Contemporary Mathematics
Volume 31, 1984

A NOTE ON THE PROPER FORCING AXIOM

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In this note we give several remarks on the following most familiar form of the proper forcing axiom (PFA):

> *If* P *is a proper poset and if* F *is a family of* $\aleph_1$ *dense subsets of* P, *then* P *has an* F*-generic filter.*

Thus PFA is obtained from $MA_{\aleph_1}$ by replacing the countable chain condition with properness, which is a much weaker restriction. It has been shown that PFA is a much stronger forcing axiom than $MA_{\aleph_1}$, and that PFA decides many problems left open by $MA_{\aleph_1}$. The consistency proof of PFA used a supercompact cardinal ([3],[6]), but nothing stronger than the consistency of a weakly compact cardinal was known to follow from PFA ([2]).

Our first result shows that under PFA, $\aleph_2$ has a certain degree of supercompactness, so that in some sense the supercompact cardinal in the consistency proof of PFA is needed. In particular, our result connects this problem with some recent results about inner models of set theory ([4]). The second part of this note discusses an approach in applying PFA. We assume the reader is familiar with some basic facts about proper forcing, which can be found in any of the sources [2], [3] and [6].

THEOREM 1. *Assume PFA. Let* $\kappa > \omega_1$ *be a regular cardinal and let* $\Gamma \subseteq \kappa$ *be a set of limit ordinals such that* $\{\delta \in \kappa\colon \mathrm{cf}\ \delta = \omega_1\} \subseteq \Gamma$. *Let* $\langle C_\alpha\colon \alpha \in \Gamma\rangle$ *be a sequence of subsets of* κ *such that*

i) C_α *is a closed and unbounded subset of* α,

ii) if β *is a limit point of* C_α, *then* $\beta \in \Gamma$ *and* $C_\beta = C_\alpha \cap \beta$.

Then there is a club C *in* κ *such that if* α *is a limit point of* C, *then* $\alpha \in \Gamma$ *and* $C_\alpha = C \cap \alpha$.

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COROLLARY 2. *If PFA holds then $\square_\kappa$ fails for any uncountable cardinal* κ.

It is interesting to note that R.Solovay ([7]) proved a similar result where ω_2 is replaced by a supercompact cardinal λ and $\{\delta<\kappa\colon \mathrm{cf}\,\delta=\omega_1\}$ by $\{\delta<\kappa\colon \omega<\mathrm{cf}\,\delta<\lambda\}$.

PROOF OF THEOREM 1. For α and β in Γ we let $\beta\prec\alpha$ iff β is a limit point of C_α. It is clear that $\prec$ is a tree ordering on Γ and that if A is a chain of $\langle\Gamma,\prec\rangle$ of size κ then

$$C = \bigcup_{\alpha\in A} C_\alpha$$

satisfies the conclusion of the theorem. So, to finish the proof it suffices to find a contradiction assuming that $\langle\Gamma,\prec\rangle$ has no chains of size κ.

In order to define a poset for an application of PFA we need to introduce some definitions which are very useful in many other constructions.

A sequence $\vec{N}=\langle N_\alpha\colon \alpha\in A\rangle$, where A is a subset of ω_1, is called an *elementary* A-*chain* iff

1) $\forall\,\alpha\in A\quad(\langle C_\gamma\colon\gamma\in\Gamma\rangle\in N_\alpha\prec H_{\kappa^+}$ and $|N_\alpha|=\aleph_0)$,
2) $\forall\,\alpha,\beta\in A\ (\alpha<\beta\Rightarrow N_\alpha\in N_\beta)$,
3) $\forall\,\alpha\in A\quad(\alpha=\sup A\cap\alpha\Rightarrow N_\alpha=\bigcup_{\beta\in A\cap\alpha}N_\beta)$.

For countable $N\prec H_{\kappa^+}$, we let $\kappa_N=\sup(N\cap\kappa)$. If $F\subseteq\Gamma$ and if $f\colon F\to\omega$, then f is called a *specializing map* iff $f(\gamma)\neq f(\delta)$ for $\gamma,\delta\in F$ with $\gamma\prec\delta$.

