THE POWER-SET OF ω_1 AND THE CONTINUUM PROBLEM

STEVO TODORCEVIC

ABSTRACT. We analyze structure theories of the power-set of ω_1 and compare them relative to Cantor's Continuum Problem. We also compare these theories with the structure theory of the power-set of ω under the assumption of the axiom of definable determinacy.

1. INTRODUCTION

The success of the large cardinal axioms in determining the true structure theory of the power-set of ω, or more precisely the true theory of $L(\mathbb{R})$, is well documented in the literature and in particular in this series of discussions. Note, however, that, unlike the case of $\mathcal{P}(\omega)$, any reasonably rich structure theory of $\mathcal{P}(\omega_1)$ must give us a solution to Cantor's Continuum Problem. In this note we expose two such structure theories of $\mathcal{P}(\omega_1)$ based on Baire category axioms and built by generations of mathematicians for more than a century. As it will be seen, substantial parts of these structure theories have been imposed on us by problems arising in other areas of mathematics and so we should take this into account whenever we propose a solution to Cantor's problem. As much as this is possible, in our overview we will try to stress the analogies between the structure theories of $\mathcal{P}(\omega_1)$ that we choose to discuss and the structure theory of $\mathcal{P}(\omega)$ under the assumption of $\text{AD}^{L(\mathbb{R})}$, since it is this theory of $\mathcal{P}(\omega)$ that is most widely accepted among all other theories about countable structures. In contrast, however, it turns out that many deep parts of the theory of $\mathcal{P}(\omega_1)$ are of low consistency strength, as measured on the scale of the current large cardinal axioms. This is due to the fact that the theories of $\mathcal{P}(\omega_1)$ that we develop are based on quite different set-theoretic principles and as a consequence we must develop another way to measure their inevitability. For example, inevitability in this context has to be measured by the relevance of these theories to the rest of mathematics not just to set theory itself. This is, of course, not to say that the methods from logic and philosophy of set theory will not be useful in guiding us toward the right theory of the power-set of ω_1. We shall elaborate more on

this in the final section of the paper. The paper will end with a brief discussion of the Continuum Problem and other known axioms of set theory that have bearing on this problem.

Acknowledgement. This paper was composed as our contribution to the final meeting of the EFI Project (Harvard University, August 30-31, 2013). We wish to thank Peter Koellner for the invitation and for the series of correspondences that help us organize our lecture.

2. Baire category principles at the level of ω_1

The Baire Category Theorem, BC_ω, is an important fact which appears in disguise as a fundamental result in many areas of mathematics. For example, it is this principle that underlies such fundamental facts as the Open Mapping Theorem, the Closed Graph Theorem, the Banach-Steinhaus Theorem, the Effros Theorem, etc. It is also a principle which in its unrestricted form (for, say, the class of all compact Hausdorff spaces) is equivalent to one of the most frequently used choice principles, the principle of Dependent Choice, DC_ω. It is therefore quite natural that when trying to develop a theory of $\mathcal{P}(\omega_1)$ that parallels that of $\mathcal{P}(\omega)$ one should try to extend this principle to the next level, the level of ω_1. In fact, there is a straightforward way to do this:

BC_{ω_1} : For every compact Hausdorff space K, any family of no more than $\aleph_1$ dense-open subsets of K has non-empty intersection.

However, when one reformulates this in its dual form[1] saying that for every poset $\mathbb{P}$ and family of no more than $\aleph_1$ dense sets has a generic filter, one sees that some restriction is needed, at least the restriction that $\mathbb{P}$ preserves ω_1. In fact, it is easily seen that $\mathbb{P}$ must preserve all stationary subsets of ω_1. Call this class of posets (compact spaces), the maximal class, $\mathcal{M}$. Thus $\mathrm{BC}_{\omega_1}(\mathcal{M})$ is the strongest Baire category principle of this sort, it is named as *Martin's Maximum*, MM, by the authors of the following fundamental result (see [12]).

2.1 Theorem (Foreman-Magidor-Shelah). *Martin's Maximum is consistent relative to the consistency of a supercompact cardinal. It implies that* NS_{ω_1} *is a saturated ideal and that* $2^{\aleph_0} = \aleph_2$.

As it is well known there are natural weaker forms of the Baire category principle, principles of the form $\mathrm{BC}_{\omega_1}(\mathcal{X})$ for some more restrictive class $\mathcal{X}$ of posets (compact spaces). For example, in the 1940's Rothberger was considering a principle equivalent to $\mathrm{BC}_{\omega_1}(\mathcal{X})$ for $\mathcal{X}$ the class of compact Hausdorff separable spaces. When $\mathcal{X}$ is the class of

---

[1]i.e., by looking at a (π-)basis of K as a poset

all compact Hausdorff spaces satisfying the countable chain condition, ccc, the Baire category principle $\mathrm{BC}_{\omega_1}(\mathcal{X})$ is Martin's Axiom, MA_{ω_1}, of Martin and Solovay [24]. This also has deep historical roots long before the invention of forcing as the following fact shows (see [45]).

2.2 Theorem (Todorcevic). *Martin's Axiom,* MA_{ω_1} *is equivalent to the statement that every compact ccc first countable Hausdorff space is separable.*

In other words, Martin's axiom, MA_{ω_1}, is nothing more than a natural extension of the Souslin Hypothesis from the class of ordered compacta to the class of first countable compacta. Another intermediate important step is the Proper Forcing Axiom, PFA, the principle $\mathrm{BC}_{\omega_1}(\mathcal{X})$ when $\mathcal{X}$ is the class of all proper posets. In this paper we shall not need a fine analysis of the variety of these principles and we prefer to use the corresponding Baire category numbers to express them (see [13]).[2] Thus, Rothberger's inequality $\mathfrak{p} > \omega_1$, is a restatement of MA_{ω_1} for σ-centered posets, the inequality $\mathfrak{m} > \omega_1$, a restatement of MA_{ω_1} itself, and finally, the cardinal inequality $\mathfrak{mm} > \omega_1$ is a restatement of MM. We shall also need the following important cardinal characteristic of the continuum

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}} \text{ and } \mathcal{F} \text{ has no upper bounds in } (\mathbb{N}^{\mathbb{N}}, <_{\mathrm{Fin}})\}.$$

Let us discuss some consequences of the Baire category principle $\mathfrak{mm} > \omega_1$. We split the discussion in the following five subsections taking samples from a rather rich theory that has been surveyed in [13] and in [52].

2.1. **Failure of Kőnig's Lemma.**

This is indeed the most important fact in the theory of $\mathcal{P}(\omega_1)$ and it is given by the following result proven some eighty years ago (see [20]).

2.3 Theorem (Aronszajn). *There is a tree of height ω_1 that has countable levels but no uncountable branches.*

The purpose of this subsection is to examine whether there is any structure in the class of these counterexamples to Kőnig's Lemma. More precisely, we will search for the structure in the class $\mathcal{AT}$ of all trees satisfying the conclusion of Theorem 2.3. It turns out that the isomorphism relation $\cong$ is too fine and that we must retreat to a coarser relation $\equiv$ given by the following quasi-ordering: $S \leq T$ iff there is a strictly increasing map $f : S \to T$. It also turns out that the standard proofs of Theorem 2.3 do not shed much light and that we need the

---

[2]The reader interested in fine fragmentation of these principles can find some information in [48].

more canonical constructions given by the author's concepts of walks on ordinals. Recall that a $\square_\omega$-sequence is a sequence C_α ($\alpha < \omega_1$) of sets such that $C_{\alpha+1} = \{\alpha\}$ and such that for a limit ordinal α the set C_α is a cofinal subset of α of order type ω. Fixing a C-sequence one defines the notion of a walk $\beta = \beta_0 > \beta_1 > \cdots > \beta_k = \alpha$ from an ordinal β to a smaller ordinal α by letting $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$, i.e., letting β_{i+1} be the minimal point of C_{β_i} that is bigger than or equal to α. By studying various characteristics of the walk one obtain a useful metric theory of countable ordinals capable of producing critical objects in a variety of contexts (see [49]). For example, if we let $\rho_2 : [\omega_1]^2 \to \omega$ count the number of steps in the walk we get a tree

$$T(\rho_2) = \{\rho_2(\cdot, \beta) \restriction \alpha : \alpha \leq \beta < \omega_1\}$$

belonging to $\mathcal{AT}$. It turns out that this is not just an ordinary element of $\mathcal{AT}$, it is a critical object in this class. To see this one first notes that if $S \leq T$ then the strictly increasing map $f : S \to T$ witnessing this can be assumed to be level-preserving and that it can be characterized by the following Lipschitz property[3]

$$\Delta(f(x), f(y)) \geq \Delta(x, y) \text{ for all } x, y \in S.$$

This leads us to the following definition.

2.4 Definition. A tree T in $\mathcal{AT}$ is *Lipschitz* if every level preserving map from an uncountable subset of T into T has the Lipschitz property when restricted to an uncountable subset of its domain.

2.5 Theorem (Todorcevic). *The tree $T(\rho_2)$ is a Lipschitz tree that can be decomposed into countably many antichains. Moreover if we let $C(\rho_2)$ denote $T(\rho_2)$ ordered lexicographically we get an uncountable linearly ordered set whose cartesian square $C(\rho_2) \times C(\rho_2)$ can be decomposed into countably many chains.*

It turns out that $T(\rho_2)$ and $C(\rho_2)$ are critical objects in the category of counterexamples to Kőnig's Lemma and the category of uncountable linear orderings, respectively.

