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Version 0.3
July 2, 2020

Notes and Exercises from Categories for the Working Mathematician

Contents

0 Preface

This document will eventually solutions to (almost) every exercise in the first edition of Mac Lane's *Categories for the Working Mathematician.* As may be seen from the version number, these notes are preliminary. When not overly onerous to typeset, it has been endeavoured to make them complete. These solutions may of course contain errors or typos. Please report any of these to `stefand@math.utoronto.ca`. The numbering of sections is sequential, and so is not in correspondence to the numbering sections in the book, as some sections contain no exercises. Some sections contain additional notes or exposition, mostly checking exercises left to the reader in the main text.

1 Categories, Functors, and Natural Transformations

1.1 Functors

1. Let R be an integral domain and $K(R)$ its field of fractions. We claim $R \mapsto K(R)$ defines a functor from the category of integral domains with monomorphisms to the category of fields. Let $\varphi\colon R \to S$ be a ring monomorphism such that $\varphi\colon 1 \mapsto 1$. Then we have $\bar{\varphi}\colon R \to K(S)$ given by $\bar{\varphi} = \iota \circ \varphi$ where $\iota\colon S \hookrightarrow K(S)$ sends $s \mapsto \frac{s}{1}$. Then $\bar{\varphi}$ is injective as it's a composition of injective maps, and by the universal property of the field of fractions, we obtain $K(\varphi)\colon K(R) \to K(S)$.

   Let $\Phi\colon G \to H$ be a morphism of Lie groups. Then we claim $\varphi := d_e\Phi$ is a morphism of Lie algebras. This will show that Lie: $G \mapsto T_eG$ is a functor from the category of Lie groups to the category of Lie algebras. Let $\Psi\colon G \to \mathrm{Aut}(G)$ be given by $\Phi(g)(h) = ghg^{-1}$. Then

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & H \\ \downarrow{\scriptstyle \Psi(g)} & & \downarrow{\scriptstyle \Psi(g)} \\ G & \xrightarrow{\Phi} & H \end{array}$$

   commutes for each $g \in G$. If $\mathrm{Ad} = d_e\Psi\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is the derivative of Ψ at $e \in G$, then by the chain rule (*i.e.* the derivative is a functor)

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi} & H \\ \downarrow{\scriptstyle \mathrm{Ad}(g)} & & \downarrow{\scriptstyle \mathrm{Ad}(\Phi(g))} \\ \mathfrak{g} & \xrightarrow{\varphi} & H \end{array}$$

   If ad $= d_e\mathrm{Ad}$, then we finally obtain (as linear maps are their own derivatives) the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi} & H \\ \Big\downarrow{\scriptstyle \mathrm{ad}\,(X)} & & \Big\downarrow{\scriptstyle \mathrm{ad}\,(\varphi(X))} \\ \mathfrak{g} & \xrightarrow{\varphi} & H \end{array}$$

by differentiating again, where $X \in \mathfrak{g}$. Therefore φ is a morphism of Lie algebras.

2. - Functors $\mathbf{1} \to C$ are equivalent to objects of C; just pick $c \in \mathrm{Obj}(C)$ to send $*$ to. Conversely, a functor chooses an object $T(*)$.
   - Functors $\mathbf{2} \to C$ are equivalent to morphisms of C. Let $\mathbf{2}$ have objects t and h, with non-identity morphism $\varphi\colon t \to h$. Then given $f\colon c_1 \to c_2$, set $T(t) = c_1$, $T(h) = c_2$ and $T(\varphi) = f$. Conversely, $T(\varphi)\colon T(t) \to T(h)$ is a morphism in C. Clearly these assignments are bijective.
   - Functors $\mathbf{3} \to C$ are equivalent to pairs of composable morphisms in C. Let $\mathbf{3}$ have objects t, m, h, and morphisms $F\colon t \to m$, $G\colon m \to h$ and $K = G \circ F$. Then given a composable pair $f\colon a \to b$, $g\colon b \to c$, we can define a functor by $T(h) = a$, $T(m) = c$ and $T(t) = c$ and $T(F) = f$, $T(G) = g$, $T(K) = g \circ f$. Conversely, we know $T(F)$ and $T(G)$ are composable because F and G are.

3. (a) Let C and B be preoders. Then we claim functors $T\colon C \to B$ are order-preserving functions $C \to B$. Indeed, it's clear that functors induce order-preserving functions as $f\colon c_1 \to c_2$ gives $T(f)\colon T(c_1) \to T(c_2)$. Conversely, reflexivity and transitivity say that an order-preserving function induces a functor $C \to B$.

   (b) We show a functor between groups is equivalent to a morphism of groups. Let $*$ and $\bullet$ be the objects in G and H, respectively, thought of as categories. For any functor we must have $T(*) = \bullet$. Then for all $g\colon * \to *$ and $h\colon * \to *$, $T(g \circ h) = T(g) \circ T(h)$, so with $g \circ h = gh$ and $\varphi(g) = T(g)$, we obtain a morphism of groups. Conversely, given a morphism of groups φ, set $T(*) = \bullet$ and $T(g) = \varphi(g)$.

   (c) For a group G, a functor $T\colon G \to \mathbf{Set}$ is a permutation representation of G: given G we obtain a set $T(*)$, and for every morphism $g\colon * \to *$ we obtain a function $T(g)\colon S \to S$. That T is a functor says that $T(h) \circ T(g) = T(hg)$, so that G acts on $T(G)$. Conversely, G acts on $S = T(*)$ by $\nu(g, s) = T(g)(s)$.

   A functor $T\colon G \to \mathbf{Matr}_K$ is equivalent to a K-representation of G. Given a dimension n K-representation of G, put $T(*) = n$ and for $g\colon * \to *$ define $T(g) = \rho(g)\colon n \to n$. Conversely, given a functor $T\colon G \to \mathbf{Matr}_K$, we want a morphism of groups $\rho\colon G \to \mathrm{GL}(V) = \mathrm{GL}_n(K)$, after a choice of basis. Set $n = T(*)$, then for $g\colon * \to *$ we see $\rho(g) = T(g)$ will be a morphism of groups; T is a functor and all $g \in G$ are composable with each other.

4. There is no functor $\mathbf{Grp} \to \mathbf{Ab}$ sending a group to its centre. Indeed, suppose there was and consider

$$\mathfrak{S}_2 \hookrightarrow \mathfrak{S}_3 \xrightarrow{\pi} \mathfrak{S}_2 \simeq \mathfrak{S}_3/\langle(123)\rangle,$$

where the first map is the inclusion. It is checked by hand that $\langle(123)\rangle$ is normal, and this composition is nontrivial, hence an isomorphism, hence is the identity map. But another calculation shows that $\mathfrak{S}_3$ has trivial centre, which says the identity morphism on $Z(\mathfrak{S}_2) = \mathfrak{S}_2$ factors through the trivial group.

5. There are many endofunctors of $\mathbf{Grp}$ with are the identity on objects. One is the identity functor. For another one, pick automorphisms ψ_G for some groups G. Let $T(G) = G$ on objects, and if $\psi\colon G \to H$ is a morphism, define $T(\varphi) = \psi_H \circ \varphi \circ \psi_G^{-1}$. This defines a functor different from the identity.

Remark 1. Recall that an *algebraic group* is a functor $k - \mathbf{Alg} \to \mathbf{Grp}$. There is a notion of the centre of an algebraic group, but excise 4 says that there is some subtly to this notion.

1.2 Natural Transformations

1. Let $X, S \in \operatorname{Obj} \mathbf{Set}$. We claim $X \mapsto X^S$ is the object function of a functor $\mathbf{Set} \to \mathbf{Set}$. Indeed, the functor is $T(-) = \operatorname{Hom}(S, -)$, which on objects sends $f\colon X \to X'$ to the pushforward f_* (see §2.2 in Mac Lane). For any $s \in S$ (or passing to the functor $X \mapsto X^S \times S$, we get a natural transformation $e^s\colon T \to \mathrm{id}$ given by evaluation $e^s_X\colon h \mapsto h(s)$ for $h\colon X \to S$ in X^S. Naturality is immediate: given a square

$$\begin{array}{ccc} X^S & \xrightarrow{e^s_X} & X \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f} \\ (X')^S & \xrightarrow{e^s_{X'}} & X', \end{array}$$

paths around it are $h \mapsto h(s) \mapsto f(h(s))$ and $h \mapsto f \circ h \mapsto (f \circ h)(s)$.

2. If H is a fixed group, then $H \times -\colon \mathbf{Grp} \to \mathbf{Grp}$ is a functor. On objects $G \mapsto H \times G$, and if $\varphi\colon G \to G'$, then $\mathrm{id}_H \times \varphi\colon H \times G \to H \times G'$. Next, if $f\colon H \to K$, then obtain a natural transformation $H \times - \to K \times -$ whose components are $f \times \mathrm{id}_G$ for a given G. The naturality is just that, if $\varphi\colon G \to G'$,

$$(f(h), \varphi(g)) = (\mathrm{id}_K \times \varphi) \circ (f \times \mathrm{id}_G)(h, g) = (f \times \mathrm{id}_{G'}) \circ (\mathrm{id}_H \times \varphi)(h, g).$$

3. Let $S, T\colon C \to B$ be functors between groups. Then we claim there is a natural transformation $S \to T$ iff there is a morphism (element of B thought of as a group) h such that $Tg = h(Sg)h^{-1}$ for all $g\colon C \to C$. Indeed, as C and B each have only one object, T and S are equal on objects and the above reduces to $T(g) \circ h = h \circ S(g)$ iff $Tg = h(Sg)h^{-1}$.

4. Let $S, T\colon C \to P$ be functors into a preorder P. Then we claim there exists a natural transformation $\sigma\colon S \to T$ (which is then unique) iff $Sc \leq Tc$ for all $c \in \operatorname{Obj} C$. Indeed, if $Sc \leq Tc$, then we obtain unique arrows $Sc \to Tc$ for all c. By uniquess of the arrow $Sc \to Tc'$, we see the square

$$\begin{array}{ccc} Sc & \longrightarrow & Tc \\ \downarrow{\scriptstyle Sf} & & \downarrow{\scriptstyle Tf} \\ Sc' & \longrightarrow & Tc' \end{array}$$

commutes, given $f\colon c \to c'$. Conversely, a natural transformation is a collection of components $Sc \to Tc$.

Remark 2. This makes the functor category (see §2.2 in these notes) P^C into a preorder in a natural way.

5. (Arrows-only description of natural transformations.) Let $\tau\colon S \to T$ be a natural transformation of functors $C \to B$. Then we claim τ determines a function taking $f\colon c \to c'$ in C to $\tau f\colon Sc \to Tc'$ in B such that

$$Tg \circ \tau f = \tau(gf) = \tau g \circ Sf.$$

Indeed, given τ define $\tau f = \tau_c' \circ S(f)$. Naturality gives the following diagram

$$\begin{array}{ccc} Sc & \xrightarrow{\tau_c} & Tc \\ \downarrow{\scriptstyle S(f)} & & \downarrow{\scriptstyle T(f)} \\ Sc' & \xrightarrow{\tau_{c'}} & Tc' \\ \downarrow{\scriptstyle S(g)} & & \downarrow{\scriptstyle T(g)} \\ Sc'' & \xrightarrow{\tau_{c''}} & Tc'' \end{array}$$

As τf and τg are the just diagonals of each of the squares, the above equation is satisfied. Conversely, we claim every natural transformation arises from such a function τ, with $\tau_c = \tau(1_c)$. We have compositions $f = 1_{c'}f$ and $f = f1_c$, so that $\tau(1'_c) \circ S(f) = \tau(f)$ and $Tf \circ \tau(1_c) = \tau(f) \circ Sf = \tau(f)$ by the above equation, so that $Tf \circ \tau_c = \tau'_c \circ Sf$ and τ is a natural transformation.

6. Let F be a field. We claim $\mathbf{Matr}_F$ is equivalent to the category $\mathbf{Vect}_F$ of finite- dimensional F-vector spaces, with morphisms all F-linear transformations. Indeed, define functors $T\colon \mathbf{Matr}_F \to \mathbf{Vect}_F$ by $T(n) = F^n$ and by letting $T(f)\colon F^n \to F^m$ be the linear transformation determined by the matrix $A\colon n \to m$. In the other direction define $S(V) = \dim V$ and let $S(\varphi)$ be the matrix for a linear transformation $\varphi\colon V \to W$, after choosing bases for V and W. Then $(S \circ T)(n) = n$ is the identity on objects. Let α and β be the chosen bases for F^n and F^m, respectively, and let γ and η be the bases in which A is written. Let τ_n be the change-of-basis matrix α to γ, and τ_m be the matrix changing β to η. Then if φ^β_α is the matrix for φ, we have $\tau_m \varphi^\beta_\alpha \tau_n^{-1} = A$ as required. In the other direction,$(T \circ S)(V) = F^{\dim V}$, and $(T \circ S)(\varphi)(v) = \varphi^\beta_\alpha v$, where α and β are bases chosen by S for V and W, respectively. On morphisms, $(T \circ S)(\varphi)\colon v \mapsto \varphi^\beta_\alpha v$, but with $v \in K^n$ and $\varphi^\beta_\alpha \in K^m$ written in two bases γ and η, respectively. Then let $\tau_V\colon K^n \to V$ be multiplication by the change of basis matrix from α toγ and $\tau_W\colon K^m \to W$ be multiplication by the change of basis matrix from β to γ. Then the necessary square commutes and we have an equivalence of categories.

1.3 Monics, Epis, and Zeros

1. Let $\iota\colon \mathbb{Q} \to \mathbb{R}$ be inclusion (with the standard topologies) in **Top**. Then ι is a monomorphism as $\iota(f(x)) = f(x)$ for all morphisms $f\colon X \to \mathbb{Q}$. As morphisms in **Top** are continuous functions, we have $f \circ \iota = g \circ \iota$ means $f = g$; let $x_n \to x$ for $x_n \in \mathbb{Q}$ and $x \in \mathbb{R}$, then $f(x) = \lim f(x_n) = \lim g(x_n)$. Therefore ι is an epimorphism and a monomorphism, but not an isomorphism.

2. Let $f\colon a \to b$ and $g\colon b \to c$ be monomorphisms in C. Then if $(g \circ f) \circ h = (g \circ f) \circ h'$, by associativity of composition, $f \circ h = f \circ h'$ whence $h = h'$. Associativity is used similarly to show $f \circ g$ is an epimorphism when f and g are.

3. We claim if $g \circ f$ is a monomorphism, then f is. Indeed, if $f \circ h = f \circ h'$, then $g \circ f \circ h = g \circ f \circ h'$ so $h = h'$, using associativity of composition. It need not be true that g is a monomorphism; consider a map in **Set** with large fibres and a section.

   Similarly, if $g \circ f$ is an epimorphism, then f is an epimorphism.

4. We claim that $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in the category **Rng** (where the morphisms preserve units). Let f and g be morphisms $\mathbb{Q} \to R$ and $f \circ \iota = g \circ \iota$. Then $f(\frac{a}{1}) = g(\frac{a}{1})$ for all $a \in \mathbb{Z}$. Now

$$f\left(\frac{a}{1}\frac{1}{a}\right) = 1 = g\left(\frac{a}{1}\frac{1}{a}\right)$$

   and as the units of R form a group we can cancel to see $f(1/a) = g(1/a)$ so that $f = g$.

5. We claim every epimorphism in **Grp** is surjective. Let $f\colon G \to H$ be an epimorphism and suppose $M = \operatorname{im}(g) \subsetneq H$. Suppose not. If $[H\colon M] = 2$, then $M \lhd H$ and if π is the quotient map, $\pi \circ f = \tau = \tau \circ f$, where $\tau\colon H \to 1$ is the trivial homomorphism. This says $\pi = \tau$ so $M = H$. If $[H\colon M] \geq 3$, let $\mathfrak{S}_H$ be the symmetric group on the set H and define $\sigma(xu) = xv$, $\sigma(xv) = xu$ and $\sigma = \mathrm{id}$ otherwise, for $x \in M$. Let $\psi\colon h \mapsto (\psi_h\colon h' \mapsto hh')$ map $H \to \mathfrak{S}_H$, and let $\psi'(h)\sigma^{-1} \circ \psi_h \circ \sigma$. Then we claim $\psi' \circ \varphi = \psi \circ \varphi$. Indeed, $(\psi \circ \varphi)(g)(h) = \varphi(g)h$, and if $h \notin Mu, Mv$, then $(\psi' \circ \varphi)(g)(h) = \varphi(g)h$ too. If $h = xu$ for $x \in M$, then

$$\sigma^{-1}(\varphi(g)\sigma(h)) = \sigma^{-1}(\varphi(g)xv) = \varphi(g)xu = \varphi(g)h.$$

   Now the claim is proved, but $\psi \neq \psi'$: if $h = xu$, then $\psi'_h(yv) = hyv$ if and only if $u \in M$ as well as x. This is a contradiction. We claim that all idempotents split in **Set**. Let f be such that $f \circ f = f\colon X \to X$. We can always factor $f = hg$ with $g\colon X \twoheadrightarrow \operatorname{im}(f)$ and $h\colon \operatorname{im}(f) \hookrightarrow X$ begin the obvious functions. As f is the identity on its image, $gh\colon \operatorname{im}(f) \hookrightarrow X \twoheadrightarrow \operatorname{im}(f)$ is the identity on $\operatorname{im}(f)$. Therefore f is split.