Now we are ready to define our poset P. Let $p\in P$ iff $p=\langle f_p,\vec{N}_p\rangle$, where:

a) $\vec{N}_p$ is an elementary A_p-chain for some finite $A_p\subseteq\omega_1$ such that $\vec{N}_p\subseteq\vec{N}$ for some elementary ω_1-chain $\vec{N}$.
b) f_p is a partial specializing map from $\langle\Gamma,\prec\rangle$ into ω such that $\mathrm{dom}(f_p)\subseteq\{\kappa_{N^p_\alpha}\colon\alpha\in A_p\}$.

P is partially ordered by: $q\leq p$ iff $f_q\supseteq f_p$ and $\vec{N}_q\supseteq\vec{N}_p$.

CLAIM. P is a proper poset.

PROOF. Let θ be a big enough regular cardinal and let $M\prec H_\theta$ be countable such that $p,P\in M$. Let $\delta=M\cap\omega_1$ and let

$$q = \langle f_p,\ \vec{N}_p\cup\{\langle\delta,\ M\cap H_{\kappa^+}\rangle\}\rangle$$

Then it is easily seen that q is a member of P and that $q\leq p$.

We shall prove that q is an $(M,\mathcal{P})$-generic condition.

Let $\mathcal{D} \in M$ be a dense open subset of $\mathcal{P}$ and let $r \leq q$ be a given condition. By extending r we assume $r \in \mathcal{D}$. Let $p_0 = r \cap M$. Then it is easily seen that $p_0 \in M$ and $p_0 \leq p$. Let $F = \mathrm{dom}(f_r) \setminus M$ and let $n = |F|$. We may assume that $n \geq 1$. Since M is an elementary submodel of H_θ we can find sequences $\langle F_\xi : \xi < \kappa \rangle$ and $\langle r_\xi : \xi < \kappa \rangle$ in M such that

c) $F_\xi \subseteq \kappa$ and $|F_\xi| = n$,

d) $\xi < F_\xi < \eta$ for $\xi < \eta < \kappa$,

e) $r_\xi \in \mathcal{D}$, $r_\xi \leq p_0$ and $\mathrm{dom}(f_{r_\xi}) = \mathrm{dom}(f_{p_0}) \cup F_\xi$.

Since $\langle \Gamma, \prec \rangle$ has no κ-chains, a standard argument shows that there exists a $\xi \in M \cap \kappa$ such that

$$\forall x \in F, \quad \forall y \in F_\xi \qquad (x \text{ and } y \text{ are } \prec\text{-incomparable}) .$$

Let $s = \langle f_r \cup f_{r_\xi}, \vec{N}_r \cup \vec{N}_{r_\xi} \rangle$. Then it is easily checked that s is a member of $\mathcal{P}$ and that $s \leq r$ and $s \leq r_\xi$. Since $r_\xi \in \mathcal{D} \cap M$ this completes the proof of the Claim.

Let

$$\bar{\mathcal{P}} = \left\{ p \in \mathcal{P} : \mathrm{dom}(f_p) = \{\kappa_{N^p_\alpha} : \alpha \in A_p\} \cap \Gamma \right\} .$$

Then $\bar{\mathcal{P}}$ is a dense subset of $\mathcal{P}$ and we shall apply PFA on $\bar{\mathcal{P}}$ rather than $\mathcal{P}$. So let G be a filter on $\bar{\mathcal{P}}$ with the property that

$$\vec{N}_G = \bigcup_{p \in G} \vec{N}_p$$

is an elementary ω_1-chain. Such a filter exists since it is possible to define a family $\mathcal{F}$ of $\aleph_1$ dense open subsets of $\bar{\mathcal{P}}$ with the property that $\vec{N}_G$ is an elementary ω_1-chain for any $\mathcal{F}$-generic filter G on $\bar{\mathcal{P}}$. For $\alpha < \omega_1$ we let

$$\kappa_\alpha = \sup(N^G_\alpha \cap \kappa) .$$

Let $\gamma = \sup\{\kappa_\alpha : \alpha < \omega_1\}$. Then $\gamma \in \Gamma$ since $\mathrm{cf}\,\gamma = \omega_1$. Let

$$f = \bigcup_{p \in G} f_p \qquad \text{and} \qquad \Delta = \Gamma \cap \{\kappa_\alpha : \alpha < \omega_1\} .$$