Let us now examine the class of $\mathcal{LT}$ of all Lipschitz trees and the way it is placed in the class $\mathcal{AT}$.

2.6 Theorem (Todorcevic). *Assume $\mathfrak{mm} > \omega_1$.*

(1) *For every $S \in \mathcal{LT}$ and every $T \in \mathcal{AT}$ either $S \leq T$ or else $T \leq S$.*

---

[3]Here, for two nodes x, y in a given tree T, by $\Delta(x, y)$ we denote the 'distance' between x and y in T, the height of the maximal node $t \in T$ that is below both x and y.

(2) *The chain $\mathcal{LT}$ is discrete, i.e., for every $T \in \mathcal{LT}$ there is $T^{(1)} \in \mathcal{LT}$ such that for all $S \in \mathcal{AT}$ either $S \leq T$ or $T^{(1)} \leq S$.*

(3) *The chain $\mathcal{LT}$ is both cofinal and coinitial in the whole class $\mathcal{AT}$ i.e, for every $T \in \mathcal{AT}$ there exist $S, U \in \mathcal{LT}$ such that $S \leq T \leq U$.*

(4) *The shift operator associated with the map $\alpha \mapsto \alpha + 1$ that gives $T \mapsto T^{(1)}$ naturally extends to other monotone maps g from ω_1 into ω_1 giving us the corresponding shift operator $T \mapsto T^{(g)}$. Then for all $S, T \in \mathcal{LT}$ there exists monotone map g such that either $S \equiv T^{(g)}$ or else $T \equiv S^{(g)}$.*

In other words, modulo shifting, there is really only one counterexample to Kőnig's lemma. It turns out also that in the class $\mathcal{LT}_k$ of Lipschitz trees of some uniform branching degree $k \leq \omega$ the equivalence relation $\equiv$ coincides with the isomorphism relation $\cong$ and so there was no need for any retreat after all. In fact, to every $T \in \mathcal{LT}$ we can associate an ultrafilter $\mathcal{U}(T)$ on ω_1 that is Σ_1-definable in the parameter T as follows,

$$\mathcal{U}(T) = \{A \subseteq \omega_1 : (\exists X \subseteq T)[|X| = \aleph_1 \wedge \Delta(X) \subseteq A]\},$$

where for a subset X of T, we let $\Delta(X) = \{\Delta(x, y) : x, y \in X, x \neq y\}$. It turns out that $T \mapsto \mathcal{U}(T)$ is a complete invariant for $\equiv$, i.e., $S \equiv T$ iff $\mathcal{U}(S) = \mathcal{U}(T)$. It also turns out if we shift $\mathcal{U}(T(\rho_2))$ to the domain ω in such a way that the image is nonprincipal, then we obtain a selective ultrafilter on ω whose Rudin-Keisler class does not depend on the $\square_\omega$-sequence we started with. So, we have the following fact (see [50]).

2.7 Theorem (Todorcevic). *Assume $\mathfrak{mm} > \omega_1$. Then there is a selective ultrafilter $\mathcal{U}$ on ω whose Rudin-Keisler class is Σ_1-definable with no parameter. In particular, there is a $L(\mathbb{R})$-generic ultrafilter for the boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$.*

It turns out that the chain $\mathcal{LT}/\mathbb{Z}$—that is, $\mathcal{LT}$ modulo the equivalence relation that relates trees that are a finite shift-distance from each other—is also a canonical object (see [26]).

2.8 Theorem (Martinez-Ranero-Todorcevic). *Assume $\mathfrak{mm} > \omega_1$. Then $\mathcal{LT}/\mathbb{Z}$ is the $\aleph_2$-saturated linear ordering, i.e., a realization of Hausdorff's η_2-set.*

2.2. **Structure theory of separable linear orderings.** Besides the isomorphism relation in this class of linear orderings it is also useful to consider the related relations $K \leq L$ (which holds whenever there

is an embedding—a strictly increasing map $f : K \to L$) and the corresponding equivalence relation $K \equiv L$ (which is given by $K \leq L$ and $L \leq K$ and is coarser than the isomorphism relation).

We start by recalling the following theorem of Cantor who invented his well known back-and-forth argument for its proof.

2.9 Theorem (Cantor). *All countable dense linear orderings are isomorphic.*

2.10 Corollary. $L \leq \mathbb{Q}$ *for every countable linear ordering L.*

The strongest structural information about the class of all countable linear orderings is given in the following result whose proof used Nash-Williams' theory of better-quasi-orderings (see [21]).

2.11 Theorem (Laver). *The class of countable linear orderings is well-quasi-ordered, i.e., for every sequence L_i $(i < \omega)$ of countable linear orderings there exist $i < j < \omega$ such that $L_i \leq L_j$.*

In other words, theory of countable linear orderings parallels very much the theory of countable ordinals even to the level that there are analogues of the notion of indecomposable ordinal, Cantor normal form, etc.

For an infinite cardinal κ, we say that a linear ordering L is κ-*dense* if all nontrivial intervals of L (including those with end-points $-\infty$ and $+\infty$) have cardinality κ. Consider the following principle:

BA(κ) : All *separable* κ-dense linear orderings are isomorphic.

By Cantor's theorem, we have that BA($\aleph_0$) is true and for large κ's we have the following facts, the second of which uses author's theory of oscillations on $\mathbb{N}^{\mathbb{N}}$, a theory that will show up again below.

2.12 Theorem (Dushnik-Miller). BA($\mathfrak{c}$) *is false.*

2.13 Theorem (Todorcevic). BA($\mathfrak{b}$) *is false.*

What about BA($\aleph_1$)? It is, in fact, this statement that belongs to the theory of $\mathcal{P}(\omega_1)$ that we are trying to describe here in some detail. We have the following fundamental result that generalizes Cantor's theorem (see [4]).

2.14 Theorem (Baumgartner). *Assume $\mathfrak{mm} > \omega_1$. Then all separable $\aleph_1$-dense linear orderings are isomorphic.*

2.15 Corollary. *Assume $\mathfrak{mm} > \omega_1$. Then every pair K and L of separable linear orderings of cardinality $\aleph_1$ are equivalent, i.e., $K \leq L$ and $L \leq K$.*

It should be also noted that the corollary holds the information of the theorem since under the assumption of $\mathfrak{p} > \omega_1$ its conclusion gives

the conclusion of Baumgartner's theorem. Baumgartner's theorem is an interesting structural result that has been analyzed by many set theorists. In particular, while analyzing its proof, the author was led to the following interesting graph-theoretic principle:

[OGA] For every open graph $\mathcal{G} = (X, E)$ on a separable metric space X either $\mathcal{G}$ is countably chromatic or else $\mathcal{G}$ has an uncountable clique.

2.16 Theorem (Todorcevic). $\mathfrak{mm} > \omega_1$ *implies* OGA.

The Open Graph Axiom, OGA, is a graph-theoretic dichotomy that is quite easy to understand and use in place of the often technically demanding applications of Baire category principles such as $\mathfrak{mm} > \omega_1$. So it is not surprising that it has been used in other areas in mathematics. The following is its typical application (see [10]).

2.17 Theorem (Farah). *Under* OGA *all automorphisms of the Calkin algebras are inner.*

This dichotomy tends to transfer problems about structures of cardinality continuum to those of cardinality $\aleph_1$ and so it is quite relevant to the discussion of this paper. In particular, OGA gains some extra power if supplemented by assumptions like $\mathfrak{m} > \omega_1$. The following (see [7]) is a typical application of such a combination which actually uses the generic version OGA$^+$ of OGA where the uncountable clique has some generic behavior, a version which is still a consequence of $\mathfrak{mm} > \omega_1$.[4]

2.18 Theorem (Woodin). *Assume* OGA$^+$ *and* $\mathfrak{m} > \omega_1$. *For every compact Hausdorff space K every algebraic norm on the Banach algebra $\mathcal{C}(K, \mathbb{C})$ is equivalent to the uniform norm.*

It turns out that OGA has also a substantial 'ZFC-shadow' as it is provable in ZFC for open graphs whose sets of vertices are Σ_1^1. In fact, in the descriptive set-theoretic context it is more natural to strengthen the second alternative to saying that $\mathcal{G}$ has a clique spanned by a perfect set of vertices. Then we have the following result where PD denotes the axiom of determinacy of projective sets of reals.

2.19 Theorem (Feng). PD *implies* OGA *for open graphs spanned by projective sets of vertices.*

Similarly, AD$^{L(\mathbb{R})}$ implies OGA for open graphs spanned by sets of reals from $L(\mathbb{R})$ (see [11]). The first use of OGA was in showing that this principle is giving us an essentially complete picture about the spectrum of gaps in the quotient boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ and that

---

[4]More precisely, OGA$^+$ is used in the proof of this result to show that the linear ordering $(2^{\omega_1}, <_{\mathrm{lex}})$ does not embed into the quotient algebra $\mathcal{P}(\omega)/\mathrm{Fin}$.

Hausdorff's (ω_1, ω_1^*)-gap in this quotient algebra is the only critical object in this category (see [18]). One statement of this result is the following (see [41]).

2.20 Theorem (Todorcevic). OGA *implies that Hausdorff's* (ω_1, ω_1^*)-*gap in* $\mathcal{P}(\omega)/\mathrm{Fin}$ *is the only gap in this quotient algebra with regular uncountable sides.*

This had led us to the following surprising consequence (see [41]).

2.21 Corollary (Todorcevic). OGA *implies* $\mathfrak{b} = \omega_2$.

This leads us to the following natural question.