6. We claim $f\colon a \to b$ is regular is f has a left or right inverse. Indeed, let g be either a left or right inverse, then either $fgf = f\mathrm{id}_a = f$ or $fgf = \mathrm{id}_b f = f$ as required. We claim next every morphism in **Set** is regular. Let $g\colon b \to a$ be defined by choosing a section $\mathrm{im}(f) \to a$ of f and a constant function $b \setminus \mathrm{im}(f) \to a$. Then g is almost a section of f and $fgf = f$.

7. todo

8. Let $T\colon C \to B$ be a faithful functor, and Tf be monic. Then we claim f is monic. Let $f \circ g = f \circ g'$ in C. Then $Tf \circ Tg = Tf \circ Tg'$ so $Tg = Tg'$. As T is faithful, $g = g'$ as required.

2 Constructions on Categories

2.1 Products of Categories

1. We claim the product of categories includes the product of monoids, the (direct) product of groups, and the product of sets.

   If G and H are groups, then the category $G \times H$ has one object $(*, *)$ and morphisms $(g, h)\colon (*, *) \to (*, *)$ are still invertible, with inverses (g^{-1}, h^{-1}). The same reasoning applies to **Set** and the category of monoids. The universal property for product categories holds in each of these specific categories, which means the product $G \times H$ of categories must be the group $G \times H$, and likewise for sets and monoids; objects solving a universal mapping product are unique up to isomorphism.

2. We claim the product of two preoders is a preoder. A morphism $(c, b) \to (c', b')$ in $C \times B$ projects to two morphisms $c \to c'$ and $b \to b'$, so if C and B are preoders, then two morphisms $(c, b) \to (c', b')$ must agree on both factors. Therefore there is at most one morphism between objects in the product. Thus the product is a preoder.

3. We define the product $C := \prod_{i \in I} C_i$ of small categories C_i indexed by a set I. Set $\mathrm{Obj}(C) := \{F\colon I \to \bigcup_i \mathrm{Obj}(C_i) \mid F(i) \in \mathrm{Obj}(C_i)\ \forall i\}$ and $\mathrm{Mor}(C) := \{\mathcal{F}\colon I \to \bigcup_i \mathrm{Mor}(C_i) \mid \mathcal{F}(i) \in \mathrm{Mor}(C_i)\ \forall i\}$. Note that $\bigcup_i \mathrm{Obj}(C_i)$ is a small set, and likewise for morphisms. Define projections $P_i\colon C \to C_i$ by $P_i(F) = F(i)$ and $P_i(\mathcal{F}) = \mathcal{F}(i)$. Then C obeys the following universal property: given a family $\{T_i\colon D \to C_i \mid i \in I\}$, there is a unique $T\colon D \to C$ such that $T_i = P_i \circ T$. Indeed, set $T(d)(i) = T_i(d)$ and $T(f)(i) = T_i(f)$.

4. The opposite category of $\mathbf{Matr}_K$ is obtained by "taking adjoints" *i.e.* viewing the dual space of K^n as a space of row vectors, where matrices multiply on the right, acting by pullback.

5. The functor $T\colon \mathbf{Top} \to \mathbf{Rng}$ sending $X \mapsto \mathcal{C}(X, \mathbb{R})$ is contravariant; if $f\colon X \to Y$ then $T(f) = f^*$ is pullback by f.

2.2 Functor categories

1. Let R be a ring and view $(R, \cdot)$ as a monoid. We claim R-**Mod** is a full subcategory of $\mathbf{Ab}^R$. Indeed, if $T\colon R \to \mathbf{Ab}$ is a functor, then we have morphisms $T(r)\colon M = T(*) \to T(*)$. If we require that $T(0)(m) = 0$ for all $m \in M$, and $T(r) \circ (m \circ n) = T(r)(m + n) = T(r)m + T(r)n$, then $T(*)$ is an R-module. A morphism in the functor category is an R-equivariant morphism of abelian groups, which is exactly a morphism of R-modules, so the above subcategory is full.

2. Let X be a finite discrete category. Then objects of B^X for any category B are functors $X \to B$, and a function $X \to \mathrm{Obj}(B)$ is a collection of at most $\#X$ objects in B. As X has on identity morphisms, the data of a natural transformation $S \to T$ is just a set of functions $\tau_x\colon S(x) \to T(x)$ for each $x \in X$, *i.e.* certain sets of morphisms from the objects $S(X)$ to the objects $T(x)$.

3. Let $\mathbb{N}$ be the discrete category of natural numbers. Then we claim $\mathbf{Ab}^{\mathbb{N}}$ is the category of graded abelian groups. A object $F\colon \mathbb{N} \to \mathbf{Ab}$ defines an abelian group $F = \bigoplus_{n\in\mathbb{N}} F(n)$. A morphism is a natural transformation, which for functors from a discrete category is a just a set of morphisms $\tau_n\colon F(n) \to G(n)$ if $\tau\colon F \to G$. This is exactly the data of a morphism of graded abelian groups, with $\tau = \bigoplus_{n\in\mathbb{N}} \tau_n$.

4. Let P and Q be preoders. Then we claim Q^P is a preoder with objects all order-preserving functions $T\colon P \to Q$, and morphisms $T \to S$ iff $T(p) \le S(p)$ for all $p \in \mathrm{Obj}(P)$. As Q is a preoder, there is at most one natural transformation $\tau\colon S \to T$ for any pair of objects. If there is a such a natural transformation, the diagram

$$\begin{array}{ccc} Sp & \xrightarrow{\tau_p} & Tp \\ \downarrow{\scriptstyle S(f)} & & \downarrow{\scriptstyle T(f)} \\ Sp' & \xrightarrow{\tau_{p'}} & Tp' \end{array}$$

says that $Sp \le Tp$ for all $p \in \mathrm{Obj}\, P$, by taking $p = p'$ and $f = \mathrm{id}_p$.

5. If $\mathbf{Fin}$ is the category of all finite sets and G is a group, we claim $\mathbf{Fin}^G$ is the category of permutation representations $\rho\colon G \to \mathfrak{S}_X$ of G. An object is a functor $T\colon G \to \mathbf{Fin}$, which provides bijections $T(g)\colon X = T(*) \to X = T(*)$. As $T(1) = \mathrm{id}$ and $T(gh) = T(g) \circ T(h)$, this is an action of G. Morphisms are natural transformation, *i.e.* G-equivariant functions.

6. Let M be the infinite cyclic monoid with morphisms $\{1, m, m^2, \dots\}$, where $m^n = m \circ m \circ \cdots \circ m\colon * \to *$. Then we claim $(\mathbf{Matr}_K)^{\mathbf{2}}$ has matrices for objects, and two objects are isomorphic iff they are equivalent, and that $(\mathbf{Matr}_K)^M$ also has matrices for objects, with objects isomorphic iff they are conjugate. Indeed, a functor $\mathbf{2} \to \mathbf{Matr}_K$ gives $T(\downarrow)\colon T(0) = n \to T(1) = m$, which is an $n \times m$ matrix. Of course, $T(\mathrm{id})$ is the identity matrix. That there exists an invertible morphism $\tau\colon S \to T$ means exactly that $S(\downarrow) = \tau_1^{-1} T(\downarrow)\tau_0$, or that S and T are similar. For $(\mathbf{Matr}_K)^M$, $S(m)\colon S(*) = n \to S(*) = n$ is square, and natural transformations have only one component, so isomorphism implies similarity by the above equation.

7. We claim there is a bijection $H \mapsto (S, \tau, T)$ from functors $H\colon C \to B^{\mathbf{2}}$ to pairs of functors $S, T\colon C \to B$ with a natural transformation $\tau\colon S \to T$. Given H, we can define $T\colon C \to B$ by $T(c) = H(c)(0)$ and $T(f) = H(f)_0$, $S(c) = H(c)(1)$ and $S(f) = H(f)_1$ for morphisms $f\colon c \to c'$. We obtain a diagram

$$\begin{array}{ccc} Tc = H(c)(0) & \xrightarrow{H(f)_0} & H(c')(0) = Tc' \\ \downarrow{\scriptstyle H(c)(\downarrow)} & & \downarrow{\scriptstyle H(c')(\downarrow)} \\ H(c)(1) & \xrightarrow{H(f)_1} & H(c')(1) \end{array} \tag{1}$$

so $H(f)_0$ really is a morphism $Tc \to Tc'$ in B. The vertical composition of natural transformations says that T respects composition of morphisms. Turning (1) on its side, we see that setting $\tau_c := H(c)(\downarrow)$ defines a natural transformation $\tau\colon T \to S$. Conversely, given (1) turned on its side, we can define H on objects by $H(c)(0) := T(c)$, $H(c)(1) = S(c)$ and $H(c)(\downarrow) = \tau_c$. Given a morphism in $f\colon c \to$ ¸ in C, we can define $H(f)\colon H(c) \to H(c')$ by $H(f)_0 = T(f)$ and $H(f)_1 = S(f)$. Clearly these assignments are bijections, which proves the claim.

8. In section 2.3 of Mac Lane we see there is a unique functor $F\colon C \times \mathbf{2} \to B$ given two functors $S, T\colon C \to B$ with a natural transformation $\tau\colon S \to T$, such that $F\mu_c = \tau_c$, where μ is the natural transformation between functors $T_0, T_1\colon C \to C \times \mathbf{2}$ given by $mu_c = (\mathrm{id}_c, \downarrow)$. We relate F to H from exercise 7.

Given $F\colon C\times\mathbf{2}\to B$ and μ, define $H\colon C\to B^{\mathbf{2}}$ by $H(c)(i) := F(c,i)$ and as $H(c)(\downarrow) = \tau_c\colon T(c)\to S(c)$, set $H(c)(\downarrow) := F\mu_c$. This defines the functor H on objects. Given $f\colon c\to c'$ in C, set $H(f)_i := T(f,\mathrm{id}_i)\colon H(c)(i)\to H(c')(i)$.

Conversely, given H, one can define F from the above equations.

2.2.1 The Interchange Law

There are two ways to compose natural transformations. Given three functors $S,T,U\colon C\to B$ and natural transformations $\tau\colon S\to T$ and $\sigma\colon T\to U$, we can define the *vertical composition* $\sigma\cdot\tau\colon S\to U$ by $(\sigma\cdot\tau)_c = \sigma_c\circ\tau_c$ where $\circ$ is composition in B. We can also define the *horizontal composition* of a natural transformation $\tau\colon S\to S'$ of functors $C\to B$ with $\sigma\colon T\to T'$ for functors $B\to A$. By definition, $(\sigma\circ\tau)_c$ is the diagonal of *e.g.* one of the small square below. The *interchange law* says that if $S,T,U\colon C\to B$ and $S',T',U'\colon B\to A$ with $\sigma\colon S\to T$ and $\tau\colon T\to U$ and likwise for τ' and σ', then

Proposition 1. *We have*

$$(\tau'\cdot\sigma')\circ(\tau\cdot\sigma) = (\tau'\circ\tau)\cdot(\sigma'\circ\sigma).$$

Proof. Consider the diagram

$$\begin{CD}
S'Sc @>{\sigma' Sc}>> T'Sc @>{\tau' Sc}>> U'Sc \\
@V{S'\sigma c}VV @V{T'\sigma c}VV @V{U'\sigma c}VV \\
STc @>{\sigma' Tc}>> T'Tc' @>{\tau' Tc}>> U'Tc \\
@V{S'\tau c}VV @V{T'\tau c}VV @V{U'\tau c}VV \\
S'Uc @>{\sigma' Uc}>> T'Uc @>{\tau' Uc}>> U'Uc
\end{CD}$$

which commutes as each of the small squares commutes. The diagonal of the outer square is (the component at c of) the left-hand side of the claim, and the composite of the dashed morphisms is the right-hand side. As these are one and the same, the claim is proven. ☐

2.3 The Category of All Categories

1. (Product-Exponential adjunction.) Let A,B and C be small categories. Then we claim $\mathbf{Cat}(A\times B,C)\simeq\mathbf{Cat}(A,C^B)$ naturally in all three arguments. This will show that $-\times B\colon\mathbf{Cat}\to\mathbf{Cat}$ has a right adjoint, namely $C\mapsto C^B$. Define the function

   $$\varphi\colon\ \mathbf{Cat}(A\times B,C)\to\mathbf{Cat}(A,C^B)$$

   By

   $$(\varphi T)(a)(b) = T(a,b) \text{ and } (\varphi T)(f)_b = T(f,\mathrm{id}_b)$$

   for $b\in B$ and morphisms $f\colon a\to a'$ in A. Define also $(\varphi T)(a)(f) = T(\mathrm{id}_a,f)$. The diagram

   $$\begin{CD}
   \varphi(T)(a)(b) @>{T(f,\mathrm{id}_b)}>> \varphi(T)(a')(b) \\
   @V{\varphi(T)(a)(g)}VV @V{\varphi(T)(a')(g)}VV \\
   \varphi(T)(a)(b') @>{T(f,\mathrm{id}_{b'})}>> \varphi(T)(a')(b')
   \end{CD}$$

   commutes, and so $(\varphi T)(f)$ is a morphism in C^B. Define a function in the opposite direction by $\psi(T)(a,b) = T(a)(b)$ on objects. If $f = (f_A,f_B)\colon(a,b)\to(a',b')$, then define

   $$\psi(T)(f) = T(a')(f_B)\circ T(f_A)_b\colon T(a)(b)\to T(a')(b').$$

   Checking that these functions are inverse is straightforward. Naturality in C is most compact to write down. We require that, if $F\colon C\to C'$, the diagram

$$\begin{array}{ccc} \mathbf{Cat}(A \times B, C) & \xrightarrow{\varphi} & \mathbf{Cat}(A, C^B) \\ \downarrow{\scriptstyle F_*} & & \downarrow{\scriptstyle F_*} \\ \mathbf{Cat}(A \times B, C') & \xrightarrow{\varphi} & \mathbf{Cat}(A, C'^B) \end{array}$$

commute. Here the map F_* on the right is $F_*(T)(a)(b) = F(T(a)(b))$. This is just a short calculation. Note that to the show the diagram commutes, one must show that the functors $A \to C'^B$ obtained via either path agree on objects *and* morphisms.

2. Let A, B and C be categories. We claim there is an isomorphism of categories

$$\varphi\colon C^{A\times B} \to (C^B)^A$$

natural in each of A, B, and C. First we define φ on objects. Given $T\colon A \times B \to C$, define a functor $\varphi(T)\colon A \to C^B$ by $\varphi(T)(a)(b) = T(a, b)$ on objects. On morphisms $f\colon b \to b'$ in B define $\varphi(T)(a)(f) = T(\mathrm{id}_a, f)\colon \varphi(T)(a)(b) \to \varphi(T)(a)(b')$. We must now define the functor $\varphi(T)$ on morphisms. Given $f\colon a \to a'$ in A, we request a natural transformation $\varphi(T)(f)\colon \varphi(T)(a) \to \varphi(T)(a')$. We therefore define $\varphi(T)(f)_b = T(f, \mathrm{id}_b)$. We can now define the functor φ on morphisms. Given a natural transformation $\tau\colon S \to T$ between functors $S, T\colon A \times B \to C$ we require a natural transformation in $(C^B)^A$. We therefore define $\varphi(\tau)_a = \tau(a, -)\colon \varphi(S)(a) \to \varphi(T)(a)$. This transformation has components $(\varphi(\tau)_a)_b = \tau_{(a,b)}$.

In the other direction, define a functor $\psi\colon (C^B)^A \to C^{A\times B}$ on objects as follows. Set $\psi(T)((a, b)) = T(a)(b)$. Given $(f, g)\colon (a, b) \to (a', b')$ in $A \times B$, set

$$\psi(T)((f, g)) = T(f)_b \circ T(a)(g)\colon \psi(T)((a, b)) = T(a)(b) \to T(a')(b') = \psi(T)((a', b')).$$

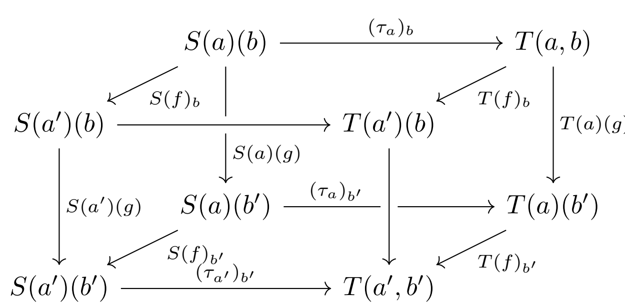
We define ψ on morphisms as follows. Given $\tau\colon S \to T$ between functors $S, T\colon A \to C^B$, set $\psi(\tau)_{(a,b)} = (\tau_a)_b$. We require that the diagram

$$\begin{array}{ccc} \psi(S)((a, b)) & \xrightarrow{\psi(\tau)_{(a,b)}} & \psi(T)(a)(b) \\ \downarrow{\scriptstyle \psi(S)((f,g))} & & \downarrow{\scriptstyle \psi(T)((f,g))} \\ \varphi(S)((a, b')) & \xrightarrow{\psi(\tau)_{(a',b')}} & \psi(T)((a', b')) \end{array} \tag{2}$$

commute. By the definition of τ we obtain a diagram

$$\begin{array}{ccc} S(a) & \xrightarrow{\tau_a} & T(a) \\ \downarrow{\scriptstyle S(f)} & & \downarrow{\scriptstyle T(f)} \\ S(a') & \xrightarrow{\tau_{a'}} & T(a') \end{array}$$

in C^B. This implies there is a commutative cube (here (f, g) is as above)



which implies that (2) commutes.