Then by the definition of $\mathcal{P}$, $f : \Delta \to \omega$ is a specializing map. Since C_γ and $\{\kappa_\alpha : \alpha < \omega_1\}$ are clubs in γ we can find a club $D \subseteq \omega_1$ such that κ_α is a limit point of C_γ for all $\alpha \in D$. By the property (ii) of $\langle C_\alpha : \alpha \in \Gamma \rangle$ it follows that $\kappa_\alpha \prec \kappa_\beta$ for $\alpha < \beta$ in D, i.e., that $\{\kappa_\alpha : \alpha \in D\}$ is an ω_1-chain of $\langle \Delta, \prec \rangle$. But this is a contradiction since $\langle \Delta, \prec \rangle$ is a special tree. This completes the proof of Theorem 1.

The construction from the previous proof is only one instance of a quite general approach in constructing proper partial orderings which we now intend to discuss in more detail. In general terms this approach can be described as follows. Suppose we want to force an uncountable subset A of a given structure E with A having some specific properties. The natural thing would be to force with the poset $\mathcal{P}$ of all finite approximations to A. We would like to prove that $\mathcal{P}$, or a certain subposet of it, is proper. So let $N \prec H_\theta$ be countable such that $p, \mathcal{P} \in N$. Let $q \leq p$ be a condition for which we would like to prove that it is $(N,\mathcal{P})$-generic. So let $\mathcal{D} \in N$ be dense open and let $r \leq q$. In most of the cases we shall not be able to show that there is an $s \in \mathcal{D} \cap N$ such that $r \sim s$, since $r \setminus N$ will be in certain "bad" places with respect to N. To avoid this we shall simply "add" N to be a "side condition", saying explicitly that $r \setminus N$ is not in a bad place with respect to N. In order to clarify this, we now present a typical case of such a construction which has its own independent interest.

Let E be a partially ordered set and let $D \subseteq E$. We say that D is a *directed subset* of E iff $\forall a,b \in D$, $\exists c \in D$, $a,b \leq c$. Recently E.Milner and K.Prikry ([5]) proved that every poset with no uncountable antichains is the union of $\leq 2^{\aleph_0}$ directed subsets, thus answering a question of F.Galvin concerning the well-known Dilworth decomposition theorem. Actually, Milner and Prikry proved a much more general result but this special case can be proved more directly using $(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}$. Namely, one first shows that if E is moreover well-founded, then every family of pairwise $\subseteq$-incomparable initial parts of E have size $\leq 2^{\aleph_0}$. Now the result follows easily.

Let σ_0 be the minimal cardinal with the property that every poset with no uncountable antichains is the union of $\leq \sigma_0$ directed subsets. It is an open problem whether $\sigma_0 \leq \aleph_1$ can be proved without additional set-theoretic assumptions. Our next result, asked by A.Hajnal and E.Milner, shows that σ_0 can have the minimal possible value $\aleph_0$. Its proof will be a very good illustration of our approach in constructing proper posets.

THEOREM 3. *Assume PFA. Then every partially ordered set with no uncountable antichains is the union of countably many directed subsets.*

PROOF. Let E be a poset with no uncountable antichains and let $\kappa = |E|$. We shall prove the result by induction on the cardinal κ. Clearly, we may assume $\mathrm{cf}\,\kappa > \omega$. Assume by way of contradiction that E is not the union of $\leq \aleph_0$ directed sets. We shall find a proper poset $\mathcal{P}$ which forces an uncountable antichain to E which will be a contradiction since PFA holds.

By going to a cofinal subset of E, we may assume that:

(1) $E = \langle \kappa, \leq_E \rangle$,

(2) $a <_E b \Rightarrow a < b$,

(3) $\forall a \in E, \; |\{b \in E: a \leq_E b\}| = \kappa$.