2.22 Question (Todorcevic). *Does* OGA *imply* $\mathfrak{c} = \omega_2$?

If this turns out to be the case this would certainly match Gödel's attempt to solve the Continuum Problem as seen from his well-known unpublished manuscript (see [17] and [5]). Recall that Gödel was inspired by Hausdorff's work on gaps in quotient structures like $\mathbb{N}^{\mathbb{N}}/\mathrm{Fin}$ and $\mathcal{P}(\omega)/\mathrm{Fin}$ and by the notions of small sets of reals, or more precisely by Borel's notion of strong measure zero for sets of reals. In particular, he wanted to decompose the reals into $\aleph_2$ sets that are both of strong measure zero and are equal to unions of at most $\aleph_1$ closed sets of reals. Using the $<_{\mathrm{Fin}}$-well-ordered and $<_{\mathrm{Fin}}$-unbounded subset A of $\mathbb{N}^{\uparrow \mathbb{N}}$ of order type ω_2 given by Theorem 2.21 and the author's oscillation theory of $\mathbb{N}^{\uparrow \mathbb{N}}$ we get that OGA is also giving us a decomposition of $\mathbb{R}$ into $\aleph_2$ sets X that are small in another very precise sense, they are small from the point of view of oscillation theory (see [40], [41], [44]). More precisely, we say $X \subseteq 2^{\mathbb{N}}$ is *small* if there is $h \in \mathbb{N}^{\mathbb{N}}$ such that every real r from X is coded by an uncountable subset B of $A(h) = \{f \in A : f \leq h\}$ in the sense that for every $f \neq g$ in B there is $k \geq \Delta(f, g)$ such that $\mathrm{osc}(f, g) = \sum_0^k r(i) 2^i$. Note that if B and B' are subsets of $A(h)$ that code two different members r and r' of X then their intersection must be finite, so the set X appears indeed as quite small. Determining whether this places a bound on its cardinality is now an interesting task. An attempt towards this is given in [27].

2.3. **Structure theory of non-separable linear orderings.** Returning to the structure theory of uncountable linear orderings, one must first note that separability is an essential assumption in Baumgartner's theorem since the class[5]

$$\mathcal{AL} = \{L \in \mathcal{LO}_{>\aleph_0} : L \perp \{\omega_1, \omega_1^*, \mathbb{R}\}\}$$

---

[5]Here $\perp$ signifies the fact that L shares no uncountable subordering with any linear ordering in the set.

of uncountable linear orderings is not empty. For example, it is occupied by any lexicographically ordered Aronszajn tree as well as any linear ordering given by the following well-known result (see [34]) which is also a consequence of Theorem 2.5 above.

2.23 Theorem (Shelah). *There is an uncountable linear ordering L whose cartesian square $L \times L$ can be decomposed into countably many chains.*

When he proved this theorem some forty years ago, Shelah has conjectured that all orderings satisfying the conclusion of this theorem must be equivalent (modulo of course taking reverses if necessary). This led him to further conjecture that every ordering from $\mathcal{AL}$ must contain a subordering with this property. This has been verified only thirty years later (see [29]).

2.24 Theorem (Moore). *Assume $\mathfrak{mm} > \omega_1$. Then every linear ordering L from the class $\mathcal{AL}$ contains an uncountable subset C whose cartesian square $C \times C$ can be decomposed into countably many chains. Moreover the class $\mathcal{AL}$ has a universal element $\mathbb{Q}_C$.*

Using this and Baumgartner's theorem we get the following result[6].

2.25 Corollary. *Assume $\mathfrak{mm} > \omega_1$. The class of uncountable linear orderings has a five-element basis, $\{\omega_1, \omega_1^*, B, C, C^*\}$.*

It follows in particular that the class $\mathcal{AL}$ has some properties similar to the class of countable linear orderings which also has a universal element $\mathbb{Q}$ and a two-element basis $\{\omega, \omega^*\}$. For example the class $\mathcal{AL}_F = \{L \in \mathcal{A} : \mathbb{Q}_C \not\leq L\}$ allows to be ranked using C-sums and C^*-sums in a similar way Hausdorff was ranking countable scattered orderings using ω-sums and ω^*-sums. It turns out that we also have the analogue of Laver's theorem and therefore a full classification of $\mathcal{AL}$ (see [25]).

2.26 Theorem (Martinez-Ranero). *Assume $\mathfrak{mm} > \omega_1$. The class $\mathcal{AL}$ is well-quasi-ordered.*

In fact we have more structural properties here. For example, for each ordinal $\alpha < \omega_1$ there are two incomparable linear orderings D_α^- and D_α^+ of rank α that split the whole class into four pieces, those equivalent to one of the D_α^- and D_α^+, those that are embeddable into both D_α^- and D_α^+ and those that embed both D_α^- and D_α^+

2.4. **Tukey classification theory.** In this category the following notion of reducibility provides an optimal classification scheme. For two partially ordered sets P and Q we say that P is *Tukey reducible* to

---

[6]Here B is any set of reals of cardinality $\aleph_1$

Q and write $P \leq_T Q$ whenever there is a mapping $f : P \rightarrow Q$ such that for every (upwards) unbounded subset X of P the image $f[X]$ is (upwards) unbounded in Q. Let $\equiv_T$ be the corresponding equivalence relation. This may appear as coarse relation but if P and Q are (upwards) directed then $P \equiv_T Q$ holds if and only if they both can be embedded as cofinal subsets of some directed set R (see [53]). It is for this reason that in the category of directed sets the equivalence classes of $\equiv_T$ are called *cofinal types*. It is easily seen that 1, ω, ω_1, $\omega \times \omega_1$, and $[\omega_1]^{<\omega}$ viewed as directed sets with their natural orderings represent different cofinal types. Are there any other cofinal types of directed sets of cardinality at most $\aleph_1$? The following results answers this question giving also the Tukey classification of the class of *all* posets of cardinality not bigger than $\aleph_1$ (see [39], [43]).

2.27 Theorem (Todorcevic). *Assume $\mathfrak{mm} > \omega_1$. Then every directed set of cardinality at most $\aleph_1$ is Tukey-equivalent to one on the list of five directed sets: 1, ω, ω_1, $\omega \times \omega_1$, and $[\omega_1]^{<\omega}$.*

Moreover letting $D_0 = 1$, $D_1 = \omega$, $D_2 = \omega_1$, $D_3 = \omega \times \omega_1$, and $D_4 = [\omega_1]^{<\omega}$, every partially ordered set of cardinality at most $\aleph_1$ is Tukey equivalent to one of these:

(a) $\bigoplus_{i<5} n_i D_i$ $(i < 5, n_i < \omega)$,
(b) $\aleph_0 \cdot 1 \oplus \bigoplus_{i=2}^{4} n_i D_i$ $(2 \leq i < 5, n_i < \omega)$,
(c) $\aleph_0 \cdot \omega_1 \oplus n_4 [\omega_1]^{<\omega}$ $(n_4 < \omega)$,
(d) $\aleph_0 \cdot [\omega_1]^{<\omega}$,
(e) $\aleph_1 \cdot 1$.

Thus we have a complete Tukey classification of posets with domain ω_1.[7] This result has triggered a renewed interest in the Tukey classification scheme in other contexts. For example, the Tukey classification appears quite natural in the context of descriptive set theory as many posets of interest to the rest of mathematics can be represented in a Borel or, more generally, in a projective way. This may seem at first impossible as the Tukey maps are not assumed to be definable but a deeper analysis shows that in many context these maps can always be replaced by definable ones. Another context where the Tukey classification scheme sheds some light is the theory of cardinal characteristics of the continuum where cardinal inequalities can very frequently be replaced by Tukey inequalities. This comes from the fact that the relation $P \leq_T Q$ implies that $\mathrm{cof}(P) \leq \mathrm{cof}(Q)$ and $\mathrm{add}(P) \geq \mathrm{add}(Q)$. For example, if $\mathcal{M}$ denotes the σ-ideal of all meager subsets of $\mathbb{R}$ and $\mathcal{N}$ the σ-ideal of all measure-zero sets of reals, then we have $\mathcal{M} \leq_T \mathcal{N}$,

---

[7]In a correspondence about Theorem 2.27 Peter Koellner mentions a similarity with the work of Woodin [56] which identifies the cardinals below $|[\omega_1]^\omega|$ using $\mathrm{AD}_{\mathbb{R}}$.

a Tukey reduction witnessed by a simply definable map. Note that CH sheds no light here as it implies that $\mathcal{I} \equiv_T \omega_1$ for *all* $\mathfrak{c}$-generated σ-ideals $\mathcal{I}$ on $\mathbb{R}$. This is one of the reasons why one retreats to the Tukey theory of structures of lower type without loosing the information. For example, we can replace $\mathcal{N}$ by the Banach lattice ℓ_1 and $\mathcal{M}$ by the ideal NWD of compact nowhere dense subsets of $\mathbb{R}$ and have the inequality NWD $\leq_T \ell_1$ that holds all the nontrivial information about the inequality $\mathcal{M} \leq_T \mathcal{N}$. For posets of higher descriptive complexity it is natural to use PD or $\mathrm{AD}^{L(\mathbb{R})}$ as the following result shows (see [14]).