We claim φ and ψ are inverse on objects. We have

$$\varphi(\psi(T))(a)(b) = \psi(T)((a,b)) = T(a)(b)$$

and

$$\psi(\varphi(T))((a,b)) = \psi(T)(a)(b) = T(a,b)$$

as required. On morphisms we have, given $f\colon b \to b'$, that

$$\psi(\varphi(T))(a)(f) = \psi(T)(\mathrm{id}_a, f) = T(\mathrm{id}_a)_{b'} \circ T(a)(f) = T(a)(f)$$

and given $(f,g)\colon (a,b) \to (a',b')$ we have

$$\psi(\varphi(T))((f,g)) = \varphi(T)(f)_{b'} \circ \varphi(T)(a)(g) = T(f, \mathrm{id}_{b'}) \circ T(\mathrm{id}_a, g) = T(f,g).$$

Therefore φ and ψ are inverse on objects. On morphisms, we have, given $\tau\colon S \to T$ a natural transformation between functors $S, T\colon A \times B \to C$,

$$(\psi\varphi)(\tau)_{((a,b))} = \psi(\varphi(\tau))_{(a,b)} = (\varphi(T)_a)_b = \tau_{(a,b)}.$$

and

$$\varphi\psi(\tau)_a = \varphi(\psi(\tau))_a = \psi(T)_{(a,-)} = \tau_a.$$

This establishes the isomorphism of categories. We do not check naturality.

3. We claim horizontal composition of natural transformations defines a functor $\circ\colon A^B \times B^C \to A^C$. On objects, one composes the functors: $\circ(S,T) = S \circ T$. Let $(\sigma, \sigma'), (\tau,' tau') \in \mathrm{Mor}(A^B \times B^C)$ be composable, *i.e.* be as in (4) in §2.5 of Mac Lane. By the interchange law,

$$\circ((\tau', \tau) \cdot (\sigma', \sigma)) = \circ((\tau' \cdot \sigma', \tau \cdot \sigma)) = (\tau' \cdot \sigma') \circ (\tau \cdot \sigma) = (\tau' \circ \tau) \cdot (\sigma \circ \sigma) = \circ(\tau', \tau) \cdot \circ(\sigma', \sigma).$$

This calculation together with the description in theorem 1 of §2.5 says that $\circ$ respects composition. The statement that identities for $\cdot$ are precisely identities for $\circ$ finishes the proof that $\circ$ is a functor.

4. Let G be a topological group (defined the usual way) with loops $\sigma, \tau, \sigma', \tau'$ based at the identity $e \in G$. Define $\sigma \circ \tau$ as the composition of loops as in $\pi_1(G,e)$ and $(\sigma \cdot \tau)(t) = \sigma(t)\tau(t)$ to be the pointwise product. We claim the interchange law holds. We calculate that

$$((\tau' \cdot \sigma') \circ (\tau \cdot \sigma))(t) = \begin{cases} \tau(2t)\sigma(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \tau'(2t-1)\sigma'(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases} = ((\tau' \circ \tau) \cdot (\sigma' \circ \sigma))(t),$$

which proves the claim.

5. (Hilton-Eckmann argument.) Let S be a set with two compositions defined everywhere denoted $\circ$ and $\cdot$ which share the same two-sided identity and satisfy the interchange law. Then we claim they agree and are commutative. Indeed, if e is the identity for both, then

$$s_1 \circ s_2 = (s_1 \cdot e) \circ (e \cdot s_2) = (s_1 \circ e) \cdot (e \cdot s_2) = s_1 \cdot s_2.$$

Now we have

$$s_2 \circ s_1 = (e \cdot s_2) \circ (s_1 \cdot e) = (e \circ s_1) \cdot (s_2 \circ e) = s_1 \cdot s_2 = s_1 \circ s_2.$$

6. Let G be a topological group. Then we claim $\pi_1(G)$ is abelian. Assume without loss of generality that G is based at the identity. By 4, multiplication in G and $\pi_1(G,e)$ obey the interchange law, and by our choice of basepoint, they share a two-sided identity. By 5, $\pi_1(G)$ is abelian.

Remark 3. Recall that smooth group schemes G_0 defined over $\mathbb{F}_q$ and base-changed to G over $\bar{\mathbb{F}}_q$ can have etale fundamental group surjecting onto $G_0(\mathbb{F}_q)$, and so the etale fundamental group can fail to be abelian. Therefore there can be no analogue of the interchange law in this setting. Even $\pi_1^{\mathrm{et}}(\mathbb{A}^1_{\bar{\mathbb{F}}_q})$ is unknown, but *e.g.* its finite quotients are know. See *e.g.* the discussion at `https://mathoverflow.net/questions/868/etale-covers-of-the-affine-line`.

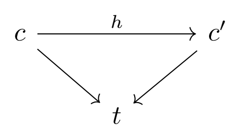
7. Let $T\colon A \to D$ be a functor. We claim the associated arrow function $T_{(a,b)}\colon A(a,b) \to D(Ta, Tb)$ defines a natural transformation between the functors $\mathrm{Mor}_A\colon A^{\mathrm{opp}} \times A \to \mathbf{Set}$ and the functor S defined by $S = \mathrm{Mor}_D \circ (T^{\mathrm{opp}}, T)$, where T^{opp} is the functor $A^{\mathrm{opp}} \to D^{\mathrm{opp}}$ induced by T.

8. finish

2.4 Comma Categories

1. Let K be a commutative ring. Then we claim the category $(K \downarrow \mathbf{CRng})$ (which we write below as $K/\mathbf{CRng}$) is the category of small commutative K-algebras. Indeed, an object is a morphism of small commutative rings $f\colon K \to R$, so that $k \cdot (ab) = f(k)ab = af(k)b$ for $a, b \in R$ because R is commutative and associative. Such a morphism is exactly the data of an algebra over a ring. A morphism in the coslice category is the obvious triangle such that $g(f_1(k)) = f_2(k)$, whence $g(f_1(k)r) = g(f_1(k))g(r) = f_2(k)g(r)$. This says g is K-linear, hence a morphism of K-algebras.

2. Let $t \in \mathrm{Obj}\, C$ be terminal for a category C. We claim the slice category C/t is isomorphic to C. Define a functor $T\colon C/t \to C$ by $T(f\colon c \to t) = c$ on objects. On morphisms, send



to $h \in \mathrm{Mor}\, C$. Define $S\colon C \to C/t$ by letting $S(c)$ be the unique morphism $f\colon c \to t$, and sending $h\colon c \to c'$ to the above triangle, which now commutes by uniqueness of morphisms to t. These functors are inverse by construction and give the desired isomorphism.

Clearly an analogous result holds for the coslice category i/C, where i is an initial object.

3. We define the functors Q, P, R on morphisms as follows. Given a morphism $(h, k) \in \mathrm{Mor}(T \downarrow S)$, set $P((h, k)) = k$, $R((h, k)) = (Tk, Sh)$ and, $Q((h, k)) = h$. The diagram (5) in Mac Lane now commutes.

4. (S.A. Huq). We claim functors $T, S\colon D \to C$ with a natural transformation $\tau\colon T \to S$ is equivalent to the data of a functor $\tau\colon D \to (T \downarrow S)$ such that $P\tau = Q\tau = \mathrm{id}_D$. Given two functors and τ as above, define $\tau\colon D \to (T \downarrow S)$ on objects by $\tau(d) = (d, d, \tau_d)$, and on morphisms as $\tau(f) = (f, f)$, given $f\colon d \to d'$. Then we have $P\tau = Q\tau = \mathrm{id}_D$ as functors. Note further that $(f, f)\colon \tau(d) \to \tau(d')$ in $(T \downarrow S)$; this square commutes by naturality.

Conversely, given $\tau\colon D \to (T \downarrow S)$ such that $Q\tau = P\tau = \mathrm{id}_D$, we have $\tau(d) = (d, d, f_d\colon Td \to Sd)$ for some f_d. If $g\colon d \to d'$ is a morphism in D, we obtain a square

$$\begin{array}{ccc} Td & \xrightarrow{Tg} & Td' \\ \downarrow{\scriptstyle f_d} & & \downarrow{\scriptstyle f_{d'}} \\ Sd & \xrightarrow{Sf} & Sd' \end{array}$$

which defines a natural transformation $T \to S$.

5. Given a diagram

$$\begin{array}{ccccccccc} & & & & X & & & & \\ & \swarrow^{P'} & & & \downarrow{\scriptstyle R'} & & & \searrow^{Q'} & \\ E & \longrightarrow & C & \longleftarrow & C^{\mathbf{2}} & \longrightarrow & C & \longleftarrow & D \end{array} \tag{3}$$

with bottom defined as in Mac Lane, we claim there is a unique functor $L\colon X \to (T \downarrow S)$ such that $P' = PL$ and likewise for Q' and R'. Set

$$L(x) = (P'(x), Q'(x), R'(x)]$$

on objects. Note that thanks to the definition of the bottom of (3), we have $R'(x)\colon T(P'(x)) \to S(Q'(x))$. Given $f\colon x \to x'$ in X, we get $P'(f)\colon P'(x)$, and after applying T we obtain a diagram

$$\begin{array}{ccc} TP'(x) & \xrightarrow{TP'f} & TP'(x'0 \\ \downarrow{\scriptstyle R'(x)} & & \downarrow{\scriptstyle R'(x')} \\ SQ'(x) & \xrightarrow{SQ'(x)} & SQ'(x') \end{array}$$

so we set $L(f) = (P'(f), Q'(f))$. We obtain the necessary factorization properties on L by construction; see 3.

Uniqueness of L follows from the fact that if $L'\colon X \to (T \downarrow S)$, then $L'(x) = (L'(x)_1, L'(x)_2, L'(x)_3)$. Now $P' = PL = PL'$, so that $L'(x)_1 = P'(x)$ and so on. This describes the comma category as a pullback.

6. (a) Let C, D, E be fixed small categories. We claim that $(T, S) \mapsto (T \downarrow S)$ is the object function of a functor $L\colon (C^E)^{\text{opp}} \times C^D \to \mathbf{Cat}$. Given natural transformations $\tau\colon T' \to T$ and $\sigma S \to S'$ defining a morphism (τ, σ) in the source category, define $L((\tau, \sigma))\colon (T \downarrow S) \to (T' \downarrow S')$ as follows. On objects put $L((\tau, \sigma))\colon (f\colon Te \to Sd) \mapsto (\sigma_d \circ f \circ \tau_e\colon T'e \to S'd)$. On morphisms, map the square

$$\begin{array}{ccc} Te & \xrightarrow{Tk} & Te' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Sd & \xrightarrow{Sh} & Sd' \end{array}$$

defined by (h, k) to the rectangle

$$\begin{array}{ccc} Te & \xrightarrow{Tk} & Te' \\ \downarrow{\scriptstyle \tau_e} & & \downarrow{\scriptstyle \tau_{e'}} \\ Te & & Te' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Sd & & Sd' \\ \downarrow{\scriptstyle \sigma_d} & & \downarrow{\scriptstyle \sigma_{d'}} \\ S'd & \xrightarrow{S'h} & S'd' \end{array}$$

which commutes, as the small square obtained by adding Tk and Sh commute, by naturality of τ and σ.

(b)

todo

2.5 Graphs and Free Categories

In this section we use “graph” to mean ”O-graph” where O is obvious or specified immediately following.

1. Given an O-graph G, we define the opposite graph G^{opp}. Set $O^{\text{opp}} = O$ and define A^{opp} by $\partial_0 f^{\text{opp}} = \partial_1 f$ and $\partial_1 f^{\text{opp}} = \partial_0 f$. We then have $U(C^{\text{opp}}) = U(C)^{\text{opp}}$ as graphs, if $U\colon$ **Cat** $\to \mathbb{G}\smallsetminus\imath\eqsim$ is the forgetful functor.

   We next define the product graph, so that the notion of products will also be preserved by U. Let G, H be O-graphs. Put $O_{G\times H} = O_G \times O_H$, the product being in **Set**. Define $A_{G\times H} = A_G \times A_H$ likewise, with $\partial_0(f_1, f_2) := (\partial_0 f_1, \partial_0 f_2)$ and likewise for ∂_1. We again have $U(C_1 \times C_2) = U(C_1) \times U(C_2)$ as graphs.

2. Let $n = \{0, 1, \ldots, n-1\}$ be a finite ordinal number. We claim n is the free category on the O-graph

$$\cdot \longrightarrow \cdot \longrightarrow \cdots \longrightarrow \cdot$$

   where the dots are labled from 0 to $n-1$. Then adding all the necessary composite arrows, we see we obtain the n-simplex, as required. More rigorously, it is clear that n satisfies the universal property in theorem 1 of this section of Mac Lane.

3. We construct the free groupoid generated by a graph G and establish its universal property. Given G, define G' by $O_{G'} = O_G$ and $A_{G'} = A_G \cup A_{G^{\text{opp}}}$ (see 1). Then $C(G') =: F$ is a groupoid after adding the relation that the string $a_i \xrightarrow{f_i} a_{i+1} \xrightarrow{f_i^{\text{opp}}} a_i$ is equal to the string (a_i), *i.e.* the identity morphism id_{a_i}. We can define $P\colon G \to UF$ via the same formula as for free categories. Given a morphism of graphs $U\colon G \to UE$ for a groupoid E, the same construction as for general categories yields a functor $D'\colon F \to E$, which is a morphism of groupoids.

   When G has a single vertex, we can identify G with the set $X := A_G$ to see that every set X generates a free group $F(X)$ with the usual universal property.

2.6 Quotient Categories

1. This is mostly notation. Let G be the given square graph, and label its vertices x, y, z, w starting from the tail of f and proceeding clockwise. Let R be the relation such that $g'f = f'g$ and let $C = C(G)$. Then $\mathrm{Obj}(C/R) = \mathrm{Obj}(C) = \{x, y, z, w\}$, and $\mathrm{Mor}(C) = \{\mathrm{id}_k, f, g, fg', g'f \mid k \in \mathrm{Obj}(C)\}$. The quotient category C/R has for morphisms the union of the hom-sets $(C/R)(k, k) = \mathrm{id}_k$, $C(C/R)(x, w) = \{f\}$ and so on, with the only change being $(C/R)(x, z)$ is now a singleton. This gives four identity and five non-identity arrows in the quotient.

2. Let $C = G$ be a groupoid, viewed as a groupoid with one object, and let R be a congruence on C. Then we claim there is $N \triangleleft G$ such that fRg iff $g^{-1}f \in N$, *i.e.* $g \equiv f \mod N$. Let F be the free group on the set $\mathrm{Mor}\, G$ (see 3 from the last section). By the universal property, we obtain a morphism of groups $Q\colon F \twoheadrightarrow G$. Let $N = \ker Q$. Then if fRg, $f \equiv g \mod N$, and conversely if $Qf = Qg$ then f and g differ by an element (strictly speaking, a morphism of C) in N.

3 Universals and Limits

3.1 Universal Arrows

1. (a) The integral group ring $\mathbb{Z}G$ of a group G has the following universal property. If $\varphi\colon G \to R^\times$ is a group homomorphism for a ring R, then there is a unique ring morphism $\bar{\varphi}\colon \mathbb{Z}G \to R$ such that the obvious triangle commutes. Therefore $(\mathbb{Z}G, r\colon G \hookrightarrow (\mathbb{Z}G)^\times)$ is a universal arrow from G to $(-)^\times\colon$ **Rng** $\to$ **Grp**.

   Changing “group” to “monoid” throughout, the same reasoning goes through.

(b) The tensor algebra has the following universal property. Given a linear map $V \to A$ from a vector space V to an associative algebra A, there is a unique map $TV \to A$ of algebras such that the obvious triangle commutes. This says that $(TV, \iota\colon V \hookrightarrow TV)$ is a universal arrow from V to $U\colon \mathbf{Alg}_K \to \mathbf{Vect}_K$, the functor forgetting the ring structure of A.

