the proof into several steps.
Step 1. We start by estimating the L^2 norm of the difference between the linear interacting evolution and the true solution. The Duhamel formula and the unitarity of the linear evolution imply that

$$\|e^{-isH}\psi(t_1) - \psi(t_1+s)\|_{L^2} = \|\int_{t_1}^{t_1+s} ds' e^{is'H} W \star |\psi(t_1+s')|^2\psi(t_1+s')\|_{L^2}.$$

Using the Strichartz estimate (10) and the fact that $W \in L^p, \;\; p \ge 2$, we have

$$\begin{aligned}
&\|\int_{t_1}^{t_1+s} ds' e^{is'H} W \star |\psi(t_1+s')|^2\psi(t_1+s')\|_{L^2} \\
&\le C\|W \star |\psi(t_1+s')|^2\psi(t_1+s')\|_{L^{\frac{1}{2p}}([0,s];L^{\frac{2p}{p+2}})} \\
&\le Cs^{\frac{1}{2p}}\|W \star |\psi(t_1+s')|^2\psi(t_1+s')\|_{L^\infty([0,s];L^{\frac{2p}{p+2}})}
\end{aligned}$$

for some constant $C > 0$. Applying Hölder's inequality yields

$$\begin{aligned}
\|W \star |\psi(t_1+s')|^2\psi(t_1+s')\|_{L^\infty([0,s];L^{\frac{2p}{p+2}})} &\le \|W\|_{L^p}\|\psi(t_1+s')\|_{L^\infty([0,s];L^2)}^2 \\
&\le C\|\psi_0\|_{L^2}^2,
\end{aligned}$$

where we have used the fact that charge is conserved in the last inequality, i.e., $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}$ for all $t \in \mathbb{R}$. Therefore,

$$\|e^{-i\cdot H}\psi(t_1) - \psi(t_1+\cdot)\|_{L^\infty([0,s];L^2)} \le Cs^{\frac{1}{2p}}\|\psi_0\|_{L^2}^2 \tag{44}$$

uniformly in $v_0 > 0$.

Step 2. Recall that as the soliton quits the pre-collision regime,

$$\psi(t_1) = u_{\sigma_0}(t_1) + w(t_1).$$

Using the Minkowski inequality,

$$\|e^{-isH}\psi(t_1)\|_{L^2} \leq \|e^{-isH}u_{\sigma_0}(t_1)\|_{L^2} + \|w(t_1)\|_{L^2}, \tag{45}$$

where we have used the fact that $\|e^{-isH}w(t_1)\|_{L^2} = \|w(t_1)\|_{L^2}$.
Step 3. We now investigate the linear evolution of the traveling solitary wave. We have the following lemma.

Lemma 3.5. Under the resonance condition, we have

$$e^{-isH}u_{\sigma_0}(t_1) = T(v_0)e^{-isH_0}u_{\sigma_0}(t_1) + R(v_0)e^{-isH_0}u_{\sigma_0}(t_1) + E_1(s) + E_2(s) + E_3(s), \tag{46}$$

with

$$\begin{aligned} &\|E_1\|_{L^2} \leq C(l + q^{-1}), \quad supp(E_1) \in [l, \infty), \\ &\|E_2\|_{L^2} \leq Cl^{1/4}, \quad supp(E_2) \in [-l, l], \\ &\|E_3(s,x)\|_{L^2} \leq Ce^{-v_0^\delta}, \end{aligned}$$

for some constant C that is independent of v_0. □

Proof. We use the spectral decomposition of the linear evolution in terms of the generalized eigenfunctions $e_\pm$. Let ϕ be an arbirary function in $L^1 \cap L^2$ with support in $(-\infty, -l)$. It follows from (5), (6), and (9) that

$$e^{-isH}\phi|_{x<-l} = e^{-isH_0}\mathcal{F}^{-1}(R\widehat{\phi})(-x) + e^{-isH_0}\phi(x).$$

and

$$e^{-isH}\phi|_{x>l} = e^{-isH_0}\mathcal{F}^{-1}(T\widehat{\phi})(x) + \tilde{E}_1(s,x),$$

where

$$\tilde{E}_1(s,x) = \frac{1}{\pi}\int_0^\infty d\lambda\, e^{-i\lambda^2}e^{i\lambda x}\widehat{\phi}(-\lambda)\mathrm{Re}(T(\lambda)\overline{R}(\lambda))\Theta(x-l)$$

and $\mathcal{F}^{-1}$ stands for the inverse Fourier transform.

The linear evolution of ϕ in the whole space can be expressed as

$$\begin{aligned} e^{-isH}\phi(x) =& e^{-isH_0}\mathcal{F}^{-1}(T\widehat{\phi})(x)\Theta(x-l) + e^{-isH_0}\mathcal{F}^{-1}(R\widehat{\phi})(-x)\Theta(-x-l) \\ &+ e^{-isH_0}\phi(x)\Theta(-x-l) + \tilde{E}_1(s,x) + \tilde{E}_2(s,x) \end{aligned} \tag{47}$$

where

$$\tilde{E}_2(s,x) := e^{-isH}\phi(x)(\Theta(x+l) - \Theta(x-l))$$

represents the contribution between the two delta potentials.