2.28 Theorem (Fremlin). *Assume* PD. *The Tukey types of directed sets of the form $\mathcal{K}(X)$ where X is a projective set of reals are well-ordered in type ω.*

Here $\mathcal{K}(X)$ is the lattice of compact subsets of X. The first few Tukey types in this classification result are easily identifiable. The Tukey Type 1 corresponds to compact X, the second Tukey type on this list corresponds to locally compact noncompact X and the third to Polish non-locally compact. More generally, the Tukey type of a $\mathcal{K}(X)$ on this list depends on the minimal n such that X is Π_n^1 at least when $n \geq 2$. It would be interesting to extend this result to sets from $L(\mathbb{R})$ using $\mathrm{AD}^{L(\mathbb{R})}$. We should also mention that some assumption is needed here since in the Gödel's constructible universe, for example, from $\mathcal{K}(X) \leq_T \mathcal{K}(\mathbb{Q})$ we can't conclude that X is Π_1^1.

The analysis of the proof of Theorem 2.27 has led us to the following interesting dichotomy (see [46], [51]).

PID: For every P-ideal $\mathcal{I}$ of countable subsets of some set S either

(1) there is uncountable $X \subseteq S$ such that $[X]^{\aleph_0} \subseteq \mathcal{I}$, or else
(2) there is a decomposition $S = \bigcup_{n<\omega} S_n$ such that $S_n \cap a$ is finite for all $n < \omega$ and $a \in \mathcal{I}$.

More precisely, we have the following result (see [46]).

2.29 Theorem (Todorcevic). $\mathfrak{mm} > \omega_1$ *implies* PID. *Moreover,* PID *is consistent with* CH *relative to the consistency of the existence of a supercompact cardinal.*

PID is an interesting set-theoretic principle saying that in the category of P-ideals of countable sets there exactly two kind of critical objects, the P-ideals $[S]^{\leq \aleph_0}$ of *all* countable subsets of some set S and the P-ideals generated by a sequence S_n ($n < \omega$) of subsets of some set S as follows

$$\{a \in [S]^{\leq \aleph_0} : a \cap S_n \text{ is finite for all } n < \omega\}.$$

It turns out that PID has some properties in common with the principle OGA discussed above. In particular, it has a similar effect on the structure of gaps in the quotient algebra and so, in particular, we have the following influence on the Continuum Problem.

2.30 Theorem (Todorcevic). PID *implies that Hausdorff's* (ω_1, ω_1^*)-*gap in* $\mathcal{P}(\omega)/\mathrm{Fin}$ *is the only gap in this quotient algebra with regular uncountable sides.*

2.31 Corollary (Todorcevic). PID *implies* $\mathfrak{b} \leq \aleph_2$.

This leads us to the following interesting question.

2.32 Question (Todorcevic). *Does* PID *imply* $\mathfrak{c} \leq \aleph_2$?

It should be noted, however, that unlike OGA, the P-ideal dichotomy has a strong influence throughout the whole universe of sets, rather than just sets of reals. Here are samples from the list of its influences (see [46], [55], [38]) that show this very clearly.

2.33 Theorem (Todorcevic). PID *implies that* $\square(\theta)$ *fails for all regular cardinals* $> \omega_1$.

2.34 Theorem (Viale). PID *implies the Singular Cardinals Hypothesis.*

2.35 Theorem (Steel). PID *implies* $\mathrm{AD}^{L(\mathbb{R})}$.

In particular, PID has a substantial large cardinal strength. It is our opinion however that the real interest in PID will come from the rest of mathematics since this is a rather simple principle that is quite easy to use and therefore accessible to the non-experts to this area. One example of such use is the following remarkable metrization result that solved a sixty year old problem of Maharam (see [23],[3])

2.36 Theorem (Balcar-Jech-Pazak). PID *implies that every complete weakly distributive algebra satisfying the countable chain condition supports a strictly positive continuous submeasure.*

So, in particular, PID implies the Souslin Hypothesis.

2.5. **A well-ordering of** $\mathcal{P}(\omega_1)$**.** The purpose of this section is to point out the well-known fact that our Baire category assumptions are sufficient to give us a definable well-ordering of the inner model $L(\mathcal{P}(\omega_1))$ as this is one of the important contrasts with the case of $L(\mathcal{P}(\omega))$ under $\mathrm{AD}^{L(\mathbb{R})}$. By now we have several descriptions of such a well-ordering, the first being that of Woodin [57] in the context of the $\mathbb{P}_{\max}$ forcing extension which also uses the least amount of the axiom of choice, the existence of a stationary and costationary subset of ω_1. Here we briefly sketch the description due to the author (see [47]) which uses a different instance of choice. We could have also

described yet another description of such a well-ordering using a bit stronger instance of choice due to Moore [28] and a bit weaker instance of the Baire category principle $\mathfrak{mm} > \omega_1$. We think that in this context the choice of parameters parameters is a feature that should not be ignored.[8] The parameter in the coding that we choose to describe here is an arbitrary one-to-one sequence r_ξ ($\xi < \omega_1$) of elements of $\{0,1\}^\omega$. This allows us to associate to every countable set of ordinals X, the real $r_X = r_{\mathrm{otp}(X)}$.

For a pair x and y of distinct members of $\{0,1\}^\omega$, set

$$\Delta(x,y) = \min\{n < \omega : \ x(n) \neq y(n)\}.$$

Note that for three distinct members x, y and z of $\{0,1\}^\omega$, the set

$$\Delta(x,y,z) = \{\Delta(x,y), \Delta(y,z), \Delta(x,z)\}$$

has exactly two elements.

2.37 Theorem (Todorcevic). *For every subset S of ω_1 there is a stationary set preserving poset $\mathcal{P}_S$ which does not add reals and forces the existence of three ordinals $\gamma > \beta > \alpha \geq \omega_1$ and an increasing continuous decomposition $\gamma = \bigcup_{\nu < \omega_1} N_\nu$ of the ordinal γ into countable sets such that for all $\nu < \omega_1$,*

$$N_\nu \cap \omega_1 \in S \quad iff \quad \Delta(r_{N_\nu \cap \alpha}, r_{N_\nu \cap \beta}) = \max \Delta(r_{N_\nu \cap \alpha}, r_{N_\nu \cap \beta}, r_{N_\nu}).$$

It turns out that while for a particular subset S of ω_1 the poset $\mathcal{P}_S$ does not add reals, if we want to have simultaneously all the corresponding instances of the Baire category principle $\mathfrak{mm} > \omega_1$ (and therefore describe a well-ordering of $\mathcal{P}(\omega_1)$), the Continuum Hypothesis must be false and, in fact, we must have Luzin's Continuum Hypothesis $2^{\aleph_0} = 2^{\aleph_1}$. This is an interesting phenomenon appearing in all known descriptions of well-orderings of $\mathcal{P}(\omega_1)$ from parameters from $\mathcal{P}(\omega_1)$.

2.6. **Permanence under forcing.** We finish this section by briefly listing some results showing that the theory of $\mathcal{P}(\omega_1)$ we have just sketched has substantial permanence properties. These are results about forcing absoluteness of certain theories under the assumption of large cardinals. The first such result is the following well-known theorem.

2.38 Theorem (Woodin). *Assume there is a proper class of Woodin cardinals. Then the theory of $L(\mathbb{R})$ is invariant under any set forcing.*

---

[8]Recall, for example, that if we assume $\mathrm{AD}^{L(\mathbb{R})}$, add an ω_1-sequence of Cohen reals, and then form $L(\mathcal{P}(\omega_1))$ in the generic extension, this inner model will have an ω_1-sequence of distinct reals, but it will not have any counterexamples to König's Lemma and so, in particular, it will not have any $\square_\omega$-sequence or any other critical objects of the theory of $\mathcal{P}(\omega_1)$ described above on the basis of such a sequence.

The relevance of this to our present discussion is given by the following fact.

2.39 Theorem (Woodin). *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Then there is a homogeneous forcing notion* $\mathbb{P}_{\max}$ *in* $L(\mathbb{R})$ *which forces (over* $L(\mathbb{R})$*!) all the consequences of the Baire category principle* $\mathfrak{mm} > \omega_1$ *about objects from the structure* $(H(\aleph_2), \in)$*, and more.*[9]

Note that the homogeneity of $\mathbb{P}_{\max}$ implies that the inner model $L(\mathbb{R})$ can see all the sentences true in the $\mathbb{P}_{\max}$-forcing extension and thus the connection with Theorem 2.38. It is unknown if there is a natural strengthening of $\mathfrak{mm} > \omega_1$ which implies that $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$-forcing extension of $L(\mathbb{R})$. That would make the theory of $L(\mathcal{P}(\omega_1))$ invariant under any set forcing that forces this version of $\mathfrak{mm} > \omega_1$. However, we do have a quite satisfactory result in that direction proved recently by M. Viale [54].

2.40 Theorem (Viale). *Assume there is a proper class of superhuge cardinals and that a natural strengthening of the Baire category principle* $\mathfrak{mm} > \omega_1$ *is true. Then the theory of* $H(\aleph_2)$ *is preserved by any stationary-set preserving poset which forces this strengthening of* $\mathfrak{mm} > \omega_1$.

3. $\mathcal{P}(\omega_1)$ UNDER CH

In this section we first list some of the well-known consequences of CH and in particular those which Godel [16] calls 'paradoxical'. What we can observe here is that CH is giving us an extremely rich array of different mathematical objects with no apparent relationships between them. We therefore must examine whether there could be an interesting structure theory of $\mathcal{P}(\omega_1)$ that is compatible with CH and that could give us some explanation of its consequences.