(c) The exterior algebra has the universal property that given a linear map $j\colon V \to A$ for an associative A, such that $j(v)j(v) = 0$ for all $v \in V$, there exists a unique arrow $\bar{j}\colon \Lambda V\colon A$ such that the obvious triangle commutes. This says that $(\Lambda V, \iota\colon V \hookrightarrow \Lambda V)$ is a universal arrow from V to the forgetful functor form the full subcategory of K-algebras with alternating multiplications to $\mathbf{Vect}_K$.

2. Let $\mathcal{P}\colon \mathbf{Set}^{\text{opp}} \to \mathbf{Set}$ be the contravariant power set functor, and let d be a set. Let $r = \{0, 1\}$ and let $f\colon r \to d$ be the characteristic function χ_x for $x \subset d$. Then if $e = \{1\}$, $f^{-1}(e) = x$ and (r, e) is a universal element for $\mathcal{P}$.

3. - We construct a universal arrow from G to the forgetful functor $U\colon \mathbf{Ab} \to \mathbf{Grp}$. Let $A = G/[G, G]$ be the abelianization of G and let u be the natural map. Then for any morphism of groups $\varphi\colon G \to B$ with B abelian, we have $\varphi = \bar{\varphi} \circ u$ uniquely, and hence $\varphi = U(\bar{\varphi}) \circ u$ in $\mathbf{Grp}$, as required.
   - todo
   - Let X be a set. We construct a universal arrow from X to the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$. Let Z be X with the discrete topology and set $f'(x) = f(x)$ given $f\colon X \to UY$ for a topological space Y. If $u(x) = x$ sends X into Z, we have $Uf' \circ u = f$ as required.
   - Let X be a set. We construct a universal arrow from X to the forgetful functor $U\colon \mathbf{Set}_* \to \mathbf{Set}$. Let Z be $X \cup \{*\}$ and define $f'(x) = \begin{cases} f(x) & \text{if } x \in X \\ *_Y & \text{if } x = * \end{cases}$ from Z to a pointed set Y, given $f\colon X \to Y$. If $u(x) = x$ sends X into Z, we have $Uf' \circ u = f$ as required.

4. The first isomorphism theorem from groups follows from the exposition in this section, namely p.57.

(a) There is a typo in this question. The correct right-hand side of the desired isomorphism is G/N. The third isomorphism theorem follows by applying the first to the surjective morphism of groups $G/M \to G/N$ given by $gM \mapsto gN$. This is well-defined as $M \leq N$.

(b) The second isomorphism theorem again follows from the first. We clearly have $S \cap N \triangleleft S$ and $N \triangleleft SN$. Define a morphism $S \to SN/N$ by $s \mapsto sN$. This is surjective as $snN \equiv sN \mod N$, and has kernel $S \cap N$.

5. We describe the quotient A/S of a K-module by a submodule in terms of universality. The quotient module has the universal property that if $f\colon A \to M$ is a morphism of K-module such that $f(S) = 0$, then f factors through A/S. This says the functor $H\colon K\text{-}\mathbf{Mod} \to \mathbf{Set}$ sending $M \mapsto \{f\colon A \to M \mid f(X) = 0\}$ has universal element $(A/S, \pi)$, where π is the natural (quotient) map. All the isomorphism theorems now follow with the above proofs.

Remark 4. All such that functors H (c.f. p. 57) act on morphisms by $Hf = f_*$, so the condition $(Hf)e = x$ is equivalent to the usual commuting triangle condition.

6. We describe two-sided ideals $I \triangleleft R$ by universality. If $I \triangleleft R$, then we have the following universal property: any morphism of rings $R \to S$ sending I to 0 factors through R/I. This says that $(R/I, \pi)$, where π is the quotient map, is a universal element for the functor $H\colon \mathbf{Rng} \to \mathbf{Set}$, defined similarly to H above.

7.

3.2 The Yoneda Lemma

3.2.1 Proof of the Yoneda Lemma

For a very detailed proof of naturality of the Yoneda lemma, see the online notes *Linear Algebraic Groups* by Tom De Medts, §3.3.

Proposition 2 (Yoneda)**.** *If $K\colon D \to \mathbf{Set}$ is a functor from a locally small category D, there is a bijection*

$$\mathrm{Nat}(D(r,-),K) \xrightarrow{\sim} Kr$$

given by

$$\alpha\colon D(r,-) \to K \mapsto \alpha_r(\mathrm{id}_r).$$

Proof. Injectivity of the above mapping follows from the diagram below. Given a morphism $f\colon r \to d$ and a natural transformation α, we have

$$\begin{array}{ccc}
\mathrm{id}_r & \longmapsto & f \\
\big\downarrow & \begin{array}{ccc} D(r,r) & \xrightarrow{f_*} & D(r,d) \\ \downarrow{\scriptstyle \alpha_r} & & \downarrow{\scriptstyle \alpha_d} \\ K(r) & \xrightarrow{K(f)} & K(d) \end{array} & \big\downarrow \\
\alpha_r(\mathrm{id}_r) & \longmapsto & K(f)(\alpha_r(\mathrm{id}_r)),
\end{array}$$

which says that for any d and f, $\alpha_d(f) = K(f)(\alpha_r(\mathrm{id}_r))$. Therefore the mapping above is injective. To show surjectivity, let $\xi \in K(r)$. Then define $\alpha_d(f) := K(f)(\xi)$ for $f \in D(r,d)$. This is natural: if $g\colon d \to d'$, we have

$$\begin{array}{ccc}
f & \longmapsto & g \circ f \\
\big\downarrow & \begin{array}{ccc} D(r,d) & \xrightarrow{g_*} & D(r,d') \\ \downarrow{\scriptstyle \alpha_r} & & \downarrow{\scriptstyle \alpha_d} \\ K(r) & \xrightarrow{K(g)} & K(d') \end{array} & \big\downarrow \\
K(f)(\xi) & \longmapsto & K(g)(K(f))(\xi) = K(g \circ f)(\xi).
\end{array}$$

□

1. Set up as in the text. The vertical composition $(\psi')^{-1} \circ \tau \circ \psi\colon D(r,-) \to D(r',-)$ is a natural transformation between functors of the form $D(s,-)$. By the corollary to the Yoneda lemma, there exists a unique morphism $h\colon r' \to r$ such that $(\psi')^{-1} \circ \tau \circ \psi = D(h,-)$, so $\tau \circ \psi = \psi' \circ D(h,-)$.

2. We state and prove the dual of the Yoneda lemma. (Of course, such a proof is not necessary given the proof of the above. Our notation is that of algebraic geometry. Note that the part of the proof of injectivity is dual to the proof of surjectivity above, and vice-versa.

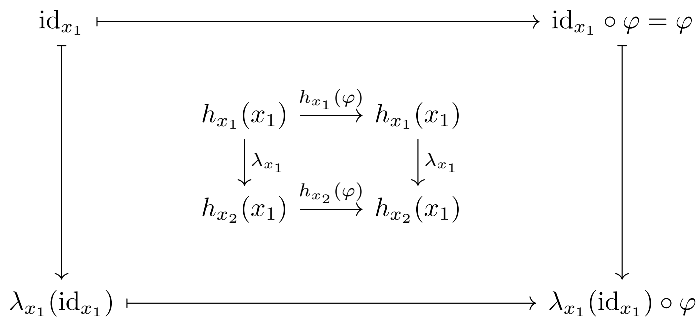
   Let C be a locally small category and consider the **Set**-valued presheaf $h_x\colon C^{\mathrm{opp}} \to \mathbf{Set}$ where $h_x\colon z \mapsto \mathrm{Hom}_C(z,x)$ and if $f\colon z_1 \to z_2$, $h_x(f)\colon \mathrm{Hom}_C(z_2,x) \to \mathrm{Hom}_C(z_1,x)$ by pulling back by f, *i.e.*, $g \mapsto g \circ f$. We claim there is a bijection

$$\mathrm{Hom}_C(x_1,x_2) \to \mathrm{Nat}(h_{X_1},h_{X_2}) \quad f \mapsto h_f.$$

Indeed, we have $\mathrm{id}_{x_1} \in \mathrm{Hom}_C(x_1, x_2)$, and $f\colon x_1 \to x_2$ gives rise to $h_f(z)\colon h_{x_1}(z) \to h_{x_2}(z)$ sending $g \mapsto f \circ g$. To recover f, put $z = x_1$ and $g = \mathrm{id}_{x_1}$, then $f = f \circ \mathrm{id}_{x_1} = h_f(x_1)$ and $f \mapsto h_f$ is injective. Given $\lambda \in \mathrm{Nat}(h_{x_1}, h_{x_2})$, we get

$$\lambda_{x_1} : h_{x_1}(x_1) \to h_{x_2}(x_1).$$

Set $\lambda_{x_1}(\mathrm{id}_{x_1}) = f$. We claim $\lambda = h_f$. Let $\varphi \in h_{x_1}(x_1)$ and consider the diagram



The bottom right corner shows that $\lambda_{x_1}(\mathrm{id}_{x_1}) \circ \varphi = \lambda_{x_1}(\varphi) = f \circ \varphi$ as required.

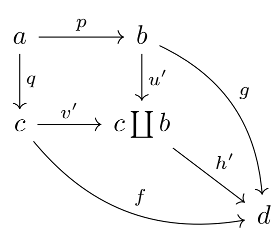
3. todo

4. (Naturality not changed by enlarging target.) Let $J\colon E \to E'$ be inclusion of a full subcategory. Let $K, L\colon D \to E$ be functors. We claim $\mathrm{Nat}(K, L) \simeq \mathrm{Nat}(JK, JL)$. Given $\tau\colon K \to L$, we obtain components $(J\tau)_s L = J(\tau_s)$. This defines $J(\tau)\colon JK \to JL$, which is natural as J is a functor. As J is inclusion, we have $J(\tau_s) = \tau_s\colon Ks \to Ls$, so $\tau \mapsto J\tau$ is injective. Given $\sigma\colon JK \to JL$ natural, we see that $\sigma_s \in \mathrm{Mor}\, E$ by fullness. This shows surjectivity.

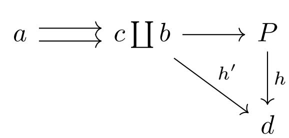
3.3 Coproducts and Colimits

1. We claim $R \to R \otimes S \leftarrow S$ is a coproduct diagram in **CRng**. Here we view $R \otimes S = R \otimes_{\mathbb{Z}} S$ as the tensor product of $\mathbb{Z}$-algebras. The maps are $r \mapsto r \otimes 1$ and likewise for S. If we have maps $f\colon R \to T$ and $g\colon S \to T$ such that the obvious triangle commutes, then define $\tilde{h}\colon R \times S \to T$ by $h((r, s)) = f(r)g(s)$. This is bilinear, so induces a map of $\mathbb{Z}$-modules $h\colon R \otimes S \to T$. We must check h is multiplicative, but we can do so on elementary tensors, using that the rings involved are commutative.

2. We claim that if a category has binary coproducts and coequalizers, it has pushouts. Consider the span below, with maps f and g out of it. We obtain a map $h\colon c \coprod b \to d$ such that the diagram minus a commutes.



Now we have, as we have coequalizers, a diagram



for some object P. It follows that P is the required pushout.

In **Set**, we have disjoint unions and quotients of disjoint unions by the equivalence relation $\{(fx, gx) \mid x \in X\} \subset Y \times Y$ for parallel arrows from X to Y, so we have pushouts in **Set**. The same goes for **Top** with the obvious topologies. In **Grp**, we have the free product (our coproduct). For coequalizers, given parallel arrows $\psi, \varphi \colon G \to H$, we let N be the normal closure of $\left\{\varphi(g)\psi(g^{-1}) \,\middle|\, g \in G\right\} \subset H$, so the coequalizer of φ and ψ is $\pi \colon H \to H/N =: E$. Therefore pushouts exist in **Grp**, where are the free product with amalgamation.

3. Let $A, B \in \operatorname{Obj} \mathbf{Matr}_K$. We describe their coequalizer as parallel arrows $n \to m$. In the category of vector spaces, after picking bases we would set $d = k^m/(\operatorname{im}(()A - B))$ and let $u \colon k^m \to d$ be the quotient map. If $s = \operatorname{rank}(A - B)$, then the coequalizer is $u \colon m \to m - s$.

4. We describe coproducts in **Cat**, **Mon** and **Grph**, and prove they exist.

   In **Cat**, define $C \coprod D$ by $\operatorname{Obj} C \coprod D = \operatorname{Obj} C \coprod \operatorname{Obj} D$ and likewise for morphisms. Note this means $C \coprod D$ is small when C and D, and likewise for being locally small. That $C \coprod D$ is a category follows from the fact that C and D are. We have functors $i \colon C \to C \coprod D \leftarrow D \colon j$ with obvious definitions. Note they are both full and faithful. The universal property is clearly satisfied.

   Now let C and D be monoids. Let M be a category with one object, and morphisms all reduced words in $\operatorname{Mor} C \cup \operatorname{Mor} D$. Define functors $i \colon C \to M$ be inclusion of morphisms as one-symbol words, and likewise for $j \colon D \to M$. It is clear M is $C \coprod D$ in **Mon**. Note that even though **Mon** is a full subcategory of **Cat**, the coproduct in **Mon** is not the coproduct in **Cat**.

   In the category of graphs, we again define the coproduct by taking disjoint unions of vertices and edges. Essentially, we draw the graphs side-by-side. The universal property is again clear.

5. Let $E \subset X \times X$ be an equivalence relation on a set X. We describe the quotient X/E as a coequalizer. Define $p_1 \colon E \to X$ by projection onto the first factor, and p_2 likewise. Then we have a diagram

$$E \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{\ \pi\ } X/E$$

   such that $h \colon X \to Y$ descends to the quotient precisely when $h(p_1(x)) = h(p_2(x))$ for $x = (x_1, x_2) \in E$, *i.e.* $x_1 E x_2$.

6. We claim the coproduct $a \coprod b$ exists in a category C if f the functor $K := C(a, -) \times C(b, -) \colon C \to \mathbf{Set}$ is representable. Let $\psi \colon C(r, -) \xrightarrow{\sim} K$ be a representation. This says immediately there is a bijection between pairs f, g and arrows h in the diagram

$$\begin{array}{ccccc} a & \xrightarrow{i} & r & \xleftarrow{j} & b \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle g} & \\ & & c, & & \end{array}$$

   where we defined $(i, j) \in K(r)$ as the image of identity in $C(r, r)$ under ψ. For h as above, naturality gives a square

$$\begin{array}{ccccc} \mathrm{id}_r & \longmapsto & & \longmapsto & (i, j) \\ \big\downarrow & C(r, r) & \xrightarrow{\psi_r} & C(a, r) \times C(b, r) & \big\downarrow \\ & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle \alpha_d} & \\ \big\downarrow & C(r, c) & \xrightarrow{\psi_c} & C(a, c) \times C(b, c) & \big\downarrow \\ h & \longmapsto & & \longmapsto & (f, g). \end{array}$$

Thus the first diagram commutes and $r = a \coprod b$ (up to unique isomorphism).

Conversely, if $a \coprod b$ exists, we again get a bijection $C(a \coprod b, -) \xrightarrow{\sim} K$. If $\varphi \colon c \to c'$, then uniqueness forces the square

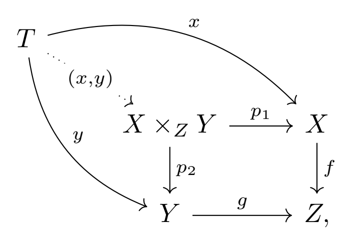
$$\begin{array}{ccc} C(a \coprod b, c) & \longrightarrow & C(a,b) \times C(b,c) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle (\varphi_*, \varphi_*)} \\ C(a \coprod b, c') & \longrightarrow & C(a,c') \times C(b,c') \end{array}$$

to commute. Thus K is representable with representing object $a \coprod b$.

7. Every abelian group is the colimit of its finitely-generated subgroups. Precisely, we claim A is the colimit of the inclusion functor $J_A \to \mathbf{Ab}$ of the preoder J_A of finitely- generated subgroups. Let B be an abelian group. Then a natural natural transformation from J_A to ΔB gives components $\tau_{\langle a \rangle}$ for every $a \in A$. Define $\eta(a) = \tau_{\langle a \rangle}(a)$; then $\eta \colon A \to B$. Because $\tau_{\langle a,b \rangle}(a+b) = \tau_{\langle a,b \rangle}(a) + \tau_{\langle a,b \rangle}(b) = \tau_{\langle a \rangle}(a) + \tau_{\langle b \rangle}(b)$ by naturality, η is a group homomorphism with the required universality. The same fact generalizes to R-modules; the only other calculation is $\eta(ra) = \tau_{\langle a \rangle}(ra) = r\tau_{\langle a \rangle}(a) = r\eta(a)$.

3.4 Products and Limits

1. We claim the pullback in **Set** is the usual fibre product. Let $g \colon Y \to Z$ and $f \colon X \to Z$, and let $P = X \times_Z Y$ be the fibre product with projections p_i. Given the solid diagram



we see $\xi := x(a)$ and $\zeta := y(a)$ are such that $g\zeta = f\xi$, so we obtain $h = (x,y)$ by $h(a) = (x(a), y(a))$. This map is clearly unique as $p_1 h = x$ etc. Therefore P is the pullback. In **Top**, the pullback as the same underlying set as above, with the subspace topology.