We say that a subset B of E is *finitely bounded* in E iff B is either a directed subset of E, or else there exist a positive integer n, a stationary subset S of $[\kappa]^{\aleph_0}$, and a sequence $\langle F_s: s \in S \rangle$ of subsets of κ of size n such that

(4) $\forall s \in S, \quad \forall a \in B \cap s, \quad \exists f \in F_s, \; a <_E f$.

CLAIM 1. E is not the union of countably many finitely bounded subsets.

PROOF. Otherwise, let $E = \bigcup_{i<\omega} B_i$ where each B_i is a finitely bounded subset of E. Clearly we may assume that B_i's are disjoint and that for each i there exists a stationary set $S_i \subseteq [\kappa]^{\aleph_0}$ and a sequence $\langle F^i_s: s \in S_i \rangle$ of subsets of κ of size n_i which together with B_i satisfy the condition (4). We may also assume that

$$\forall i < \omega, \quad \forall j \in \{1,\ldots,n_i\}, \quad \exists k(i,j) < \omega, \quad \{f^i_s(j): s \in S_i\} \subseteq B_{k(i,j)},$$

where $\{f^i_s(j): 1 \leq j \leq n_i\}$ is the increasing enumeration of F^i_s. For each $i < \omega$, we fix an ultrafilter $\mathcal{U}_i$ on S_i which extends the club filter on S_i. Then for each $a \in B_i$ there exists $j_a \in \{1,\ldots,n_i\}$ such that

$$\{s \in S_i: \; a <_E f^i_s(j_a)\} \in \mathcal{U}_i .$$

For $i < \omega$ and $j \in \{1,\ldots,n_i\}$ we define

$$B^j_i = \{a \in B_i: j_a = j\} .$$

Then $B_i = \bigcup_{j=1}^{n_i} B^j_i$. Since $\mathcal{U}_i$ is an ultrafilter on S_i, for each $j \in \{1,\ldots,n_i\}$ we can find $\ell(i,j) \in \{1,\ldots,n_{k(i,j)}\}$ such that

$$\left\{s \in S_i: \; f^i_s(j) \in B^{\ell(i,j)}_{k(i,j)}\right\} \in \mathcal{U}_i .$$

Finally, for $i < \omega$ and $j \in \{1,\ldots,n_i\}$ we define

$$D^j_i = B^j_i \cup B^{\ell(i,j)}_{k(i,j)} \cup B^{\ell(k(i,j),\ell(i,j))}_{k(k(i,j),\ell(i,j))} \cup \ldots .$$

It is not hard to show that each D^j_i is a directed subset of E. Since $E = \bigcup\{D^j_i: i < \omega, \; j \in \{1,\ldots,n_i\}\}$, we get a contradiction which finishes the proof of Claim 1.

Throughout the rest of the proof, we assume that any submodel of H_{κ^+} in our consideration contains E as an element. Let $A \subseteq \kappa$ and let $\mathcal{N}$ be an $\in$-chain of countable elementary submodels of H_{κ^+}. Then we say that $\mathcal{N}$ *separates* A iff

i) $\forall \alpha,\beta \in A\ (\alpha < \beta \Rightarrow \exists N \in \mathcal{N}(\alpha \in N \text{ and } \beta \notin N))$,

ii) $\forall \alpha \in A\quad \forall N \in \mathcal{N}(\alpha \notin N \Rightarrow \forall B \in N\ (B \text{ finitely bounded in } E \Rightarrow \alpha \notin B))$.

Now we are ready to define our poset $\mathcal{P}$. Let $p \in \mathcal{P}$ iff $p = \langle A_p, \mathcal{N}_p \rangle$, where

a) $\mathcal{N}_p$ is a finite $\in$-chain of countable elementary submodels of H_{κ^+},
b) A_p is a finite antichain of E separated by $\mathcal{N}_p$,
c) $\{b \in E\colon A \cup \{b\}$ is an antichain in E$\}$ is not the union of $\leq\aleph_0$ finitely bounded sets.