3.1. **Consequences of** CH. As it is well known, the early survey of consequences of CH in various parts of mathematics appeared in Sierpinski's book [37] which has been commented upon both by Luzin [22] and Gödel [16]. For example, Gödel makes his well-known comments on the following consequences.

3.1 Theorem (Luzin). *Assume* CH. *Then there is an uncountable set of reals which has countable intersection with every meager set of reals.*

3.2 Theorem (Brown-Sierpinski). *Assume* CH. *Then there is an uncountable subset* X *of* $[0,1]^{\mathbb{N}}$ *such that for every uncountable* $Y \subseteq X$ *and for all but finitely many* n, *the* nth *projection maps* Y *onto* $[0,1]$.

---

[9]More precisely, all of the consequences of Woodin's axiom $(*)$

3.3 Theorem (Hurewicz). *Assume* CH. *Then there is an infinite-dimensional subset X of $[0,1]^{\mathbb{N}}$ such that all uncountable subsets Y of X have infinite dimension.*

Sets satisfying the conclusion of Theorem 3.1 are called *Luzin sets*. The same result is true for the σ-ideal of measure zero sets of reals but this follows from the following 'Duality Principle'.

3.4 Theorem (Erdős-Sierpinski). *There is a bijection $f : \mathbb{R} \to \mathbb{R}$ such that $f^2 = $ id and such that a subset X of $\mathbb{R}$ is meager if and only if its image $f[X]$ is of measure zero.*

Some other typical uses of CH in mathematics that appeared after Sierpinski's book [37] are as follows.

3.5 Theorem (Parovichenko). *Assume* CH. *Then $\mathcal{P}(\omega)/$Fin is a saturated boolean algebra and so, in particular, a universal object in the class of boolean algebras of cardinality at most $\aleph_1$.*

Note how discretely this avoids the structural result of Hausdorff that $\mathcal{P}(\omega)/$Fin has (ω_1, ω_1^*)-gap. In fact this manages to avoid even so basic object such as a one-to-one sequence r_ξ ($\xi < \omega_1$) of reals (elements of 2^ω)! To see this consider the following two orthogonal families in $\mathcal{P}(2^{<\omega})/$Fin which can't be separated,

$$\mathcal{C}^0 = \{c_\xi^0 : \xi < \omega_1\} \text{ and } \mathcal{C}^1 = \{c_\xi^1 : \xi < \omega_1\},$$

where $c_\xi^i = \{r_\xi \restriction n : n < \omega \text{ and } r_\xi(n) = i\}$ ($\xi < \omega_1, i < 2$).

We mention another (similar) use of CH but in another area of mathematics (see [6], [9]).

3.6 Theorem (Dales, Esterle). *Assume* CH. *Then for every infinite compact Hausdorff space K there exists a discontinuous algebraic monomorphism from $\mathcal{C}(K, \mathbb{C})$ into a Banach algebra. So, in particular, there is an algebraic norm on $\mathcal{C}(K, \mathbb{C})$ that is not equivalent to the uniform norm.*

We mention also the following result (see [32]) that is similarly related to a structural result of Section 2 (Theorem 2.17).

3.7 Theorem (Phillips-Weaver). *Assume* CH. *Then the Calkin algebra has $2^{\aleph_1}$ automorphisms and so, in particular, there is one which is not inner.*

More typical uses of CH involve diagonalization procedures of length ω_1. CH is used to ensure that during this procedure enough requirements have been met so that the resulting structure will have no uncountable substructures with a particular property. Here is a typical result of this sort (see [31]).

3.8 Theorem (Kunen). *Assume* CH. *Then there is a non-metrizable scattered compact Hausdorff space K such that the function space $\mathcal{C}(K)$ has no uncountable biorthogonal systems.*

Other typical uses of CH go through the *weak diamond principle* of Devlin and Shelah [8]. Here are typical such consequence which should be compared with the corresponding results from Section 2 above.

3.9 Theorem (Todorcevic). *Assume* CH. *The class of Lipschitz trees is not totally ordered under $\leq$. In fact, there is a family of cardinality $2^{\aleph_1}$ of pairwise incomparable Lipschitz trees.*

3.10 Theorem (Martinez-Ranero). *Assume* CH. *Let $\mathcal{CL}$ be the class of all uncountable linear orderings whose cartesian squares can be decomposed into countably many chains. Then $\mathcal{CL}$ is not well-quasi-ordered under the relation $\leq$ of isomorphic embedding. In fact, $\mathcal{CL}$ contains a subfamily of cardinality $2^{\aleph_1}$ of pairwise incomparable orderings.*

It can be seen that many consequences of CH can be expressed as Σ_1^2-sentences. The following result shows that, in some sense,[10] CH is the most powerful sentence of this complexity.

3.11 Theorem (Woodin). *Assume that there exist unboundedly many measurable Woodin cardinals. Then if one Σ_1^2-sentence is true in one forcing extension then it is also true in all forcing extensions satisfying the Continuum Hypothesis.*

One common feature seen in the many applications of CH is the fact that they give us an immense quantity of objects with no apparent relationships between each other, or better said, no theory that would explain their existence. So we are left to search for additional set-theoretic principles that would give us some of the structure theory comparable to that from Section 2. We discuss this in the following subsection.

3.2. **Baire category principles compatible with** CH. We are looking for a Baire category principle $\mathrm{BC}_{\omega_1}(\mathcal{X})$ consistent with CH and where $\mathcal{X}$ is maximal relative to that requirement. The deepest part of Shelah's iteration theory (see [36]) was invented for the purpose of finding this class of posets $\mathcal{X}$. It has been realized quite early that a poset $\mathbb{P}$ from such $\mathcal{X}$ must be more than just proper and not add reals—$\mathbb{P}$ must be complete relative to some simply definable 'completeness system' which itself must be at least '2-complete'. In fact, to preserve not adding reals $\mathbb{P}$ must be 'α-proper' for every countable ordinal α. For example, the P-ideal dichotomy is a consequence of $\mathrm{BC}_{\omega_1}(\mathcal{X})$ for $\mathcal{X}$ the class of posets that are α-proper for all $\alpha < \omega_1$ and are complete

---

[10]We refer the reader to the paper [19] of Koellner that makes this precise.

relative to some simple σ-complete completeness system, a principle proved by Shelah [36] to be consistent with CH relative to the consistency of the existence of a supercompact cardinal. The problem of using PID in the context of CH is that there are not many P-ideals around to apply this principle.

More precisely, PID becomes powerful only if joined by additional assumptions like $\mathfrak{p} > \omega_1$ or $\mathfrak{m} > \omega_1$ since they turn PID into a dual dichotomy that applies to arbitrary ideals of countable sets that are in some precise sense $\aleph_1$-generated.

In fact, this was the original way these ideal-dichotomies were stated before the realization that the dual dichotomy PID is consistent with CH. It turns out that any natural variation on PID requires iteration theory for a class $\mathcal{X}$ of posets that fail to satisfy one of the two key requirements above. In particular, Shelah asked whether $\mathrm{BC}_{\omega_1}(\mathcal{X})$ is consistent when $\mathcal{X}$ is the class of *all* posets that are complete relative to some simple σ-complete completeness system, i.e., with no requirement of α-properness for all countable ordinals α. Recently, the difficulty was in part explained by the following result, which hints towards the non-existence of the maximal class $\mathcal{X}$.

3.12 Theorem (Asperó-Larson-Moore). *There exist two Π_2 sentences ψ_1 and ψ_2 of $L(\mathcal{P}(\omega_1))$ such that*

    (1) *ψ_1 is true in a forcing extension by proper posets that does not add new reals,*

    (2) *if there is an inaccessible limit of measurable cardinals then ψ_2 is true in a proper forcing extension that does not add new reals,*

    (3) *the conjunction of ψ_1 and ψ_2 implies Luzin's hypothesis $2^{\aleph_0} = 2^{\aleph_1}$.*

The point here is that ψ_1 and ψ_2 require two different iteration theorems for proper forcing that do not add new reals, two different Baire category principles $\mathrm{BC}_{\omega_1}(\mathcal{X}_1)$ and $\mathrm{BC}_{\omega_1}(\mathcal{X}_2)$ that can't be joined together if we are to keep CH (see the discussion in [1]). Thus, here we have something quite different from the case described above in Section 2. In other words, we are still far from any structure theory of $\mathcal{P}(\omega_1)$ compatible with CH. An analogous analysis of this sort that uses Ω-logic has been given by Koellner [19].

3.3. **Well-orderings.** Recall that in Subsection 2.5 we have seen that the Baire category assumption $\mathfrak{mm} > \omega_1$ was giving us a well-ordering of $\mathcal{P}(\omega_1)$ (and of $\mathcal{P}(\omega)$) that is definable in $L(\mathcal{P}(\omega_1))$ so it is natural to ask the following.

3.13 Question. *Are there any interesting Baire category assumptions compatible with* CH *that would give give us a well-ordering of* $\mathcal{P}(\omega_1)$ *which belongs to* $L(\mathcal{P}(\omega_1))$?

Note that in the context of CH well-orderings of $\mathcal{P}(\omega)$ must be of lengths shorter than the lengths of well-orderings of $\mathcal{P}(\omega_1)$ and in fact subjects of the theory of $\mathcal{P}(\omega_1)$ that we are searching for. This is an important distinction from the case of Section 2. The importance of well-ordering of $\mathbb{R}$ of length ω_1 in any analysis of CH was first stressed by Luzin [22] in his early writing about the Continuum Problem. Luzin asks that they be 'effective'.[11] Well-orderings of $\mathbb{R}$ in this context are used to derive other objects which the theory must explain, so perhaps it is more to the point to ask that they be 'canonical' enough so that the analysis is possible. The purpose of this section is to expose the analysis of Moore [30] which sheds some light in this direction.