2. It is obvious the Cartesian product is the categorical product in **Set** and **Top**.

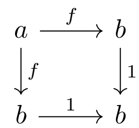
3. Let J have an initial object s. Then we claim all functors $F \colon J \to C$ have limits, namely $F(s)$. Indeed, let $\tau \colon c \to F$ be a cone. As $s \in \operatorname{Obj} J$, there is $\tau_s \colon c \to s$. There is a cone $\sigma \colon F(s) \to F$. Let $t = \tau_s$. We have

$$\begin{array}{ccc} F(s) & \xleftarrow[t]{} & c \\ \downarrow{\scriptstyle \sigma} & \swarrow{\scriptstyle \tau} & \\ F. & & \end{array}$$

The universal property says that, within the cone τ, any $\tau_j \colon c \to F(j)$ factors as $F(\varphi) \circ \tau_s$, where $\varphi \colon s \to j$. Therefore the diagram above commutes, so $F(s) = \lim F$.

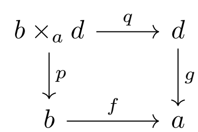
Dually, we see any functor from a category with a terminal object has a colimit.

4. We claim that, in any category, $f \colon a \to b$ is an epimorphism iff the square



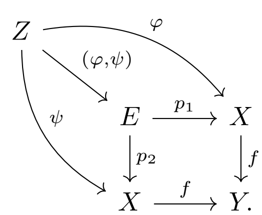
is a pushout. If f is an epimorphism and $\varphi\colon b \to c$ and $\psi\colon b \to c$ such that obvious diagram commutes, then $\varphi = \psi$, so we can define $h\colon b \to c$ to be either and see that the bottom copy of b in the above diagram is a pushout of the rest. Conversely, if the diagram above is a pushout, then given φ and ψ as above we have a unique $h\colon b \to c$ such that $h = \varphi = \psi$. Thus $\varphi f = \psi f$ implies $\varphi = \psi$ and f is an epimorphism.

5. Let

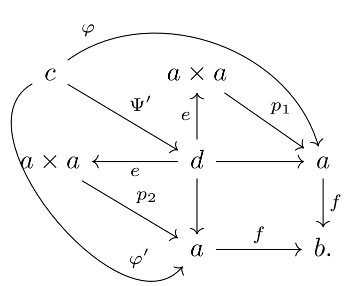


be a pullback square. Then we claim if f is monic, q is monic. Let f be monic and let $\varphi, \psi\colon e \to b \times_a d$ such that $q\varphi = q\psi$. Then $gq\varphi = gq\psi$ so that $fp\varphi = fp\psi$ whence $p\varphi = p\psi$. Therefore we have a cone from X into $b \longrightarrow a \longleftarrow d,$ such that ψ and φ both make the diagram for the universal property of the pullback commute. By the uniquness of this morphism, $\varphi = \psi$. Therefore q is monic.

6. Let $f\colon X \to Y$ be a morphism in **Set**. We claim the kernel pair is $E := \{(x, x') \mid fx = fx'\}$, and projections $p_i\colon E \to X$. The small square in the below commutes by construction. If we have φ, ψ as indicated, then $f\varphi x = f\psi x$ so (φ, ψ) is into E. This says that E is the pullback. The diagram is

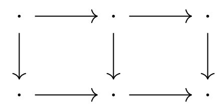


7. Consider a category with finite products and equalizers. Then we can realize the kernel pair of $f\colon a \to b$ in terms of projections $p_i\colon a \times a \to a$ and the equalizer (d, e) of $fp_1, fp_2\colon a \times a \to b$. Consider the diagram



The inner square on its own commutes as $fp_1e = fp_2e$ by the definition of an equalizer. Now φ and φ' define a morphism Ψ into $a \times a$ by $\Psi = (\varphi, \varphi')$ such that $fp_1\Psi = fp_2\Psi$, whence the existence of Ψ'. Everything commutes by construction, and we have the kernel pair.

8. Consider the diagram



(4)

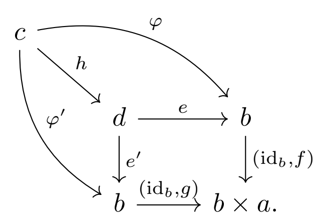
   (a) Suppose both small squares are pullbacks. Then we claim the large square is a pullback. If we have morphisms into the top right and bottom left objects of (4), we obtain a unique morphism into the top middle object. We then obtain a unique morphism into the top left object. We check this new diagram commutes, which proves the claim.

   (b) Suppose the outer and small right squares are pullbacks. Then we claim the left-hand square is too.

   Remark 5. It can happen that the outer and left squares are pullbacks, but not the right square.

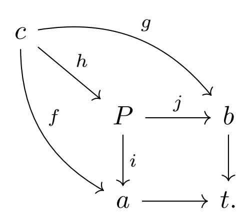
   Consider a cone over the left cospan. Composition then gives a cone over the outer cospan, so we obtain a unique morphism into the top left object. Using uniqueness arguments and the fact that the right-hand square is a pullback, we see this diagram commutes, which proves the claim.

9. Let $f, g\colon b \to a$ be parallel morphisms in a category with products and pullbacks. Then we claim the equalizer of f and g can be realized as a pullback of a product. Let d be a pullback of the following diagram:



The commutativity of the small square and the fact that the rightmost morphisms have a component which is id_b implies that $e = e'$. Taking $\varphi = \varphi'$ we see the cone condition implies $f\varphi = g\varphi$, so d is the equalizer.

10. Let C have all pullbacks and a terminal object t. Then we claim C has all products and equalizers. Let $a, b, c \in \operatorname{Obj} C$. Let P be the pullback in the following diagram



As P is a pullback, h exists and is unique. As t is terminal, any morphisms (f, g) suffice to make the diagram commute. Therefore $P = a \times b$.

Problem 9 now implies that C has equalizers.

3.4.1 The p-adic integers

We give an example of a limit (frequently termed a *projective limit*) in the literature. See the essay of Bill Casselman at `http://www.math.ubc.ca/~cass/research/pdf/Profinite.pdf` for a very nice description in more generality.

The p-adic intergers $\mathbb{Z}_p$ are the limit in the category of rings with 1 (and if the $\mathbb{Z}/p^k\mathbb{Z}$ are given the discrete topology, in the category of topological rings) of the diagram

$$\cdots \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^{n-1}\mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z}/p\mathbb{Z} \tag{5}$$

where the morphism $\pi_{k+1,k}$ into $\mathbb{Z}/p^k\mathbb{Z}$ is reduction modulo p^k. We define $\mathbb{Z}_p \subset \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ as the closure of $\mathbb{Z}$ with respect to the p-adic norm $|\cdot|_p$. The sequences (x_n) such that $\pi_{n+1,n}(x_{n+1}) = x_n$ are Cauchy in this metric, so lie in $\mathbb{Z}_p$.

The limiting cone has morphisms given by projection onto the k-th coordinate. The universal mapping property follows from the fact that given a unital ring R, a cone over (5) implies $\varphi_{k+1}(r) = \pi_{k,k+1}\varphi_k(r)$ for all r. This defines a morphism $R \to \mathbb{Z}_p$ via $r \mapsto (\varphi_k(r))_k$ which satisfies the universal property.

3.5 Categories with Finite Products

1. We claim the diagonal $\delta\colon \mathrm{id}_C \to \times$ is a natural transformation. By definition, we have components $\delta_c\colon c \to c \times c$ defined by $p_1\delta_c = \mathrm{id}_c = p_2\delta_c$. Let $f\colon c \to c'$. We claim

$$\begin{array}{ccc} c & \xrightarrow{\delta_c} & c \times c \\ \big\downarrow f & & \big\downarrow f\times f \\ c' & \xrightarrow{\delta_{c'}} & c' \times c' \end{array}$$

commutes. As $f_1 \times f_2$ is defined by $p_i(f_1 \times f_2) = f_i p_i$, the path around the top of the square is

$$(fp_1\delta_c, fp_2\delta_c) = (f, f)\colon c \to c' \times c'$$

by definition of δ_c. The other path has components $p_i(\delta_{c'} f) = f$, so the diagram commutes. Thus δ_c is natural in c.

2. Consider the diagrams defined on p. 74. The first commutes by uniqueness: both paths are natural isomorphisms $a \times (b \times (c \times d))) \xrightarrow{\sim} ((a \times b) \times c) \times d$. Uniqueness again implies the bottom-left diagram commutes. To apply this reasoning to the final diagram, we need only show that $a \times (t \times c)$ satisfies the universal property for $a \times c$. But this is clear.

3. (a) We claim **Cat** has all pullbacks. Assume the setup of question 5 from §2.6. Let Π be the full subcategory of $(T \downarrow S)$ such that $Te = Sd$. That is, such that that the square

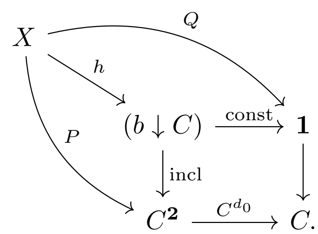
$$\begin{array}{ccc} (T \downarrow S) & \xrightarrow{Q} & B \\ \big\downarrow P & & \big\downarrow T \\ A & \xrightarrow{S} & C \end{array}$$

commutes. Given any commutative square

$$\begin{array}{ccc} X & \xrightarrow{Q'} & B \\ \big\downarrow P' & & \big\downarrow T \\ A & \xrightarrow{S} & C, \end{array}$$

we can define R' as in (5), by $R'x = \mathrm{id}_{T(Q'x)}\colon T(Q'x)) \to S(P'x)) = T(Q'x)$ ($C^{\mathbf{2}}$ being the category of morphisms in C) on objects and $R'(\varphi)_x = TQ'f$. By (5) we obtain $L\colon X \to (T \downarrow S)$ such that $P' = PL$ and $Q' = QL$. Therefore we see that the image of L lies in Π, so Π is the desired pullback.

(b) We show the (co)slice categories $(b \downarrow C)$ and $(C \downarrow a)$ are pullbacks. This requires a different argument than (a), as the corresponding subcategories Π are clearly proper in general. Consider the diagram



i.e. Px is an arrow in C with codomain b. Therefore we can define $h = P$, and as the vertical morphism is inclusion, this is the only h we can pick. Therefore $(b \downarrow C)$ is a pullback. The same diagram with $(C \downarrow a)$ and C^{d_1} shows that $(C \downarrow a)$ is a pullback.

4. Our description of binary coproducts of small categories above involves only disjoint unions of sets. A disjoint union of small sets taken over a small index set is small, so **Cat** has all small coproducts.

   Remark 6. See the nLab article on *Grothendieck universes*.

5. Let B have all finite products, and let C be any category. We claim B^C has all finite products, with calculations done "pointwise." Say we have a diagram

$$G \xleftarrow{\tau_1} F \xrightarrow{\tau_2} H.$$

   Define $\Psi \in \operatorname{Obj} B^C$ by $\Psi c = Gc \times Hc \in \operatorname{Obj} C$, and $\Psi f := Gf \times Hf \colon Gc \times Hc \to Gc' \times Hc'$ if $f \colon c \to c'$. Define further natural transformations $i \colon \Psi \to G$ by $i_c \colon \Psi c \to Gc$ by projection $Gc \times Hc \to Gc$ and likewise define j_c. Then set $\sigma \colon F \to \Psi$ by $\sigma_c = (\tau_{1c} \times \tau_{2c}) \circ \delta_c \colon Fc \to Fc \times Hc$, where δ_c is the diagonal morphism. The obvious product diagram now commutes, so B^C has products.

3.6 Groups in Categories

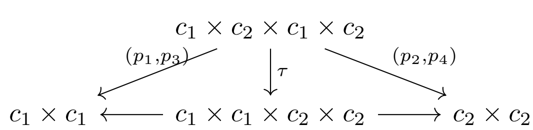
Let C have finite products and let t be a terminal object in C.

1. We describe the category of monoids in C. The objects are monoids (c, μ) in C, and morphisms $(c, \mu, \eta) \to (c', \mu', \eta')$ are morphisms $f \colon c \to c'$ in C such that $f \circ \mu = \mu' \circ (f \times f) \circ \delta$, where δ is the diagonal morphism. We must also require that $f \circ \eta = \eta'$.

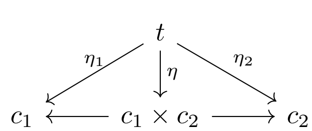
   Let M_1 and M_2 be monoids in C, and define $M = (c, \mu, \eta)$ with $c = c_1 \times c_2$ and mu the composite (suppressing parentheses in the codomain product)

$$c_1 \times c_2 \times c_1 \times c_2 \xrightarrow{\tau} c_1 \times c_1 \times c_2 \times c_2 \xrightarrow{\mu_1 \times \mu_2} c_1 \times c_2,$$

   where the twist τ is defined by the diagram



   The morphisms (p_i, p_j) are defined, as we can first map to $c_1 \times t \times c_1 \times t$, and then use the isomorphisms $a \times t \to a$. The unit η is induced defined by the diagram



The two diagrams for "associativity of multiplication" and η commute because morphisms of products are uniquely determined by their components, and the diagrams commute for M_1 and M_2 by hypothesis.

2. We now claim the category of groups in C have all finite products. By 1, it is enough to show a suitable morphism ξ exists. Given groups G_i in C, define the monoid $G = G_1 \times G_2$ as above, and define ξ by the diagram

$$\begin{array}{ccccc} & & G_1 \times G_2 & & \\ & \swarrow^{\xi_1} & \downarrow \xi & \searrow^{\xi_2} & \\ G_1 & \longleftarrow & G_1 \times G_2 & \longrightarrow & G_2. \end{array}$$

   Again, the necessary diagram for "inverses" in $G \times G$ will commute as its morphisms are defined uniquely by their components, for which the corresponding diagrams commute.

3. Let $T\colon B \to \mathbf{Set}$ be a functor. We claim T is a group in $\mathbf{Set}^B$ iff Tb is a group (in the classical sense) for all $b \in \operatorname{Obj} B$ and Tf is a morphism of groups for all morphisms f in B.

   Say the Tb are groups and the Tf are morphisms of groups. Then define $\mu\colon T \times T \to T$, a natural transformation, by letting $\mu_b\colon Tb \times Tb \to Tb$ be multiplication in Tb. That μ is natural is the statement that the Tf are morphisms of groups. Define $\eta\colon t \to T$, where t is the constant functor to $\{*\}$ by $\eta_b\colon \{*\} \to Tb$, the function selecting the identity in Tb. Naturality follows for the same reason as above. We define ξ by defining each ξ_b in the same way. The necessary diagrams again commute, because the compositions are vertical compositions of natural transformations and the diagrams commute when specialized to their components for every $b \in B$ (this is the statement that the Tb are groups).

   Conversely, say T is a group in $\mathbf{Set}^B$. Then the components of the three diagrams endow the sets Tb with a group structure. Naturality of μ says that Tf is a morphism of groups for all $f \in \operatorname{Mor} B$.

4. (a) Let A be an abelian group in **Set**. Then we claim its multiplication, inverse, and unit maps make it a group in **Grp**. It is enough to show that these morphisms are morphisms of groups, and not just functions. We have

$$\mu((a,b)(a',b')) = \mu((aa',bb')) = aa'bb' = aba'b' = \mu((a,b))\mu((a',b'))$$

   to start, and if $1 = \{e\}$ is the trivial group, then $\eta\colon 1 \to A$ we have $\eta(ee) = \eta(e) = \eta(e)\eta(e)$. Finally, we have

$$\xi(ab) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = \xi(a)\xi(b),$$

   so μ, η, and ξ are morphisms of groups. The three necessary diagrams already hold in **Set**, and so they hold now in **Grp**. Therefore A is a group in **Grp**.

   (b) Now we claim all groups G in **Grp** are abelian groups in **Set**, with μ being multiplication in G (as a set) and so on. It is enough to show that μ and multiplication in G as a set obey the interchange law; η is a morphism of groups, so $\eta(1)$ is the identity in G and then the Hilton-Eckmann argument will show the two agree and that G is abelian. Then (2) from §3.6 will show ξ is inversion in G. That the interchange law is satisfied follows from the fact that $\mu\colon G \times G \to G$ is a morphism of groups, so the claim follows.

4 Adjoints

4.1 Adjunctions

1. This question appears to just ask for a proof of corollary 2 from the previous page. We claim $G\colon A \to X$ has a left adjoint iff for all $x \in \operatorname{Obj} X$, the functor $X(x, G-)\colon A \to \mathbf{Set}$ is representable. If

$\varphi\colon A(F_0x,-)\xrightarrow{\sim}X(x,G-)$ is a representation, then F_0 is the object function of a functor which is left adjoint to G, and φ gives the adjunction.