We now apply this decomposition to the solitary wave $u_{\sigma_0}(t_1)$, centered at distance of order $-v_0^\delta$ away from the origin and exponentially localized. From (47), we have that

$$\begin{aligned} e^{-isH}u_{\sigma_0}(t_1)(x) =& e^{-isH_0}\mathcal{F}^{-1}(T\widehat{u_{\sigma_0}})(x)\Theta(x-l) + e^{-isH_0}\mathcal{F}^{-1}(R\widehat{u_{\sigma_0}})(-x)\Theta(-x-l) \\ &+ e^{-isH_0}u_{\sigma_0}(x)\Theta(-x-l) + E_1(s,x) + E_2(s,x) + E_3(s,x), \end{aligned} \tag{48}$$

where

$$E_1(s,x) = \frac{1}{\pi}\int_0^\infty d\lambda\, e^{-i\lambda^2}e^{i\lambda x}\widehat{u_{\sigma_0}(t_1)}(-\lambda)\mathrm{Re}(T(\lambda)\overline{R}(\lambda))\Theta(x-l), \tag{49}$$

$$E_2(s,x) := e^{-isH}u_{\sigma_0}(t_1,x)(\Theta(x+l) - \Theta(x-l)). \tag{50}$$

and

$$\|E_3(s,x)\|_{L^2} \leq Ce^{-v_0^\delta} \tag{51}$$

for some constant C which is independent of v_0. The correction term E_3 comes form the exponential decay of u_σ in the considered region.

We now use the fact that u_{σ_0} is localized and that $\partial_\lambda T(\lambda)$ is of order $O(max(l,q^{-1}))$. Applying the mean value theorem, we have

$$\|(T(v_0)-T(\lambda))\widehat{u_{\sigma_0}(t_1)}\|_{L^2} \le C\,(l+q^{-1}) \tag{52}$$

for some positive constant $C>0$ that is independent of v_0. Similarly,

$$\|(R(v_0)-R(\lambda))\widehat{u_{\sigma_0}(t_1)}\|_{L^2} \le C\,(l+q^{-1}). \tag{53}$$

At resonance, $R(v_0)=0$, and $|T(v_0)|=1$, which yields

$$\|E_1\|_{L^2} \le C\,(l+q^{-1}). \tag{54}$$

To estimate E_2, note that

$$\begin{aligned}
\|E_2\|_{L^2}^2 &= \langle u_{\sigma_0}(t_1), e^{isH}(\Theta(x+l)-\Theta(x-l))e^{-isH}u_{\sigma_0}(t_1)\rangle \\
&= \langle \eta_{\mu_0}, T^{-1}_{\sigma_0(t_1)} e^{is\mathrm{ad}_H}(\Theta(x+l)-\Theta(x-l))T_{\sigma_0(t_1)}\eta_{\mu_0}\rangle \\
&= \langle \eta_{\mu_0}, e^{\mathrm{ad}_{-(a_0+v_0t_1)\partial_x}} e^{\mathrm{ad}_{iv_0x}} e^{is\mathrm{ad}_H}(\Theta(x+l)-\Theta(x-l))\eta_{\mu_0}\rangle \\
&\le \|\eta_{\mu_0}\|_{L^2}\|e^{\mathrm{ad}_{-(a_0+v_0t_1)\partial_x}} e^{\mathrm{ad}_{iv_0x}} e^{is\mathrm{ad}_H}(\Theta(x+l)-\Theta(x-l))\eta_{\mu_0}\|_{L^2},
\end{aligned}$$

where $\mathrm{ad}_A f=[A,f]$. It follows that

$$\begin{aligned}
\|E_2\|_{L^2} &\le C\|e^{\mathrm{ad}_{-(a_0+v_0t_1)\partial_x}} e^{\mathrm{ad}_{iv_0x}} e^{is\mathrm{ad}_H}(\Theta(x+l)-\Theta(x-l))\eta_{\mu_0}\|_{L^2}^{1/2} \\
&\le C\|(\Theta(x+l)-\Theta(x-l))\|_{L^2}^{1/2} \\
&\le Cl^{1/4},
\end{aligned} \tag{55}$$

where we have used the unitarity of $e^{\mathrm{ad}_{-(a_0+v_0t_1)\partial_x}} e^{\mathrm{ad}_{iv_0x}} e^{is\mathrm{ad}_H}$ in the second line. The claim of the lemma follows from (48) - (55). ■

Step 4. We now use the fact that the free linear evolution generated by H_0 is close to the *nonlinear* evolution in the absence of the external potential over a short time interval of interaction. The analysis is almost identical to Step 1 above. We have

$$\|e^{-i\cdot H_0}u_{\sigma_0}(t_1)-u_{\sigma_0}(t_1+\cdot)\|_{L^\infty([0,s];L^2)} \le Cs^{\frac{1}{2p}}\|\psi_0\|_{L^2}^2 \tag{56}$$

uniformly in $v_0>0$.