Fix a ω_1-sequence $\vec{r} = (r_\alpha : \alpha < \omega_1)$ of distinct reals (elements of $\{0,1\}^\omega$). We shall associate to it in a canonical way a sequence $T_\xi(\vec{r})$ ($\xi < \xi(\vec{r})$) of some length $\xi(\vec{r}) \leq \omega^2$ using an index function $\text{ind} : [\omega_1]^{\leq \aleph_0} \to \omega$ such that $\sup(x) < \text{ind}(x)$ and a C-sequence C_α ($\alpha < \omega_1$) as parameters which could of course be read from $\vec{r}$ if this is an enumeration of all the reals. Elements of each $T_\xi(\vec{r})$ will be closed countable subsets of ω_1 with the end-extension as a tree ordering. We shall also have that for s and t in some $T_\xi(\vec{r})$, if $\sup(s) = \sup(t) = \sup(s \cap t)$ then $s = t$. This provides the uniqueness of an ω_1-branch if there is one. Other properties of this functor of our interest here are summarized as follows.

3.14 Theorem (Moore). *The functor* $\vec{r} \mapsto (T_\xi(\vec{r}) : \xi < \xi(\vec{r}))$ *has the following properties:*

(1) $\xi(\vec{r}) = 0$ *exactly when* $(\omega_1)^{L[\vec{r}]} < \omega_1$;
(2) $T_\xi(\vec{r})$ *has an uncountable branch if and only if* $\xi \neq \xi(\vec{r}) - 1$;

---

[11]Here is the quote from page 130 of [22]: "Le seule preuve de la vérité de l'hypothèse de Cantor consisterait à donner une correspondance univoque et réciproque Z, *effective*, c'est-à-dire décrite d'une manière précise et sans ambiguïté possible, entre les points d'une ligne droite et les nombres transfinis de seconde classe. Cette effectivité aurait un très grand intérêt et une grande importance, puisque, dans ce cas, elle serait une source d'un très grand nombre d'importantes relations arithmétiques, algébriques, geométriques et analytiques. Or, on sait que non seulement nous ne pouvons pas attendre que les progrès de la Science nous amènent à une telle correspondance Z effective, mais que, au contraire, c'est le fait inverse qui est beaucoup plus probable: un jour les ressources de la théorie de M. Hilbert seront *peut-être* si avancées qu'on pourra tenter avec succès une démonstration de la *non-existence* d'aucune correspondance Z effective, bien que l'existence d'une correspondance Z non effective soit non contradictoire."

(3) *if $\xi = \xi(\vec{r}) - 1$ then either $\mathbb{R} \nsubseteq L[\vec{r}]$ or else $T_\xi(\vec{r})$ is completely proper[12] in an any outer model with the same reals;[13]*
(4) *if $\xi(\vec{r}) = \omega^2$ then $\mathbb{R} \nsubseteq L[\vec{r}]$;*
(5) *for every $\xi < \omega^2$ the (partial) function $\vec{r} \mapsto T_\xi(\vec{r})$ is Σ_1 definable, so in an outer model for a given $\vec{r}$ the associated sequence of trees may increase in length but it must maintain the entries from the old model.*

3.15 Corollary (Moore). *If $\mathcal{X}$ is the class of all completely proper posets then the corresponding Baire category principle $\mathrm{BC}_{\omega_1}(\mathcal{X})$ implies the negation of the Continuum Hypothesis.*

Thus we have here a functor which to every well-ordering $<_w$ of the reals of order type ω_1 associates an ordinal $\xi(<_w) < \omega^2$ which is canonical enough to expect that any important structure theory of $\mathcal{P}(\omega_1)$ should explain it. Unfortunately, the following consequence of an unpublished result of Shelah shows that if such a theory exists it will not be invariant under forcing extensions of completely proper posets.

3.16 Theorem (Shelah). *For every well-ordering $\vec{r} = (r_\alpha : \alpha < \omega_1)$ of $\mathbb{R}$ and every ordinal $\xi < \omega^2$ there is a forcing extension with the same reals such that $\xi(\vec{r}) \geq \xi$.*

4. FURTHER REMARKS

The previous two sections show that while the Baire category assumptions like $\mathfrak{mm} > \omega_1$ reveal a fine structure theory of $\mathcal{P}(\omega_1)$ that could also address problems coming from different areas of mathematics, nothing comparable to this is known if we require CH to be true. The purpose of this section to further speculate on this.

4.1. **Theory of $\mathcal{P}(\omega)$ versus the theory of $\mathcal{P}(\omega_1)$.** Let us recall some of the known facts about the structure theory of $\mathcal{P}(\omega)$ under $\mathrm{AD}^{L(\mathbb{R})}$.

(a) $\mathrm{AD}^{L(\mathbb{R})}$ provides a structure theory of $L(\mathbb{R})$ which is a natural extension of the structure theory that can be established for sets of reals of lower complexity in ZFC.
(b) $\mathrm{AD}^{L(\mathbb{R})}$ follows from the structure theory of $L(\mathbb{R})$ that it yields.
(c) $\mathrm{AD}^{L(\mathbb{R})}$ follows from Large Cardinal Axioms.

---

[12]i.e., proper and compete relative to some simple σ-complete completeness system.

[13]In other words, if $\vec{r}$ enumerates all the reals, then $\xi(\vec{r})$ is a successor ordinal $< \omega^2$ (see (4)) and the last tree in the sequence, as a forcing notion, is completely proper although it has no uncountable branches. In particular, this shows that if CH holds then we can't have $\mathrm{BC}_{\omega_1}(\mathcal{X})$ for $\mathcal{X}$ the class of all completely proper forcing notions.

(d) The Large Cardinal Axioms give an Ω-complete theory of $L(\mathbb{R})$.

(e) $\mathrm{AD}^{L(\mathbb{R})}$ is equivalent to the existence of inner models of certain Large Cardinal Axioms.

(f) $\mathrm{AD}^{L(\mathbb{R})}$ is implied by any other statement of sufficiently strong consistency strength (measured on the scale of the Large Cardinal Axioms).

Thus, there is a very close relationship between $\mathrm{AD}^{L(\mathbb{R})}$, the structure theory of $L(\mathbb{R})$ it yields and the standard Large Cardinal Axioms. As it has been observed quite early, the standard Large Cardinal Axioms are quite insensitive to the Continuum Problem. So the structure theory of $\mathcal{P}(\omega_1)$, since it must give an answer to the Continuum Problem, cannot be so closely tied with standard Large Cardinal Axiom. In particular, we can't have (c), (e) and (f) in this context. In Section 2, we have seen that both OGA and PID have substantial 'ZFC-shadows', so to some extent we do have the analogue of (a). For the analogue of (b) we can take either Woodin's (*) (see [57]) or Fuchino's Potential Embedding Principle with perhaps some adjustments (see [15]). Some work is needed to achieve something comparable to (d) but the work of Viale [54] is a start. Finally we mention that, since PID (which is compatible both with CH and its negation) implies $\mathrm{AD}^{L(\mathbb{R})}$, we have the compatibility between the two structure theories.

4.2. **The set-theoretic universe.** At this stage it is difficult to predict the picture of the set theoretic universe that would accommodate the right structure theory of $\mathcal{P}(\omega_1)$ and so, in particular, solve the Continuum Problem, but we can still speculate. For example, recent work of Woodin [58] on the ultimate version of Gödel's constructible universe L gives us a hint towards the ultimate culmination of the Inner Model Program whose goal is to give us a fine analysis of the standard Large Cardinal Axioms. To give a comparable analysis of the right structure theory of $\mathcal{P}(\omega_1)$, and therefore give still a more precise picture of the universe of sets, one would need to invest in yet another program that would analyze possible maximal (forcing?) extensions of this Ultimate-L. In correspondence Peter Koellner mentions the possibility that the universe of sets is equal to the Ultimate-L and that the right structure theory of $\mathcal{P}(\omega_1)$ holds in an inner model, having in mind the historical retreat from AD to $\mathrm{AD}^{L(\mathbb{R})}$ in order to accommodate AC whose role in the case of $\mathcal{P}(\omega_1)$ would be played by the CH. However, the retreat from AD to $\mathrm{AD}^{L(\mathbb{R})}$ was 'forced on us' by the enormous success of AC in the rest of mathematics where it gave us key representation and duality theorems making this axiom generally acceptable (true). We do