If we have a representation $\varphi\colon A(F_0x,-)\xrightarrow{\sim}X(x,G-)$, then φ is natural in a. We therefore want a universal arrow from x to G for each x, and we will be done by theorem 2 part (ii). Put $a = F_0x$, let η_x be the image of id under the representation φ. Given $h\colon x \to Gy$, we obtain $\varphi^{-1}(h)\colon F_0x \to y$. By naturality of φ, $h = Gh \circ \eta_x$, which says that η_x is universal. Conversely, an adjunction gives a representation.

2. (Lawvere.) We describe an adjuction (F, G, φ) has an isomorphism $\theta\colon (F \downarrow I_A) \xrightarrow{\sim} (I_X \downarrow G)$ of comma categories such that

$$\begin{array}{ccc} (F\downarrow I_A) & \xrightarrow{\theta} & (I_X\downarrow G) \\ \big\downarrow{\scriptstyle (P,Q)} & & \big\downarrow{\scriptstyle (P,Q)} \\ X\times A & \xrightarrow{\text{id}} & X\times A \end{array} \tag{6}$$

commutes. Given an adjunction, send $f\colon Fx \to a$ to $\varphi(f)\colon x \to Ga$ thus defining an isomorphism θ on objects. That is, $\theta\colon (x, a, f) \mapsto (x, a, varphi(f))$. Next, given a morphism (k, h):

$$\begin{array}{ccc} Fx & \xrightarrow{Fk} & Fx' \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} \\ a & \xrightarrow{h} & a', \end{array}$$

consider the square

$$\begin{array}{ccc} x & \xrightarrow{k} & x' \\ \big\downarrow{\scriptstyle \varphi(f)} & & \big\downarrow{\scriptstyle \varphi(f)'} \\ Ga & \xrightarrow{Gh} & Ga'. \end{array}$$

By naturality of φ in x and a the first square commuting implies the second square does, so θ is defined on morphisms. Doing the same for φ^{-1} shows θ is an isomorphism of categories. Now it is also clear that (6) commutes.

3. We claim the unit $\delta_c\colon c \to c \times c$ of the adjunction $(\Delta, \times, \varphi)$ is the unique arrow in C such that

$$\begin{array}{ccccc} & & c & & \\ & {\scriptstyle 1}\swarrow & \big\downarrow{\scriptstyle \delta_c} & \searrow{\scriptstyle 1} & \\ c & \xleftarrow[p]{} & c\times c & \xrightarrow[q]{} & c \end{array}$$

commutes. Indeed, $\delta_c = \varphi(1)$ is defined by the fact that — as φ is an isomorphism — $\varphi^{-1}\colon \delta_c \mapsto (p\delta_c, q\delta_c) = (1, 1) \in C(c, c) \times C(c, c)$ (c.f. p.82). Therefore the diagram commutes. In **Set**, this means $\delta_c\colon x \mapsto (x, x)$.

4. todo

4.2 Examples of Adjoints

1. Let EV be the exterior algebra of a vector space V. We claim $E\colon \mathbf{Vect}_K \to C$ is left-adjoint to the forgetful functor, where C is the category of graded-commutative associative unital (note this does not meet "graded and all elements commute") K-algebras. The functor $U\colon C \to \mathbf{Vect}_K$ sends an algebra to

its underlying vector space and morphisms to their degree zero component. The universal property of the exterior algebra (which follows from its construction as a quotient of the tensor algebra of V) now says that $\mathrm{Hom}_C(EV, A) \simeq \mathrm{Hom}_{\mathbf{Vect}_K}(V, UA)$. Checking naturality is similar to all the examples below.

2. We show that the forgetful functor $U\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ has a right adjoint $\mathrm{Hom}_{\mathbb{Z}}(R, -)$, where $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ is an R-module via $(r'\cdot\varphi)(r) = \varphi(rr')$. We define a map $\mathbf{Ab}(UM, A) \to R\text{-}\mathbf{Mod}(M, \mathrm{Hom}_{\mathbb{Z}}(R, A))$ by sending
$$\varphi \mapsto m \mapsto (\psi_m\colon 1 \mapsto \varphi(m)).$$
This determines ψ_m completely as $\psi_m(r) = (r\cdot\psi_m)(1) = \varphi(r\cdot m) = \psi_{rm}(1)$. This shows that $m \mapsto \psi_m$ is R-linear, and is totally determined by φ, so the map is injective. We can send
$$(m \mapsto \psi_m) \mapsto (m \mapsto \psi_m(1))$$
in the other direction. We see that
$$(m \mapsto \psi_m(1)) \mapsto (m \mapsto (1 \mapsto \psi_m(1)),$$
so our map is also surjective and an isomorphism. To show naturality we use (ii) of theorem 2 of this section. Indeed, given $f\colon M \to \mathrm{Hom}_{\mathbb{Z}}(R, N)$ in $R\text{-}\mathbf{Mod}$, consider the morphism $g\colon UM \to N$ given by $m \mapsto f(m)(1)$, so that the right-down path of the diagram
$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & \mathrm{Hom}_{\mathbb{Z}}(R, UM) \\ & \searrow^{f} & \downarrow{\scriptstyle g_*} \\ & & \mathrm{Hom}_{\mathbb{Z}}(R, N) \end{array}$$
sends $m \mapsto rm \mapsto f(rm)(1) = (r\cdot f(m))(1) = f(m)(r)$. Therefore the diagram commutes. This says η_M is universal and the hypotheses of (ii) of theorem 2 are met. Therefore we have an adjunction.

Having gong to the trouble of writing down the bijection explicitly, we describe the unit and counit of this important adjunction. The unit $\eta_m\colon m \mapsto (1 \mapsto m)$, os that $\eta_m(r) = rm$ recovers the R-module structure on M. The counit is the image of $\mathrm{id}\colon \varphi \to \varphi$, and sends $\varphi \mapsto \psi_\varphi(1) = \varphi(1)$, *i.e.* ϵ_A is evaluation at $1 \in R$.

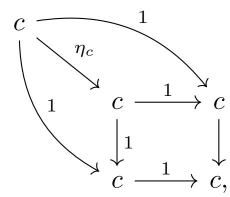
3. Let $U\mathfrak{g}$ be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ over K, so that $U\colon \mathbf{Lie}_K \to \mathbf{Alg}_K$ is a functor. Consider also the functor $V\colon \mathbf{Alg}_K \to \mathbf{Lie}_K$ sending A to the Lie algebra with vector space A and bracket $[x, y] = xy - yx$. Then the construction of $U\mathfrak{g}$ as a quotient of the tensor algebra provides an isomorphism $\mathrm{Hom}_{\mathbf{Alg}_K}(U\mathfrak{g}, A) \simeq \mathrm{Hom}_{\mathbf{Lie}_K}(\mathfrak{g}, VA)$. The PBW theorem provides a convenient basis to check naturality, but we do not do that here.

4. Let $\mathbf{Rng}'$ be the category of rings not necessarily containing a multiplicative identity. We claim that the functor $R \mapsto R^\wedge$ which adjoins 1 to a ring is left adjoint to the forgetful functor $\mathbf{Rng} \to \mathbf{Rng}'$. Define $R^\wedge = \mathbb{Z} \times R$ with multiplication $(a, r)(b, r') = (ab, ar' + br + rr')$. Then $(1, 0)$ is a multiplicative identity in $R^\wedge$. Then given f, there is a unique $\tilde{f}$ such that the diagram
$$\begin{array}{ccc} R & \xrightarrow{\iota} & R^\wedge \\ & \searrow^{f} & \downarrow{\scriptstyle \tilde{f}} \\ & & S \end{array}$$
commutes, where $\iota\colon r \mapsto (0, r)$. Indeed, define $\tilde{f}(a, r) = (a1_S) + f(r)$. Further, $\tilde{f}$ is unique as it must send $(1, 0) \mapsto 1_S$ and $(0, r) \mapsto f(r)$. This gives a bijection, and checking each naturality condition is routine. We conclude the two functors are adjoint.

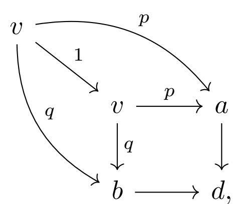
5.

todo

6. We describe the units and counits for the (vertex of) pushout and pullback functors. Let $\mathbb{V}$ be "vertex of pullback" functor, so that $\mathrm{Hom}_{C^{\to\cdot\leftarrow}}(\Delta c, T) \simeq \mathrm{Hom}_C(c, \mathbb{V}T)$. When $T = \Delta c$, we get a diagram



so $\eta_c = 1_c$. If we take $v = \mathbb{V}T$ for a functor (cospan) T, we obtain a diagram



so $\epsilon_T = (p, q)$, the projections from the pullback.

Dually, we have $\eta_c = (i, j)$, the inclusions into the pushout, and $\epsilon = \mathrm{id}_c$ for the "vertex of pushout" functor which is left adjoint to Δ.

7. (a) Let $J = \coprod_k J_k$ be a coproduct of connected categories, and let $I_k : J_k \to J$ be the injection functors. Let $F_k = FI_k$ if $F : J \to C$. If $\lim F_k$ exists for each k, we claim $\lim F \simeq \prod_k \lim F_k$. Indeed, define a natural transformation $r : \Delta \prod_k F_k \to F$ by $r_j = R_j \circ \Delta P_k : \Delta \prod_k \lim F_k \to \Delta \lim F_k \to F(j)$, where k is such that that $j \in \mathrm{Obj}\, J$ is in the connected component labelled by k, and R_j is the natural transformation guaranteed by the universal property of $\lim F_k$. Let $\tau : \Delta c \to F$ be a natural transformation, and define $\nu : \Delta c \to \Delta \prod_k \lim F_k$ by $p_k(\nu)_i = (\nu_k)_i$, where ν_k is the natural transformation guaranteed by $\tau_k : \Delta_c \to F_k$ with components $(\tau_k)_j$ for $j \in J_k$. Conversely, $p_k\nu : \Delta c \to F_k$ such that that the obvious diagram commutes, so ν is unique. Therefore $\prod_k \lim F_k$ is another limit of F, and so is naturally isomorphic to $\lim F$.

   (b) We claim every category is a disjoint union of connected categories. To prove this it suffices to note that "connected to" in the sense of p.86 is an equivalence relation.

   (c) We claim all limits can be obtained from products and limits over connected index categories. Let $F : J \to C$ and keep the above notation. By (b), $J = \coprod_k J_k$, and by (a), $\lim F = \prod_k \lim F_k$. This proves the claim.

8. (a) Let J be a connected category. We claim $\lim \Delta c = \operatorname{colim} \Delta c = c$ for any $c \in \mathrm{Obj}\, C$. If J is connected, then the image of Δc is connected with all arrows the identity. Therefore a cone from the base of Δc to d (dually, from d to the base of Δc) is equivalent to a morphism $c \to d$ ($d \to c$). Therefore $(c, 1)$ is a limit and colimit of Δc.

   (b) We describe the unit for the right adjoint to the diagonal $\Delta : C \to C^J$. We have $\mathrm{Hom}_{C^J}(\Delta c, F) \simeq \mathrm{Hom}_C(c, \lim F)$, and if $F = \Delta c$, we get $\lim \Delta c = c^K$ as the unit if $J = \coprod_{k \in K} J_k$ by (b) and problem 7.

9. We claim the functor $O : \mathbf{Cat} \to \mathbf{Set}$ taking $C \mapsto \mathrm{Obj}\, C$ has left adjoint $D : \mathbf{Set} \to \mathbf{Cat}$ sending X to the discrete category with object set X. Indeed, DX is discrete, so a functor $F : DX \to C$

is totally determined by its values on objects, *i.e.* $\varphi\colon F \mapsto F_0$, the object function, is a bijection. Indeed, given $\varphi\colon X \to OC$, setting $F(x) = f(x)$ and $F(\mathrm{id}_x) = \mathrm{id}_{f(x)}$ defines an inverse. Therefore $\varphi\colon \mathbf{Cat}(DX, C) \xrightarrow{\sim} \mathbf{Set}(X, OC)$. Naturality in X follows from the fact that the object function of a composite functor in the composite of the object functions, and the object function of Df if f. Naturality in C follows similarly, except now we use that the object function of T is OT.

We claim that D also has a left adjoint $T\colon \mathbf{Cat} \to \mathbf{Set}$, sending C to the set of its connected components. As DX is discrete, a functor $C \to DX$ is constant on objects on connected components of C (as $T\varphi = \mathrm{id}$ on all morphisms). Therefore given $F\colon C \to DX$, we get $f\colon TC \to X$ by $f(K) = f(k)$ for any $k \in K \subset \operatorname{Obj} C$ a connected component. Conversely, we can define such a functor by picking the image of each connected component. Therefore this defines a bijection $\varphi\colon \mathbf{Set}(TC, X) \xrightarrow{\sim} \mathbf{Cat}(C, DX)$. Naturality in X is as follows. Around the obvious square, $g \in \mathbf{Set}(TC, X)$ is sent along the top-down path to $(Df) \circ G$ defined by $((Df) \circ G)(k) = (f(g(k)))$, and along the down-bottom path to the functor H defined by $H(k) = (f \circ g)(k)$. Naturality is also straightforward.

Finally, we claim O has a right adjoint given by $S\colon \mathbf{Set} \to \mathbf{Cat}$ sending X to the category with objects X and singleton Hom-sets. In this category, all triangles commute. Therefore sending $f\colon OC \to X$ to the functor F such that $F(c) = f(c)$ on objects and setting $F(\varphi)$ is the unique element of $(SX)(f(c), f(c'))$. As there is no choice for $F\varphi$, this is a bijection $\mathbf{Set}(OC, X) \xrightarrow{\sim} \mathbf{Cat}(C, SX)$. Naturality is again routine.

10. Let C be a category with cokernel pairs and equalizers. Let $K\colon C^{\mathbf{2}} \to C^{\downarrow\downarrow}$ be the functor sending an arrow to its cokernel pair, and $E\colon C^{\downarrow\downarrow} \to C^{\mathbf{2}}$ be the functor sending a pair of parallel arrows tot their equalizer. We define a map $\varphi\colon (C^{\mathbf{2}})(f, EG) \to C^{\downarrow\downarrow}(Kf, G)$. Note that a morphism in $C^{\downarrow\downarrow}$ is a pair of commuting squares, and given a morphism

$$\begin{array}{ccc} a & \xrightarrow{h'} & c \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle EG} \\ f & \xrightarrow{k} & d. \end{array}$$

consider the diagram

$$\begin{array}{ccc} a & \xrightarrow{h'} & c \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle EG} \\ b & \xrightarrow{k} & d \\ {\scriptstyle u}\downarrow\downarrow{\scriptstyle v} & & {\scriptstyle G_1}\downarrow\downarrow{\scriptstyle G_2} \\ r & \xrightarrow{R} & e. \end{array}$$

The map R is induced as follows. We have $G_1 E G h' = G_2 E G h'$, so that $G_1 k f = G_2 k f$. Therefore we obtain a unique map $R\colon r \to e$ such that that the lower squares formed of the rightmost and leftmost vertical arrows on each side each commute. Uniqueness of r says this map φ is injective. For surjectivity, suppose we are given

$$b \underset{u}{\overset{v}{\rightrightarrows}} r$$

and

$$d \underset{G_1}{\overset{G_2}{\rightrightarrows}} e$$

such that the diagrams

$$\begin{array}{ccccc} a & \xrightarrow{f} & b & \xrightarrow{h} & d \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle G_1} \\ & & r & \xrightarrow{g} & d. \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} a & \xrightarrow{f} & b & \xrightarrow{h} & d \\ & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle G_2} \\ & & r & \xrightarrow{g} & d. \end{array}$$

commute. Then we have $G_1hf = guf$ and $G_2hf = gvf$, so because Kf is the cokernel pair of f, $uv = vf$ and $G_1hf = G_2hf$. Therefore we get a unique morphism $h'\colon a \to c$ such that $h'EG = hf$, so

$$\begin{array}{ccc} a & \xrightarrow{h'} & c \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle EG} \\ f & \xrightarrow{h} & d. \end{array}$$

commutes. Therefore φ is a bijection. We do not check naturality here.