Step 5. Using the fact that the traveling wave is exponentially localized in space, we have

$$\|u_{\sigma_0}(t_1+s)\Theta(-x-l)\|_{L^2} \le Ce^{-\tilde{\delta}|a_0+v_0t+l|}, \tag{57}$$

Similarly,

$$\|u_{\sigma_0}(t_1+s)(\Theta(x+l)-\Theta(x-l))\|_{L^2} \le Ce^{-\tilde{\delta}(|a_0+v_0s+l|+|a_0+v_0s-l|)}. \tag{58}$$

The claim of the proposition follows noting that $|s|\le v_0^{\delta-1}$ and using (44), (45), (46), (56), (57) and (58). ■

3.3 Proof of main theorem

Proof. Since $l, q^{-1}=O(v_0^{-1})$, the main theorem follows from Propositions 3.3 and 3.4. ■

4 General nonlinearities and potential

The resonant tunneling phenomenon described in the above example of an external potential in the form of double delta potential can be extended to more general potentials and nonlinearities. In this section, we list sufficient assumptions for our analysis to hold. We then remark how it is applied to another simple form of external potential, namely the box potential.

We assume that the potential $V\in L^\infty_{comp}\cup L^1_{comp}$, such that the resolvent $R_V(\lambda)=(H-\lambda^2)^{-1}$ has no poles for $\mathrm{Im}\lambda>0$. Also, for $\lambda\in\mathbb{R}\backslash\{0\}$, the equation

$$(H-\lambda^2)u=0$$

has unique solutions $e_\pm(x,\lambda)$ satisfying

$$e_\pm(x,\lambda) = \begin{cases} e^{\pm i\lambda x} + R(\lambda)e^{\mp i\lambda x}, & \pm x < -l, \\ T(\lambda)e^{\pm i\lambda x}, & \pm x > l, \end{cases}$$

for some $l > 0$. The transmission and reflection coefficients $T(\lambda)$ and $R(\lambda)$ respectively, satisfy the unitary condition

$$|T(\lambda)|^2 + |R(\lambda)|^2 = 1.$$

We assume in addition that T and R are differentiable in $\lambda \in \mathbb{R}\backslash\{0\}$, and

$$|\partial_\lambda T|, |\partial_\lambda R| = O(l + q^{-1}),$$

where l is a length scale that depends on the potential V, and q is a measure of the potential's size.

The important hypothesis concerns *resonant tunneling* : We assume that there exists λ_0 that depends on V such that $R(\lambda_0) = 0$.

The hypotheses on the nonlinearity are those essentially sufficient for existence of a global smooth solution to the NLS equation as well as the existence - in the absence of external potential - of an orbitally stable family of exponentially localized solitary waves with the usual symmetries (translation, gauge transformation and Galilean boost). For the application of Strichartz estimates, we require that for $r \in [2,\infty]$,

$$\|f(u_\sigma + v) - f(u_\sigma)\|_{L^{r'}} \leq C\|v\|_{L^2},$$

$$\|f(u_\sigma) - f(u_\sigma) - f'(u_\sigma)v\|_{L^{r'}} \leq C\|v\|_{L^2}^2.$$

Remark 1. The above assumptions are satisfied by a box potential

$$V(x) = q(\Theta(x+l) - \Theta(x-l))$$

where Θ is the Heaviside step function. The corresponding transmission and reflection coefficients are

$$T(\lambda) = \frac{e^{-2il\sqrt{\lambda^2-q}}(1 + \tan^2(l\sqrt{\lambda^2-q}))}{(1 - \frac{i\lambda}{\sqrt{\lambda^2-q}}\tan(l\sqrt{\lambda^2-q}))(1 - \frac{i\sqrt{\lambda^2-q}}{\lambda}\tan(l\sqrt{\lambda^2-q}))}$$

and

$$R(\lambda) = -\frac{e^{-2il\sqrt{\lambda^2-q}}q\tan(l\sqrt{\lambda^2-q})}{\lambda\sqrt{\lambda^2-q}(1 - \frac{i\lambda}{\sqrt{\lambda^2-q}}\tan(a\sqrt{\lambda^2-q}))(1 - \frac{i\sqrt{\lambda^2-q}}{\lambda}\tan(l\sqrt{\lambda^2-q}))}$$

respectively. In this case, resonant tunneling occurs when $l = \frac{n\pi}{\sqrt{v_0^2-q}}$, for some $n \in \mathbb{N}$.

Remark 2. We note that the case of power nonlinearity $f(\psi) = -|\psi|^{p-1}\psi$ $(p < 5)$ clearly satisfies all the assumptions.

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