not have anything resembling this in case of CH so the retreat seems
doubtful at this stage of our knowledge. While at this point, note that
important parts of the theory exposed in Section 2 are not really about
just structures from $L(\mathcal{P}(\omega_1))$ but about structures that could not be
captured by inner models of the form $L(X)$ for X a set. So it is quite
unclear that the theory could be captured by a homogeneous forcing
extension of an inner model of determinacy. It is true that only part of
the theory seen at the level of $L(\mathcal{P}(\omega_1))$ is important for the solution of
the Continuum Problem. But cutting off a large part of the theory for
such a reason in not acceptable if we are serious about discovering the
true set-theoretic universe. Nevertheless, we think that the analogies
between the theory of $L(\mathbb{R})$ under $\mathrm{AD}^{L(\mathbb{R})}$ and the theory of $L(\mathcal{P}(\omega_1))$
under $\mathfrak{mm} > \omega_1$ should be further analyzed before we could make a
proper philosophical analysis. The principles like PD and $\mathrm{AD}^{L(\mathbb{R})}$ are
really helping us with countable objects, or more precisely, objects that
could be coded by a countable amount of information. On the other
hand, the structure theory of $L(\mathcal{P}(\omega_1))$ is about objects of cardinality
at most $\aleph_1$ and this is why the Baire category principles at the level
ω_1 are useful. So indeed there is some analogy here. Looking more
closely, one observes that $\mathrm{AD}^{L(\mathbb{R})}$ gives us the fine structure theory for
objects living in the inner model $L(\mathbb{R})$ (i.e., the pleasant Wadge hi-
erarchy, the structure theory of cardinals, etc) avoiding us having to
say something about structures that depend on a well-ordering of the
reals. The inner model $L(\mathcal{P}(\omega_1))$, on the other hand, has a definable
well-ordering and OGA, PID, and eventually $\mathfrak{mm} > \omega_1$, have to treat
the structures that depend in it. Note that the short well-ordering of
$\mathcal{P}(\omega_1)$ of order type ω_2 joined with the assumption $\mathfrak{mm} > \omega_1$ is giv-
ing us the non-structure situation at the level of *subsets* of $\mathcal{P}(\omega_1)$,[14] so
we too have to retreat to simpler objects like 'open graphs, 'P-ideals',
etc. So, if there is a stronger form of analogy here it should perhaps
be between the power-set of $\mathbb{R}$ and the power-set of $\mathcal{P}(\omega_1)$. The max-
imality considerations in the large cardinal hierarchy has given us the
structure theory of $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ and the maximality consideration in
the hierarchy of the Baire category principles at the level ω_1 has given
us the structure theory of $\mathcal{P}_{\mathrm{simple}}(\mathcal{P}(\omega_1))$ partly exposed above in Sec-
tion 2. Note that if we want to move from $\mathcal{P}_{\mathrm{simple}}(\mathcal{P}(\omega_1))$ to $\mathcal{P}(\mathcal{P}(\omega_1))$
we would need to perform yet another maximality consideration but
now among the Baire category principles at the level of ω_2, those that
are analogous to the Baire category principles at level ω_1 compatible
with CH. When discovering more about the set-theoretic universe we

---

[14]i.e., the situation quite analogous to that discussed above in Section 3.

will be faced with the mathematical as well as philosophical problem of determining the proper order and the precise relationships between these different maximality considerations.

4.3. **Other axioms.** One of the earliest alternatives to CH found in the literature is Luzin's hypothesis $2^{\aleph_0} = 2^{\aleph_1}$ found in his paper [22]. Looking closely one sees that Luzin was really after the following principle:

L: All sets of reals of cardinality $\aleph_1$ are co-analytic.

Today we know that this statement could be adjusted as follows:

B: All sets of reals of cardinality $\aleph_1$ are pairwise equivalent as linear orderings.

When one postulates some properties of large cardinals to hold at the level of small cardinals like ω_1 or ω_2 one sometimes gets statements sensitive to the Continuum Problem. Here is a typical example of such a result (see [35]).

4.1 Theorem (Shelah). *If the quotient algebra $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ has a dense subset of cardinality $\aleph_1$ then Luzin's Continuum Hypothesis $2^{\aleph_0} = 2^{\aleph_1}$ is true.*

Unfortunately, this assumption contradicts even a weak form of the Baire category principle $\mathfrak{m} > \omega_1$ but the following result gives a natural recovery (see [57]).

4.2 Theorem (Woodin). *If there is a measurable cardinal and if the quotient algebra $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ satisfies the $\aleph_2$-chain condition then $2^{\aleph_0} > \aleph_1$.*

Recall that Kőnig's Lemma has also been generalized to a notion of compactness which gives us the notions of weakly compact and strongly compact cardinals. However, there is another way to generalize the notion of compactness, which was discovered by Richard Rado, who was motivated by his early results about intersection graphs on families of intervals of linearly ordered sets. In particular Rado [33] states the following conjecture.

RC: Suppose that an intersection graph $\mathcal{G}$ of a family of intervals of some linearly ordered set is not countably chromatic. Then $\mathcal{G}$ has a subgraph of cardinality $\aleph_1$ which is also not countably chromatic.

It is still not clear what is the largest class of graphs for which such a compactness principle can hold at this level. For example, it is not

known if the conjecture can be extended to the class of all incomparability graphs on posets. It definitely cannot be extended to the class of all graphs since for example there is a graph on the vertex set of cardinality $\mathfrak{c}^+$ which is not countably chromatic but all subgraphs of smaller cardinality are countable chromatic.

The influence of RC on the cardinality of the continuum is given by the following result (see [42]).

4.3 Theorem (Todorcevic). RC *implies* $2^{\aleph_0} \le \aleph_2$.

Unfortunately, RC contradicts the consequence of $\mathfrak{m} > \omega_1$ saying that all trees of cardinality at most $\aleph_1$ with no uncountable branches can be decomposed into countably many antichains, and therefore, it does not allow much of the structure theory of $\mathcal{P}(\omega_1)$ from Section 2. However, in this case, if necessary, the structure theory could be recovered using the following result.

4.4 Theorem (Todorcevic). *If for every stationary subset S of $[\omega_2]^{\aleph_0}$ there is $\alpha < \omega_2$ such that $S \cap [\alpha]^{\aleph_0}$ is stationary in $[\alpha]^{\aleph_0}$ then $2^{\aleph_0} \le \aleph_2$.*

The point here is that if $\mathrm{WRP}(\omega_2)$ denotes the hypothesis of this theorem then both RC and $\mathfrak{mm} > \omega_1$ imply it.

Finally, we could also speculate about some truly new large cardinal axioms which on the one hand talk only about sets of very high rank and which, on the other hand, have an effect on the Continuum Problem. Their eventual discovery could revolutionize the whole of set theory, not just the study of Cantor's Continuum Problem. In the paper [2] of Joan Bagaria the reader can find a discussion of some other natural axioms of set theory that we do not cover here.

4.4. **Structure theories for higher power-sets.** Currently there is an ongoing research program on the Baire category principle at levels higher than ω_1 and the theory of its consequences have yet to be developed. For example, it remains to be seen whether they have influence on the Continuum Problem at the corresponding level. I think that it would be important to know if the new Baire category principles at a given level κ can give us well-orderings of $\mathcal{P}(\kappa)$ that are described by formulas that use only parameters from $\mathcal{P}(\kappa)$ to match the case of Baire category principle $\mathfrak{mm} > \omega_1$ at level ω_1. Note also that if we are to discover the true universe of sets that incorporates structure theories of various power-sets then we might need to constantly move from a theory of the power-set of κ^+ that forces $2^\kappa > \kappa^+$ to a theory of the power-set of κ^{++} that allows $2^{\kappa^+} = \kappa^{++}$. For example, if we are to have the fine structure theory (of Section 2) at the level of ω_1, because of the well-ordering of $\mathcal{P}(\omega_1)$ of order-type ω_2, at the

level $\mathcal{P}(\omega_2)$ we are in a situation which at first sight looks analogous to the situation of $\mathcal{P}(\omega_1)$ with CH holding. This is, however, not quite the case since in this situation we lack the existence of $\square_{\omega_1}$-sequences that helps us in constructing objects of cardinality $\aleph_2$ (analogously to the way we would use $\square_\omega$-sequences to describe objects of cardinality $\aleph_1$) and so a well-ordering of $\mathcal{P}(\omega_1)$ of order-type ω_2 that gives us $\Diamond(\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\})$ could not be used for constructing pathologies like, for example, the existence of an ω_2-Souslin tree or a witness that Kőnig's Lemma at this level is false.

4.5. **Conclusion.** We have seen above that the two Baire category theories of the power-set of ω_1 behave quite differently when we perform some maximality and permanence tests. These tests point out that from the point of view of our current knowledge the theory of $\mathcal{P}(\omega_1)$ that implies the negation of the Continuum Hypothesis has a clear advantage confirming thus the intuitions of both Luzin [22] and Gödel [16]. However, one may still ask the following questions.

4.5 Question. *What other tests we should take in order to determine the true structure theory of $\mathcal{P}(\omega_1)$?*

4.6 Question. *What is the true structure theory of $\mathcal{P}(\omega_1)$? Is CH or its negation a part of this theory?*

4.7 Question. *In order to have the true structure theory of $\mathcal{P}(\omega_1)$ do we really need to retreat to an inner model of the universe of sets?*

We believe that the tests that will prove crucial are those coming from the rest of mathematics. The combined experience coming from different areas of mathematics might eventually give us a hint as to which of the two theories of $\mathcal{P}(\omega_1)$ is more useful and should be kept: a CH-theory with an immense quantity of unrelated mathematical structures, or a fine structure theory of $\mathcal{P}(\omega_1)$ that contradicts CH and that resembles the structure theory of $\mathcal{P}(\omega)$ under $\mathrm{AD}^{L(\mathbb{R})}$. This need for further tests coming from the rest of mathematics seems in agreement with the following insight from Luzin [22] that is also implicit in the often cited paragraph from Gödel [16]:

"Alors, la nécessité s'imposera à nous de *choisir* entre les diverses hypothèses du continu, toutes exemptes de contradiction, et ce choix devra être dicté par l'observation seule *des faits*."

References

[1] David Asperó, Paul Larson, and Justin Tatch Moore. Forcing axioms and the continuum hypothesis. *Acta Math.*, 210(1):1–29, 2013.