$\mathcal{P}$ is partially ordered by: $q \leq p$ iff $A_q \supseteq A_p$ and $\mathcal{N}_q \supseteq \mathcal{N}_p$.

CLAIM 2. $\mathcal{P}$ is a proper poset.

PROOF. Let θ be a big enough regular cardinal and let $M \prec H_\theta$ be countable such that $p, \mathcal{P} \in M$. Let

$$q = \langle A_p,\ \mathcal{N}_p \cup \{M \cap H_{\kappa^+}\} \rangle$$

Clearly $q \in \mathcal{P}$ and $q \leq p$. We shall prove that q is an $(M,\mathcal{P})$-generic condition. So let $\mathcal{D} \in M$ be a dense open subset of $\mathcal{P}$ and let $r \leq q$ be arbitrary. We may assume that $r \in \mathcal{D}$. Let $p_0 = r \cap M$. Then $p_0 \in \mathcal{P} \cap M$ and $r \leq p_0 \leq p$. We may also assume that $A_r \setminus M \neq \emptyset$. Let $\{a_1, \ldots, a_n\}$ be the increasing enumeration of $A_r \setminus M$.

By induction on $i \leq n$ we define formulas $\Phi_{n-i}(b_1, \ldots, b_{n-i})$, where $b_1 < \ldots < b_{n-i} < \kappa$, as follows.

$\Phi_n(b_1, \ldots, b_n)$ iff $\exists s \leq p_0\ (s \in \mathcal{D}$ and $A_s \setminus A_p = \{b_1, \ldots, b_n\})$,

$\Phi_{n-1}(b_1, \ldots, b_{n-1})$ iff $\{b < \kappa\colon \Phi_{n-i}(b_1, \ldots, b_{n-i}, b)$ holds$\}$ is not finitely bounded in E ,

for $0 < i \leq n$. We would like to show that Φ_0 holds and a natural way to show this would be to prove that $\Phi_{n-i}(a_1, \ldots, a_{n-i})$ holds for each $i < n$. Exactly for this reason the properties (b) and (c) of our poset $\mathcal{P}$ have been introduced. The only difficulty is that models from $\mathcal{N}_r$ cannot talk about Φ_{n-i}'s. So we do the following construction.

Let $H \in M$ be an elementary substructure of $\langle H_{\kappa^+}, \varepsilon, \mathcal{P}, \mathcal{D} \rangle$ such that $\kappa \cup \{p_0\} \subseteq H$ and $|H| = \kappa$. Let $\mathcal{H} = \langle H, \varepsilon, \mathcal{P} \cap H, \mathcal{D} \cap H \rangle$. Then $\mathcal{H} \in M \cap H_{\kappa^+}$ and so $\mathcal{H} \in N$ for all $N \in \mathcal{N}_r$ with $M \cap H_{\kappa^+} \subseteq N$. For each $i \in \{1, \ldots, n\}$, let N_i

be the $\in$-maximal $N \in \mathcal{N}_r$ with the property $a_i \notin N$, and let $H_i = N_i \cap H$. Then $H_i \prec \mathcal{H}$ and so H_i can talk about formulas Φ_{n-j}'s. Since $\{a_1,\ldots,a_n\}$ is separated by $\{H_1,\ldots,H_n\}$ it follows easily that $\Phi_{n-i}(a_1,\ldots,a_{n-i})$ holds for each $i \leq n$. Thus in particular Φ_0 holds.

Let

$$B_1 = \{b < \kappa : \Phi_1(b) \text{ holds}\} .$$

Then $B_1 \in M$ and B_1 is not finitely bounded since Φ_0 holds. We claim that

$$\exists\, b \in B_1 \cap M, \quad \forall\, i \in \{1,\ldots,n\}, \quad b \not<_E a_i .$$

Otherwise let

$$S = \left\{ s \in [\kappa]^{\aleph_0} :\ \exists\, F_s \in [\kappa]^n \quad \forall\, b \in B_1 \cap s \quad \exists\, f \in F_s \quad b <_E f \right\} .$$

Then $S \in M$ and $M \cap \kappa \in S$ since we could put $F_{M\cap\kappa} = \{a_1,\ldots,a_n\}$. Thus S is a stationary subset of $[\kappa]^{\aleph_0}$. But this contradicts the fact that B_1 is not a finitely bounded set.