11. Let C have finite products. Then we claim that the projection $Q\colon (a \downarrow C) \to C$ sending $f\colon a \to c \mapsto c$ has left adjoint $T\colon c \mapsto (a \to a \coprod c)$. We define a map $(a \downarrow C)(Tc, f) \to C(c, Qf)$ by sending a triangle

$$\begin{array}{ccc} & a & \\ {\scriptstyle Tc}\swarrow & & \searrow{\scriptstyle f} \\ a \coprod c & \xrightarrow{h} & b \end{array}$$

to $\varphi(h) = hj\colon c \to b$. Here j is the canonical inclusion $c \to a \coprod c$. Conversely, send $h'\colon c \to b$ to the triangle

$$\begin{array}{ccc} & a & \\ {\scriptstyle Tc}\swarrow & & \searrow{\scriptstyle f} \\ a \coprod c & \xrightarrow{h} & b, \end{array}$$

where h is unique such that

$$\begin{array}{ccccc} a & \xrightarrow{i} & a \coprod c & \xleftarrow{j=Tc} & a \\ & {\scriptstyle h'}\searrow & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle f} & \\ & & b & & \end{array}$$

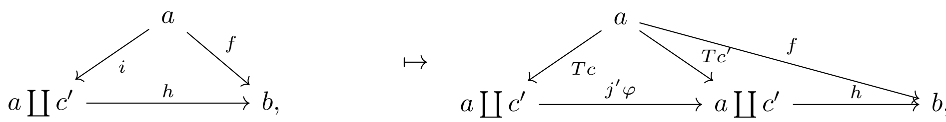
commutes. These assignments are inverse, so we have defined a bijection. For naturality in c, say $\varphi\colon c \to c'$ and let $T\varphi$ be the morphism induced by Tc and $j' \circ \varphi\colon c \mapsto a \coprod c'$. Then we have the square

$$\begin{array}{ccc} (a \downarrow C)(Tc, f) & \longrightarrow & C(c, Qf) \\ {\scriptstyle (T\varphi)^*}\uparrow & & \uparrow{\scriptstyle \varphi^*} \\ (a \downarrow C)(Tc', f) & \longrightarrow & C(c', Qf). \end{array}$$

The right-up path sends

$$\begin{array}{ccc} & a & \\ {\scriptstyle i}\swarrow & & \searrow{\scriptstyle f} \\ a \coprod c' & \xrightarrow{h} & b, \end{array} \quad \mapsto \quad \begin{array}{c} c' \\ \downarrow{\scriptstyle hj'} \\ b \end{array} \quad \overset{\varphi^*}{\mapsto} \quad \begin{array}{c} c \\ \downarrow{\scriptstyle \varphi} \\ c' \\ \downarrow{\scriptstyle hj'} \\ b, \end{array}$$

whereas the up-right path sends



which is then sent to $h \circ (j' \circ \varphi)$, whence naturality. Given $h\colon f \to f'$ in the comma category, showing naturality in f is similar.

12. We claim the copower is left-adjoint to the power, in categories where both exist. We note the relevant exposition in Mac Lane furnishes bijections $C(X \cdot c, d) \xrightarrow{\sim} C(c, d^X)$ for a set X, but we do not check naturality here.

4.3 Reflective Subcategories

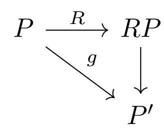
1. We find the dual statements corresponding to the statements
   (a) $S, T\colon C \to B$ are functors;
   (b) T is full;
   (c) T is faithful;
   (d) $\eta\colon S \to T$ is a natural transformation;
   (e) (F, G, φ) is an adjunction;
   (f) η is the unit of (F, G, φ).

   As the statement "$\iota = 1_c$" is self-dual, and composite morphisms are sent to composite morphisms upon reversing arrows, we see the dual statement to (a) is is (a). That T is full is the statement "$\forall g\colon Tc \to Tc'$ in B, $\exists g'\colon c \to c'$ such that $g = Tg'$." The dual sentence is "$\forall g\colon Tc \leftarrow Tc'$ in B, $\exists g'\colon c \leftarrow c'$ such that $g = Tg'$," and therefore the dual of (b) is (b). Likewise, the dual statement for (c) is (c). The arrows in the square for each component of η will reverse but still commute, so the dual of (d) is (d). If (F, G, φ) is an adjunction, then we have a bijection $\varphi\colon A(Fx, a) \to X(x, Ga)$, so dualizing we obtain $\varphi^{-1}\colon X(Ga, x) \to A(a, Fx)$. Therefore the dual of (e) is "(G, F, φ^{-1}) is an adjunction." The statement "this morphism is the identity" is its own dual, and so by the above it follows that the dual of (f) is "η is the counit of (G, F, φ^{-1})."

2. We claim the full subcategory C of torsion-free abelian groups is reflective in **Ab**. Let $F\colon \mathbf{Ab} \to C$ be given on objects by $A \mapsto A/A_{\text{tors}}$. As the order of the image of an torsion element divides the order of the element, F is a functor. This also implies that $f\colon A \to B$ must be trivial on A_{tors} if $B \in \operatorname{Obj} C$. Therefore we define $\varphi\colon \mathbf{Ab}(A/A_{\text{tors}}, B) \to \mathbf{Ab}(A, B)$ for torsion-free B by inflation: $\varphi(f)(a) = f([a])$, where $[a]$ is the image of a in the quotient FA. By the last sentence again, any $g\colon A \to B$ descends to FA. Therefore φ is a bijection. Naturality is routine.

3. todo

4. We claim the following two categories are reflective:
   (a) *The full category C of* **Preord** *of partial orders.* Let $K\colon C \to \mathbf{Preord}$ be the inclusion functor. In any preoder, the relation $a \sim b$ iff $a \leq b$ and $b \leq a$ is an equivalence relation on objects. Form therefore the quotient FP of a preorder P with $\operatorname{Obj} FP = P/\sim$, and say $[a] \leq [b]$ in the quotient iff $a' \leq b'$ for all a' and b'. One checks this is well-defined, so F is a functor **Preord** $\to C$. Say $f\colon FP_2 \to P_2$ is monotone in C. Then set $(\varphi f)(a) = \varphi([a])$. In the other direction, inflate: set $(\psi f)([a]) = f(a)$. This is well- defined, as if $[a'] = [a]$ then $a' \leq a$ and $a \leq a'$ whence $f(a) = f(a')$ by monotonicity and the fact that KP_2 is a preoder. Assignments φ and ψ are inverse, so $\varphi\colon C(FP_1, P_2) \xrightarrow{\sim} \mathbf{Preord}(P_1, KP_2)$. To avoid checking naturality, we note that defining $R = IF$ we obtain a universal arrow as follows. Let $g\colon P \to P'$ with P' a partial order. Then g descends to the quotient RP and the diagram



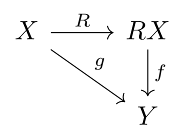
commutes. This shows the inclusion functor has a left adjoint, as the quotient map is universal from P to RP.

(b) *The full category C of* **Top** *of T_0 spaces.* It is enough to produce an endofunctor R of **Top** with values in C, and a universal arrow from X to RX for all spaces X. Define a relation $x \sim y$ for points in X if every neighbourhood of x is a neighbourhood of y and vice versa. This is an equivalence relation. Let RX be the quotient space and π the quotient map. We claim RX is T_0. Let $[x] \neq [y]$ in RX. Thus $x \not\sim y$ and without loss of generality there is $U \ni x$ open in X with $U \not\ni y$. Then $\pi^{-1}(\pi(U)) = \coprod_{u \in U}[u]$. Suppose this set has a boundary point in X. Let $u' \in [u] \subset \pi^{-1}(\pi(U))$ be such that any neighbourhood of u' intersects

$$\{x' \in X \mid x \sim u \text{ for some } u \in U\}^C = \{x'' \in X \mid x'' \not\sim u \ \forall u \in U\}.$$

But $u' \sim x''$ for such x'', and $u' \sim u$ by definition. Therefore $\pi^{-1}(\pi(U))$ is open, so $\pi(U)$ is open in RX. Now $\pi(U) \not\ni [y]$ and it contains no y' from $[y]$; U is not a neighbourhood of y so it cannot be a neighbourhood of any $y' \in [y]$ by definition of $\sim$. Therefore RX is T_0.

Given $g\colon X \to Y$ with Y a T_0 space, we have



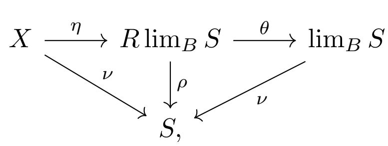
with $f([x]) = g(x)$. This is well-defined, as if $[x] = [x']$, let U be a neighbourhood of $g(x)$. Then $g^{-1}(U)$ is a neighbourhood of x, hence of x' and vice versa, taking a neighbourhood V of $g(x')$. Then $U \ni g(x')$ and $V \ni g(x)$ so $g(x) = g(x')$. The diagram commutes and f is continuous, we have a universal arrow from any X as required. Reflectivity follows.

Remark 7. The space RX is called the *Kolmogorov quotient* of X.

5. Let $(F, G, \varphi)\colon X \to A$ be an adjunction. We claim G is faithful iff φ^{-1} carries epimorphisms to epimorphisms. Let φ^{-1} carry epimorphisms to epimorphisms. Let $x = Fa$, so that $\varphi^{-1}\colon X(Ga, Ga) \to A(FGa, a)$ sends id_{Ga} to $\epsilon_a\colon FGa \to a$, which is than an epimorphism for all a. By theorem 1 of this section, G is faithful. Conversely, let G be faithful. Let $f\colon X \to Ga$ be an epimorphism and say $g \circ \varphi^{-1}(f) = h \circ \varphi^{-1}(f)\colon Fx \to a \to a'$. Then by naturality we have that $Gg \circ f = \varphi(g \circ \varphi^{-1}(f))$, so $Gg \circ f = Gh \circ f$ and $Gg = Gh$. Then $g = h$ and $\varphi^{-1}(f)$ is an epimorphism.

6. todo

7. Let A be a full, reflective subcategory of B. Let $S\colon J \to A$ be a functor with a limit in B. Then we claim S has a limit in A, and moreover, the two limits are isomorphic. Let $\lim_B S$ be the limit of S in B, and let $\varphi\colon A(Rb, a) \xrightarrow{\sim} B(b, a)$. Applying φ^{-1} with $b = \lim_B S$ and $a = Sj$, we obtain morphisms $R\nu_j$ from $R \lim_S B$ to S from the limiting cone with base S. By naturality of φ^{-1} we see this is a cone $\rho\colon \Delta R \lim_B S \to S$. We claim $R \lim_B S$ is isomorphic to $\lim_B S$, so this cone will be limiting. In B, we obtain a unique morphism θ such that $\rho = \nu\theta$ for the original limiting cone nu. Letting $a = R \lim_B S$ in the adjuntion above, we get $\eta = \varphi(\mathrm{id})\colon \lim_B S \to R \lim_B S$ in B. We have a diagram



and by uniqueness $\nu\theta\eta = \nu$ implies $\theta\eta = \mathrm{id}_{\lim_B S}$. Now we claim $\eta\theta$ is the identity in A. We have $\eta\theta\eta = \eta$ by uniqueness, so $\eta\theta = \mathrm{id}_{R\lim_B S}$ by universality of η. Therefore $\eta\theta$ is a morphism in A by fulllness, so η is an isomorphism. Therefore $R\lim_B S$ is a limit in A.

Remark 8. This shows limits in a full reflective subcategory are calculated in the same way as in the ambient category. This should be able to be used to show that some subcategories are not reflective, by demonstrating two non-isomorphic limits.

4.4 Equivalence of Categories

1. (a) We claim that any two skeletons of a category are isomorphic. Let A and A' be skeleta of of C. Then for all $a \in \operatorname{Obj} A$ let $\theta_a^{A'}$ be the isomorphism to an object a' of A'. We can define $\theta^{A'}(f)$ uniquely such that the square

$$\begin{array}{ccc} a & \xrightarrow{\theta_a^{A'}} & a' \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \theta^{A'}(f)} \\ b & \xrightarrow{\theta_b^{A'}} & b' \end{array}$$

commutes. Therefore we obtain a functor $\theta^{A'} \colon A \to A'$, and in the same way a functor $\theta^{A} \colon A' \to A$. Thus the composite $\theta^{A'}\theta^{A} \colon A \to A$ is an isomorphism on objects. As skeleta are skeletal subcategories, this composite must be the identity. Likewise $\theta^{A'}\theta^{A}$ is the identity, and $A \simeq A'$.

(b) Let $A_0 \subset C$ be a skeleton, and $A_0 \subset C_0$ also. Then we claim C is equivalent to C_0 if and only if A_0 is isomorphic to A. Say that $A_0 \simeq A$, and let S, T be functors giving the isomorphism. Let $c \in C$ and define $\tilde{S}c$ by $S(\theta_c(c))$ for the unique isomorphism $\theta_c \colon c \xrightarrow{\sim} a$. Define $\tilde{T}$ the same way. Then

$$\tilde{T}\tilde{S}c = \tilde{T}S\theta_c(c) = T\theta^0_{Sa}S\theta_c(c),$$

but $Sa \in A_0$, so $\theta^0_{Sa} = 1$, so on objects we have

$$c \xrightarrow{\theta_c} a \xrightarrow{S} Sa \xrightarrow{T} TSa = a \xrightarrow[\sim]{\theta_c^{-1}} c.$$

Thus $\theta^{-1} \colon \tilde{T}\tilde{S} \xrightarrow{\sim} I$ is a natural isomorphism. Likewise θ^0 provides an isomorphism in the opposite direction. Therefore the two categories are equivalent.

Remark 9. This tilde construction works kind of like lifting into covering spaces.

Say now C and C_0 are equivalent. Let $S \colon C \to C_0$ be the equivalence. By theorem 1 of this section, S is full, faithful, and all $c_0 \in \operatorname{Obj} C_0$ are isomorphic to Sc for some $c \in \operatorname{Obj} C$. Suppose therefore that we have a triangle

$$\begin{array}{ccc} c_0 & \xrightarrow{\sim} & Sc \\ \Big\downarrow{\scriptstyle \sim} & \nearrow{\scriptstyle g} & \\ Sc', & & \end{array}$$

where g is the inverse of the vertical isomorphism followed by the horizontal isomorphism. Then there is a unique $f \colon c \to c'$ such that $Sf = g$, and f is an isomorphism in C. Thus the image of A under G is a skeleton of C_0. By (a), we have $S(A) \simeq A_0$, so it suffices to show that $S(A)$ is isomorphic to A. We likewise obtain a skeleton $T(A_0)$ of C, so composing with an invertible functor $T(A_0) \to A$, we can assume T takes A_0 to A. Then we have $T(Sa) \xrightarrow{\sim} a$, so $TSa = a$ as both sides of the isomorphism are in A. Conversely, $S(A)$ and $S(TSa) = (ST)(Sa) \simeq Sa \in S(A)$, so $(ST)(Sa) = Sa$. This says the skeleta are isomorphic.

2. (a) We claim the composite of two equivalences $S\colon D \to C$ and $S'\colon C \to A$ is an equivalence $D \to A$. By theorem 1, it is enough to show that $S'S$ is full, faithful, and every $a \in \operatorname{Obj} A$ is isomorphic to some $S'Sd$. The first two statements follow from the corresponding hypotheses about each functor. Let $a \in A$. Then $a \simeq S'c$ for some c and $c \simeq Sd$. Therefore $a \simeq S'Sd$. Thus $S'S\colon D \to A$ is an equivalence.

   (b) We claim the composition two adjoint equivalences (as adjunctions) is again an adjoint equivalence. This follows most easily by applying theorem 1 from §8 of this chapter, using theorem 1 of this section to show that the unit and counit of the composite adjunction are still natural isomorphisms.

3. Let $S\colon A \to C$ be fully faithful and surjective on objects. Then we claim there is an adjoint equivalence $(T, S; 1, \epsilon)\colon C \to A$. By theorem 1 of this section, S is an equivalence of categories, so we have an adjunction
$$A(Tc, Tc) \xrightarrow{\sim} C(c, STc).$$
In the proof of theorem 1, adding surjectivity of S as a hypothesis means we have take the isomorphism η_c to be $1_c\colon c \to S(T_0 c)$ by taking T_0 to be a section of the object function of S. Then $\eta = 1$ is the identity natural transformation. Thus T is a left-adjoint-right-inverse of S.

4. Let $G\colon A \to X$ be a functor. Then we claim the following are equivalent:

   (a) G has a left-adjoint-left-inverse;

   (b) G has a left adjoint, is fully faithful and is injective on objects;

   (c) There exists a full reflective subcategory Y of X with an isomorphism $H\colon A \to Y$ such that $G = KH$, where $K\colon Y \to X$ is the inclusion functor.

   Assume (a). We have $(F, G; \eta, 1)$ an adjunction, so there is a natural bijection $\varphi^{-1}\colon X(Ga, Ga) \to A(FGa, a)$ such that $\varphi^{-1}(1) =: \epsilon_a = 1$. Thus there exists F such that $a = Ia = FGa = (F \circ G)(a)$, so G is injective on objects. As ϵ_a is an isomorphism for all a, G is fully faithful by theorem 1 from §3 of this chapter. Thus (a) implies (b).

   Assume (b). Then $G(A)$ is a category, and let $Y := G(A)$. Then G defines a functor $H\colon A \to G(A)$, which is an isomorphism thanks to the hypothesis on G. Let $T\colon G(A) \to A$ be the inverse functor to H. Thus $Y \simeq G(A)$ and Y is a full subcategory such that $G = KH$ by construction. We have an adjunction
$$A(Fx, a) \to X(x, Ga) = X(x, KHa)$$
and we want to show there is a functor T and a natural isomorphism
$$Y(Tx, y) \xrightarrow{\sim} X(x, Ky).$$
Let $T = HF\colon X \to Y$ and let $y = Ha$. Then define $\psi\colon Y(HFx, y) \to X(x, Ky)$ by $\psi(f) = \varphi(H^{-1}(f))$. It is a natural bijection, and (b) implies (c).