[2] Joan Bagaria. Natural axioms of set theory and the continuum problem. *http://www.crm.cat/Paginas/Publications/04/Pr591.pdf.*

[3] Bohuslav Balcar and Thomas Jech. Weak distributivity, a problem of von Neumann and the mystery of measurability. *Bull. Symbolic Logic*, 12(2):241–266, 2006.

[4] James E. Baumgartner. All $\aleph_1$-dense sets of reals can be isomorphic. *Fund. Math.*, 79(2):101–106, 1973.

[5] J. Brendle, P. Larson, and S. Todorcevic. Rectangular axioms, perfect set properties and decomposition. *Bull. Cl. Sci. Math. Nat. Sci. Math.*, (33):91–130, 2008.

[6] H. G. Dales. A discontinuous homomorphism from $C(X)$. *Amer. J. Math.*, 101(3):647–734, 1979.

[7] H. G. Dales and W. H. Woodin. *An introduction to independence for analysts*, volume 115 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1987.

[8] Keith J. Devlin and Saharon Shelah. A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$. *Israel J. Math.*, 29(2-3):239–247, 1978.

[9] J. Esterle. Solution d'un problème d'Erdős, Gillman et Henriksen et application à l'étude des homomorphismes de $\mathcal{C}(K)$. *Acta Math. Acad. Sci. Hungar.*, 30(1-2):113–127, 1977.

[10] Ilijas Farah. All automorphisms of the Calkin algebra are inner. *Ann. of Math. (2)*, 173(2):619–661, 2011.

[11] Qi Feng. Homogeneity for open partitions of pairs of reals. *Trans. Amer. Math. Soc.*, 339(2):659–684, 1993.

[12] M. Foreman, M. Magidor, and S. Shelah. Martin's maximum, saturated ideals, and nonregular ultrafilters. I. *Ann. of Math. (2)*, 127(1):1–47, 1988.

[13] D. H. Fremlin. *Consequences of Martin's axiom*, volume 84 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1984.

[14] D. H. Fremlin. Families of compact sets and Tukey's ordering. *Atti Sem. Mat. Fis. Univ. Modena*, 39(1):29–50, 1991.

[15] Sakaé Fuchino. On potential embedding and versions of Martin's axiom. *Notre Dame J. Formal Logic*, 33(4):481–492, 1992.

[16] Kurt Gödel. What is Cantor's continuum problem? *Amer. Math. Monthly*, 54:515–525, 1947.

[17] Kurt Gödel. *Collected works. Vol. III*. The Clarendon Press Oxford University Press, New York, 1995. Unpublished essays and lectures, With a preface by Solomon Feferman, Edited by Feferman, John W. Dawson, Jr., Warren Goldfarb, Charles Parsons and Robert M. Solovay.

[18] Felix Hausdorff. Die graduierung nach dem endverlauf. *Abhandl. der Königlich Säachsishen. Gesell. Wiss. Math.-Phys. Kl.*, 31:296–334, 1909.

[19] Peter Koellner. The continuum hypothesis. *Stanford Encyclopedia of Philosophy*, September 12, 2011.

[20] Georges Kurepa. Endembles ordonnés et ramifiés. *Publications Mathématiques de l'Université de Belgrade*, 4:1–138, 1935.

[21] Richard Laver. On Fraïssé's order type conjecture. *Ann. of Math. (2)*, 93:89–111, 1971.

[22] Nicolas Luzin. Sur les ensembles analytiques nuls. *Fund. Math.*, 25:109–131, 1935.

[23] Dorothy Maharam. An algebraic characterization of measure algebras. *Ann. of Math. (2)*, 48:154–167, 1947.

[24] D. A. Martin and R. M. Solovay. Internal Cohen extensions. *Ann. Math. Logic*, 2(2):143–178, 1970.

[25] Carlos Martinez-Ranero. Well-quasi-ordering Aronszajn lines. *Fund. Math.*, 213(3):197–211, 2011.

[26] Carlos Martinez-Ranero and Stevo Todorcevic. Gap structure of coherent Aronszajn trees. *Math. Res. Lett.*, 18(3):565–578, 2011.

[27] Justin Tatch Moore. Open colorings, the continuum and the second uncountable cardinal. *Proc. Amer. Math. Soc.*, 130(9):2753–2759 (electronic), 2002.

[28] Justin Tatch Moore. Set mapping reflection. *J. Math. Log.*, 5(1):87–97, 2005.

[29] Justin Tatch Moore. A five element basis for the uncountable linear orders. *Ann. of Math. (2)*, 163(2):669–688, 2006.

[30] Justin Tatch Moore. Forcing axioms and the continuum hypothesis. Part II: transcending ω_1-sequences of real numbers. *Acta Math.*, 210(1):173–183, 2013.

[31] S. Negrepontis. Banach spaces and topology. In *Handbook of set-theoretic topology*, pages 1045–1142. North-Holland, Amsterdam, 1984.

[32] N. Christopher Phillips and Nik Weaver. The Calkin algebra has outer automorphisms. *Duke Math. J.*, 139(1):185–202, 2007.

[33] Richard Rado. Theorems on intervals of ordered sets. *Discrete Math.*, 35:199–201, 1981.

[34] Saharon Shelah. Decomposing uncountable squares to countably many chains. *J. Combinatorial Theory Ser. A*, 21(1):110–114, 1976.

[35] Saharon Shelah. *Around classification theory of models*, volume 1182 of *Springer Lecture Notes in Mathematics*. Springer-Verlag, Heidelberg, 1986.

[36] Saharon Shelah. *Proper and improper forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

[37] W. Sierpinński. *Hypothese du Continu*, volume IV of *Monografje Matematiczne*. PWN-Polish Scientific Publlishers, Warszawa, 1934.

[38] John R. Steel. PFA implies $\mathrm{AD}^{L(\mathbb{R})}$. *J. Symbolic Logic*, 70(4):1255–1296, 2005.

[39] Stevo Todorcevic. Directed sets and cofinal types. *Trans. Amer. Math. Soc.*, 290(2):711–723, 1985.

[40] Stevo Todorcevic. Oscillations of real numbers. In *Logic colloquium '86 (Hull, 1986)*, volume 124 of *Stud. Logic Found. Math.*, pages 325–331. North-Holland, Amsterdam, 1988.

[41] Stevo Todorcevic. *Partition problems in topology*, volume 84 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989.

[42] Stevo Todorcevic. Conjectures of Rado and Chang and cardinal arithmetic. In *Finite and infinite combinatorics in sets and logic (Banff, AB, 1991)*, volume 411 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 385–398. Kluwer Acad. Publ., Dordrecht, 1993.

[43] Stevo Todorcevic. A classification of transitive relations on ω_1. *Proc. London Math. Soc. (3)*, 73(3):501–533, 1996.

[44] Stevo Todorcevic. Comparing the continuum with the first two uncountable cardinals. In *Logic and scientific methods (Florence, 1995)*, volume 259 of *Synthese Lib.*, pages 145–155. Kluwer Acad. Publ., Dordrecht, 1997.

[45] Stevo Todorcevic. Chain-condition methods in topology. *Topology Appl.*, 101(1):45–82, 2000.

[46] Stevo Todorcevic. A dichotomy for P-ideals of countable sets. *Fund. Math.*, 166(3):251–267, 2000.

[47] Stevo Todorcevic. Generic absoluteness and the continuum. *Math. Res. Lett.*, 9(4):465–471, 2002.

[48] Stevo Todorcevic. Localized reflection and fragments of PFA. In *Set theory (Piscataway, NJ, 1999)*, volume 58 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 135–148. Amer. Math. Soc., Providence, RI, 2002.

[49] Stevo Todorcevic. *Walks on ordinals and their characteristics*, volume 263 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

[50] Stevo Todorcevic. Walks on countable ordinals and selective ultrafilters. *Bull. Cl. Sci. Math. Nat. Sci. Math.*, (35):65–78, 2010.

[51] Stevo Todorcevic. Combinatorial dichotomies in set theory. *Bull. Symbolic Logic*, 17(1):1–72, 2011.

[52] Stevo Todorcevic. *Notes on Forcing Axioms*, volume 26 of *Lecture Notes Series of the Institute for Mathematical Sciences, National University of Singapore*. World Scientific Publlishers, Singapore, 2014.

[53] John W. Tukey. *Convergence and Uniformity in Topology*. Annals of Mathematics Studies, no. 2. Princeton University Press, Princeton, N. J., 1940.

[54] Matteo Viale. Category forcing, mmm, and generic absoluteness for the theory of strong forcing axioms. *J. Amer. Math. Soc., to appear.*

[55] Matteo Viale. Forcing axioms, supercompact cardinals, singular cardinal combinatorics. *Bull. Symbolic Logic*, 14(1):99–113, 2008.

[56] W. Hugh Woodin. The cardinals below $|[\omega_1]^{<\omega_1}|$. *Ann. Pure Appl. Logic*, 140(1-3):161–232, 2006.

[57] W. Hugh Woodin. *The axiom of determinacy, forcing axioms, and the non-stationary ideal*, volume 1 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter GmbH & Co. KG, Berlin, revised edition, 2010.

[58] W. Hugh Woodin. Suitable extender models II: beyond ω-huge. *J. Math. Log.*, 11(2):115–436, 2011.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO,
STEVO@MATH.TORONTO.EDU

INSTITUT DE MATHÉMATIQUES DE JUSSIEU, CNRS, PARIS, FRANCE
TODORCEVIC@MATH.JUSSIEU.FR