So, let $b_1 \in B_1 \cap M$ be such that $\forall\, i \in \{1,\ldots,n\}$, $b_1 \not<_E a_i$. Since $\Phi_1(b_1)$ holds, the set

$$B_2 = \{b < \kappa : \Phi_2(b_1, b) \text{ holds}\}$$

is not finitely bounded in E, and so working as above we can pick $b_2 \in B_2 \cap M$ such that $\forall\, i \in \{1,\ldots,n\}$, $b_2 \not<_E a_i$. Proceeding in this way we define $b_1 < \ldots < b_n$ in $M \cap \kappa$ such that $\Phi_n(b_1,\ldots,b_n)$ holds, and such that $\{a_1,\ldots,a_n\} \cup \{b_1,\ldots,b_n\}$ is an antichain of E. Pick an $s \in \mathcal{D} \cap M$ such that $s \leq p_0$ and $A_s = A_{p_0} \cup \{b_1,\ldots,b_n\}$. To show that r and s are compatible conditions it suffices to show that $\langle A_r \cup A_s, \mathcal{N}_r \cup \mathcal{N}_s \rangle$ satisfies the condition (c) from the definition of $\mathcal{P}$. Suppose not, and let $i \in \{1,\ldots,n\}$ be minimal with the property that $A = (A_s \cup A_r) \cap a_i$ satisfies (c) but $A_s \cup \{a_1,\ldots,a_i\}$ does not satisfy (c). For $X \subseteq E$ let $I(X)$ be the set of all elements of E which are incomparable with every member of X. Let

$$B = \left\{ b \in I(A) :\ I(A \cup \{b\}) \text{ is the union of } \leq \aleph_0 \text{ finitely bounded sets} \right\} .$$

Then $B \in N_i$ and by our assumption on i, $a_i \in B$. It follows that B itself is not the union of $\leq \aleph_0$ finitely bounded sets. A simple argument shows that there exists a $b \in B$ such that $I(\{b\}) \cap B$ is also not the union of $\leq \aleph_0$ finitely bounded sets. Since $I(A \cup \{b\}) \supseteq I(\{b\}) \cap B$ this contradicts the fact that $b \in B$. This completes the proof that $\mathcal{P}$ is a proper poset, and also the proof of Theorem 3.

The proof of Theorem 3 also gives the following decomposition theorem for partially ordered sets.

THEOREM 4. *Assume PFA. Then every partially ordered set* E *with no uncountable antichains is the union of countably many sets* E_n *such that each countable subset of* E_n *has an upper bound in* E.

PROOF. By Theorem 3 we may assume that E is a directed poset. Let B be a subset of E such that for some stationary $S \subseteq [E]^{\aleph_0}$ and a sequence $\langle F_s : s \in S \rangle$ of finite subsets of E we have that $\forall s \in S \quad \forall a \in B \cap s \quad \exists f \in F_s$ $a <_E f$. Since E is directed, we may assume that $|F_s| = 1$ for each $s \in S$. Since S is stationary in $[E]^{\aleph_0}$ it follows that each countable subset of B has an upper bound in E. Now we follow the proof of Theorem 3.

Let us also mention the following weak form of Theorem 4, which shows that $MA_{\aleph_1}$ also has some effect on posets with no uncountable antichains. This result is also connected with some problems of F.Galvin; its proof will appear elsewhere.

THEOREM 5. *Assume* $MA_{\aleph_1}$. *Then every uncountable poset* E *with no uncountable antichains contains an uncountable set, each countable subset of which has an upper bound in* E.

The approach which has been illustrated in the proof of Theorem 3 is quite general in the sense that it covers most of the known constructions of proper partial orderings. It also has some advantages over the old constructions. For example, it does not need any preliminary forcing extensions. As an example, let us show this by considering the well-known problem of specializing arbitrary trees which are not necessarily without uncountable chains. We refer the reader to [2] for the original construction due to J.Baumgartner.

We say that a tree T is *essentially special* iff there is a mapping $f: T \to \omega$ such that if $s \leq t,u$ and if $f(s) = f(t) = f(u)$, then t and u are comparable in T. Let κ be a regular cardinal such that $T \in H_\kappa$. The poset $\mathcal{P}$ which will introduce an essentially specializing map $f: T \to \omega$ is defined as follows: $p \in \mathcal{P}$ iff $p = \langle f_p, \mathcal{N}_p \rangle$, where:

(1) f_p is a finite partial essentially specializing map from T into ω.

(2) $\mathcal{N}_p$ is a finite $\in$-chain of countable elementary submodels of H_κ which contain T.

(3) If $N \in \mathcal{N}_p$, ~~and if~~ then $t \in dom(f_p) \setminus N$ is not in any maximal chain of T which is a member of N ~~then~~ iff $\forall s \in dom(f_p) \cap N$ $(s <_T t \Rightarrow f_p(s) \neq f_p(t))$.

We order $\mathcal{P}$ by: $q \leq p$ iff $f_q \supseteq f_p$ and $\mathcal{N}_q \supseteq \mathcal{N}_p$. Again we claim that $\mathcal{P}$ is made in such a way to be proper from simple reasons, and that if $M \prec H_\theta$ is countable such that $p, \mathcal{P} \in M$, then $q = \langle f_p, \mathcal{N}_p \cup \{M \cap H_\kappa\} \rangle$ is an $(M,\mathcal{P})$-generic condition.

The proof of this fact is left to the interested reader.

The same method can be used in essentially specializing $\aleph_1$-trees without adding reals. In this case as side conditions we use elementary $(\alpha+1)$-chains for $\alpha < \omega_1$, while the first part of a condition is now an essentially specializing map with countable domain.

Clearly, the side conditions which have been used in the above constructions collapse cardinals. This is necessary in some but not in all cases that we would like to use this method. In many cases this is repaired easily by letting N_p now be $\{N_p^1,\ldots,N_p^k\}$, where each N_p^i is a finite set of isomorphic countable elementary submodels of H_κ such that $\forall N \in N_p^i \;\; \exists M \in N_p^j \;\; N \in M$ for $1 \le i < j \le k$. The posets obtained in such a way satisfy one of the strong $\aleph_2$-chain conditions of S.Shelah ([6: Ch.VIII]) which are preserved under countable support iteration of length $< \omega_2$.

We finish this note with a remark which shows that the forcing only with side conditions from the above approach is still a nontrivial forcing.

THEOREM 6. *If* $\kappa > \omega_1$ *and if* $[\kappa]^{\aleph_0}$ *has a stationary subset of size* κ, *then there is a poset* P *of size* κ *which preserves* ω_1 *such that* $\Vdash_P |\kappa| = \aleph_1$.

PROOF. Let $S \subseteq [\kappa]^{\aleph_0}$ be stationary such that $|S| = \kappa$. Fix a one-to-one map $i: S \to \kappa$. Let P be the set of all finite $p \subseteq S$ such that $\forall x,y \in p(x \neq y \Rightarrow (x \subseteq y \text{ and } i(x) \in y) \vee (y \subseteq x \text{ and } i(y) \in x))$. The ordering on P is $\supseteq$. Then P is as required since if $M \prec H_\theta$ is countable such that $p,P \in M$ and $M \cap \kappa \in S$, then $p \cup \{M \cap \kappa\}$ is an (M,P)-generic condition.

The problems of this type were first considered by U.Abraham ([1]) using a different approach; we refer the reader to that work for further information. It is well-known that for each positive integer n there exists a stationary subset of $[\omega_n]^{\aleph_0}$ of size $\aleph_n$. Some information concerning the size of stationary subsets of $[\kappa]^{\aleph_0}$ for $\kappa > \aleph_\omega$ can be found in [6; Ch.XIII].

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