   Assume (c). Then $G = KH$ with K and H as supposed is fully faithful and injective on objects. Reflectivity of Y says there is a functor T such that
$$A(H^{-1}Tx, a) \simeq Y(Tx, y) = Y(Tx, Ha) \xrightarrow{\sim\ \varphi} X(x, KHa) = X(x, Ga).$$
Therefore $H^{-1}T$ is a left adjoint to G. Therefore (b) implies (c).

   b implies a. or c implies a.

5. Let J be connected and $\Delta\colon C \to C^J$ have a left adjoint, *i.e.* a colimit. We claim this left adjoint is actually a left-adjoint-left-inverse. Let $T\mathcal{F} = \operatorname{colim} \mathcal{F}$ be the adjoint. We have $\varphi\colon C(T\Delta c, c) \to C^J(\Delta c, \Delta c)$ such that $\varphi^{-1}(1) = \epsilon_{\Delta c}\colon T\Delta c \to c$. By question 8 (a) of §2 of this chapter, $\operatorname{colim} \Delta c = c$, so we get $\epsilon_{\Delta c}\colon c \to c$. The data of a natural transformation $\eta\colon \Delta c \to \Delta c'$ is just a morphism $c \to c'$, so Δ is fully faithful and clearly isomorphic on objects as if $\Delta c = \Delta c'$, then $(\Delta c)(j) = c = (\Delta c')(j) = c'$ for any $j \in \operatorname{Obj} J$. By question 4 above, the colimit is a left-adjoint-left-inverse.

4.5 Adjoints for Preorders

1. Let H be an inner product space and let $P = Q = \mathcal{P}(H)$ ordered by inclusion, with $LS = RS = S^\perp$ for all $S \in \mathcal{P}(H)$. We claim this gives a Galois connection. Suppose that $S \le LS' = (S')^\perp$ or in other words that $S \subseteq (S')^\perp$. We must show that $S' \subset S^\perp$. indeed, we have $(s', s) = 0$ because $s \in (S')^\perp$. Suppose next that $S' \subset S^\perp$, so again we have $(s, s') = 0$ because $s' \in S^\perp$.

   In Mac Lean p.94 *Galois connection* is specified to mean monotone Galois connection, but the above connection is in fact antitone. Indeed, if $A \subset B$, then $B^\perp \subset A^\perp$ because if $b \in B^\perp$ then $(a, b) = 0$, as $a \in A \subset B$.

2. todo

3. d

4.5.1 Examples of Galois Connections

Hilbert's Nullstellensatz gives a Galois connection between the preorder of algebraic subsets of $\mathbb{A}^n_k$ and the preoder of radical ideals of $k[T_1, \ldots, T_n]$. If X is a path-connected, locally path- connected and locally simply-connected topological space (essentially, when the universal covering space $\tilde{X}$ of X exists) there is a Galois connection between covers of X and subgroups of $\pi_1(X)$. It is this feature of covering space theory that was chosen to guide the definition of the etale fundamental group of a scheme. The fundamental theorem of Galois theory is perhaps the prototypical example of a Galois connection, between subgroups of $\mathrm{Gal}(K/k)$ and intermediate extensions K/k_1, where $k_1 \supset k$.

4.6 Cartesian Closed Categories

1. (a) Let U be a set. Then we claim $C := \mathcal{P}(U)$ is a Cartesian closed preorder. Indeed, define functors $0 \mapsto t := U$ from $\mathbf{1} \to C$, $(a, b) \mapsto a \times b := a \cap b$ and finally $c \mapsto c^b := C \cup (U \setminus b)$. Then we have isomorphisms $\mathrm{Hom}_{\mathbf{1}}(0, 0) \to \mathrm{Hom}_C(V, U)$ because $V \subset U$, $\mathrm{Hom}_{C \times C}((V, V), (X, Y)) \to \mathrm{Hom}_C(V, X \cap Y)$ as the left hand side says that $V \hookrightarrow X$ and $V \hookrightarrow Y$. Finally we have an isomorphism $\mathrm{Hom}_C(A \cap B, V) \to \mathrm{Hom}_C(A, V^B)$ because the left hand side says that $A \cap B \hookrightarrow V$ so if $x \in A$, then $x \in V$ or $x \notin B$. Naturally is trivial in all cases as all hom-sets are singletons.

   (b) Next we claim any Boolean algebra is Cartesian closed as a preoder. The proof is the same as (a) with $\cap$ for **and**, $\cap$ for **or** and complements for **not**.

2. skip

3. finish

5 Limits

5.1 Creation of Limits

1. We claim the projection functor $P\colon (x \downarrow C) \to C$ creates limits. Let $\tau\colon y \to PF$ be a limiting cone. Thus y is a limit in C. Here $F\colon J \to (x \downarrow C)$ is any functor. Therefore we have a cone $x \to F$, hence a unique morphism $x \to y$. Thus in $(x \downarrow C)$ a cone over F naturally factors uniquely through $(x \to y) \to F$. Therefore P creates limits.

2. We claim the forgetful functor $U\colon \mathbf{CompHaus} \to \mathbf{Set}$ creates limits (so by theorem 1 of this section, **CompHaus** has all small limits). Consider $F\colon J \to \mathbf{CompHaus}$. Then UF has a limit, which is $\mathrm{Cone}(*, F)$. This is a subset of $\prod_j F_j$, and so is a subset of a compact Hausdorff space. The conditions on being a cone are closed conditions, and so $\mathrm{Cone}(*, F)$ is a compact Hausdorff space. This topology is the *initial topology*, and by definition $\mathrm{Cone}(*, F)$ is a limiting cone in **CompHaus** with this topology.

   Remark 10. A more detailed solution is actually given on p. 121.

   Remark 11. We would not have this uniqueness in **Top**; for example we could give $\mathrm{Cone}(*, F)$ is the discrete topology.

3. We claim the functor $G\colon X^{\mathbf{2}} \to X \times X$ given by $(f\colon x \to y) \mapsto (x, y)$ and on morphisms by

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ x' & \xrightarrow{f'} & y' \end{array} \mapsto (g, h)\colon (x, y) \to (x', y').$$

   Note that a limit of GF is a product (L_1, L_2) of limits in X. Thus in the arrow category $X^{\mathbf{2}}$, we have a cone from L_1 to all the tails in the diagram $F(J)$, and from L_2 to all the heads in the diagram $F(J)$. This gives a cone $L_1 \to F$ by definition of morphisms in $X^{\mathbf{2}}$. Thus we get a unique morphism $L_1 \to L_2$, and hence a unique cone over F in $X^{\mathbf{2}}$.

   We claim it is limiting. Given a cone from $x \to y$ to F in $X^{\mathbf{2}}$, and we get unique morphisms $x \to L_1$ and $y \to L_2$ considering diagrams of tails and heads only. By the uniqueness parts of the definition of limit, this is a factorization in $X^{\mathbf{2}}$.

4. We know the limit in **Set** of a functor $F\colon J \to \mathbf{FinSet}$ is $\mathrm{Cone}(*, F)$. It suffices to show this is finite when $\mathrm{Obj}\, J$ is finite. But there are at most $\prod_j (\# Fj)$ many cones to F, and this number is finite.

5. We claim **Cat** is small-complete. This is a great abstract nonsense argument. Recall that **Cat** has pullbacks (section 3.5) so if it has a terminal object, it has all products and equalizers (section 3.4). Therefore by corollary 2 of this chapter, **Cat** would be small-complete. But **1** is a terminal object in **Cat**.

6. See also subsubsection 3.4.1.

   todo

7. We claim $k[[x]]$ is the limit of the diagram composed of quotient maps $k[x]/(x^{n+1}) \to k[x]/(x^n)$. A cone from a ring R to this diagram is just the data of a morphism $R \to k$ for each natural number n, and these assemble to a morphism of rings $R \to k[[x]]$ with the correct properties.

5.2 Limits by Products and Equalizers

1. (a) (Manes.) We claim the arrows f, g from the theorem in this section have a common left inverse. This will prove the claim. For each $i \in \mathrm{Obj}\, J$ consider $p_{\mathrm{id}_i}\colon \prod_{u\colon j \to k} F_k \to Fi$. Taken together these projections induce a morphism $h\colon \prod_{u\colon j \to k} F_k \to \prod_i Fi$. The morphisms $p_{id_i} \circ f$ and $p_{\mathrm{id}_i} \circ g$ are both induced by the projections $p_i\colon \prod_i Fi$, so the induced maps $h \circ f, h \circ g\colon \prod_i Fi$ and must be the identity by uniqueness.

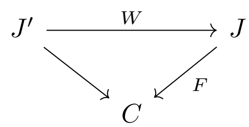
   (b) In **Set**, if $f, g\colon X \to Y$ have a common left inverse h, then consider $y \in Y$ and $hy \in X$, and $(hy, hy) \in X \times X$. Then $(fhy, ghy) = (y, y) \in Y \times Y$, so Δ_Y is in $\mathrm{im}(()f \times g)$. Conversely, if $\mathrm{im}(()f \times g) \supset \Delta_Y$, then for all $y \in Y$, $(y, y) = (fx, gx)$ for some x. Then map $h\colon y \mapsto x$ and check that h is a common left inverse.

2. It is obvious that $C \times C'$ is (co)complete when C and C' both are.

3. This is clear. In diagram in the statement of the question, $\beta\mu$ is a cone to F', and $\nu'\beta$ is a cone from F, so the morphisms from the (co)limits in question are induced.

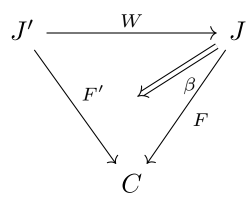
4. todo

5. (a) We define lim as a functor $(\mathbf{Cat} \downarrow C)^{\text{opp}} \to C$. Given $F\colon J \to C$, send this to the object $\lim F$. Given a triangle



we must provide a morphism $\lim F \to \lim FW$ in C. Taking $C = C'$, $H = \mathrm{id}$ then the required arrow is t from exercise 4.

   (b) Consider the *super comma category* $(\mathbf{Cat}^\bullet \downarrow {}^\bullet C)^{\text{opp}}$. Map objects $F\colon J \to C$ as in (a). Given a morphism



we need a morphism $\lim F \to \lim F'$. By 3, we get a morphism $\lim \beta\colon \lim(FW) \to \lim F'$. By (a), we have a morphism $\lim F \to \lim FW$, and composing them, we are done.

5.3 Preservation of Limits

1. We claim the composite of two continuous functors is continuous. Let $\epsilon > 0$ be given.... Let $F\colon J \to C$ and let $C \overset{G}{\to} D \overset{H}{\to} E$ be continuous functors. Let $\nu\colon b \to F$ be a limiting cone in C. Then (being slightly hieroglyphic with notation)

$$HG(b \overset{\nu}{\to} F) = H(G(b \overset{\nu}{\to} F)) = H(Gb \overset{G\nu}{\to} GF)$$

is a limiting cone in E because H is continuous and $Gb \overset{G\nu}{\to} GF$ is a limiting cone in D by continuity of G.

Dually, the composite of cocontinuous functors is cocontinuous.

2. Immediate by the results of §2.

3. We claim that the free abelian group functor $F\colon \mathbf{Set} \to \mathbf{Ab}$ is not continuous. It must therefore fail to preserve products or equalizers. Observe that $F(\{*\} \times \{*\}) = \mathbb{Z}$ as $\{*\} \times \{*\}$ is a singleton. Therefore the limiting cone

$$\{*\} \longleftarrow \{*\} \times \{*\} \longrightarrow \{*\}$$

is sent to to the diagram

$$\mathbb{Z} \longleftarrow \mathbb{Z} \longrightarrow \mathbb{Z}$$

which is not a limiting cone by uniqueness of limits and the fact that $\mathbb{Z} \not\cong \mathbb{Z} \oplus \mathbb{Z}$. Therefore F is not continuous.

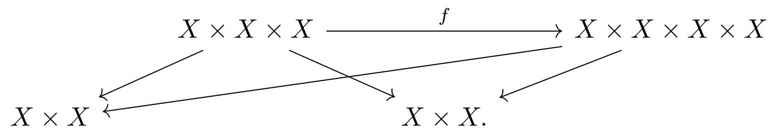
4. We claim that $X \times -\colon \mathbf{Set} \to \mathbf{Set}$ preserves colimits. Recall the adjunction

$$\mathbf{Set}(X \times Y, Z) \simeq \mathbf{Set}(Y, \mathbf{Set}(X, Z))$$

given as a prototype of adjunctions in the Adjoints chapter. By §5 of this chapter, any functor with a right adjoint preserves colimits. Alternatively, it is easy to see that $X \times -$ preserves coproducts and coequalizers.

5.4 Adjoints on Limits

1. We claim the functor $X \times -\colon \mathbf{Set} \to \mathbf{Set}$ cannot have a left adjoint unless X is the one-point set, in which case the functor is the identity, which obviously had a left adjoint. Otherwise we claim that $X \times -$ does not preserve products. Consider $(X \times -)(X \times X) = X \times X \times X$. It comes with a cone to the discrete diagram of two copies of $X \times X$, but this cone is not limiting. Indeed, consider the diagram



The projections from $X \times X \times X$ are given by $(a, b, c) \mapsto (a, b)$ and $(a, b, c) \mapsto (a, c)$, respectively. Therefore $f\colon (a, b, c) \mapsto (a, b, a, c)$. By uniqueness of limits and uniqueness of f, f must be an isomorphism, but the above assignment is not surjective. Therefore $X \times -$ does not preserve products, so cannot have a left adjoint.

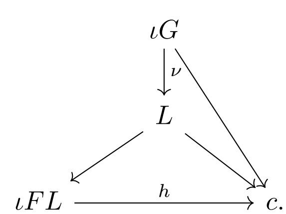
2. We claim the functor $D\colon \mathbf{Vect}^{\mathrm{opp}} \to \mathbf{Vect}$ given by $DV = V^*$ and if $f\colon W \to V$ in $\mathbf{Vect}^{\mathrm{opp}}$, then $Df\colon W^* \to V^*$ by pullback has no right adjoint. Indeed, otherwise D would preserve colimits, and in particular, coproducts. Let J be a countably infinite discrete category and consider the coproduct $\prod_j V_j$ in $\mathbf{Vect}^{\mathrm{opp}}$ for some vector spaces V_j. The colimit of the discrete diagram of V_j^* is $\bigoplus_j V_j^*$ which is not isomorphic to $(\prod_j V_j)^*$. Indeed, the components of the cone to the dual space of the product

$$V_i^* \to (\prod_j V_j)^*$$

$$\phi \mapsto ((v_j)_j \mapsto \phi(v_i))$$

and so the induced morphism $h\colon \bigoplus_j V_j^* \to (\prod_j V_j)^*$ is the inclusion of the linear functionals which are nonzero on only finitely-many components of the product. By uniqueness of colimits and of h, h must be an isomorphism, but clearly h is not surjective.

3. We claim that a full reflective subcategory $\iota\colon C \to D$ of a small-cocomplete category is small-cocomplete. Let F be the reflector and $G\colon J \to C$. Then ιG has a colimit $\nu\colon G \to L$. There is only one candidate for $\operatorname{colim} G$. Let $c \in \operatorname{Obj} C$ be the vertex of a cone with base G. By fulness the components of this cone are morphisms of C. Then we have the following diagram in D:



The lower triangle exists and commutes by definition of reflectivity of C in D (see p. 89). Again by fullness, both the components of $G \to L \to FL$, and h lie in C, and so $FL = \operatorname{colim} G$.

4. Recall that to be Cartesian closed, the functor $X \coprod - = (X \times -)^{\mathrm{opp}}\colon \mathbf{Set}^{\mathrm{opp}} \to \mathbf{Set}^{\mathrm{opp}}$ must have a particular right adjoint *i.e.* the functor $X \times -\colon \mathbf{Set} \to \mathbf{Set}$ must have a particular left adjoint. By exercise 1, this fails whenever X is not the one-point set. Therefore $\mathbf{Set}^{\mathrm{opp}}$ is not Cartesian closed.

5.5 Freyd's adjoint functor theorem

1. The fact that projection $(x \downarrow G) \to A$ creates all small limits follows from the lemma and the following general fact.

   Lemma 1. *If $G\colon C \to D$ creates small products and equalizers, then G creates all small limits.*

   Proof. Let J be small and consider $F\colon J \to C$ such that GF has a limit in D. Then in the notation of the proof of theorem 1 in §2, we have $G(F_i) = (GF)_i$ and $G(F_u) = (GF)_u$, so following the rest of the proof, the fact that G creates small products and limits means that $\lim F$ exists in C. □

2. We use the adjoint functor theorem to find left adjoints to the indicated forgetful functors U. In the cases of **Rng** $\to$ **Set**, we know that the forgetful functor creates limits, and using completeness of **Set** and theorem 2 of §4, we have that U preserves all small limits. First note that all these categories are small-complete.

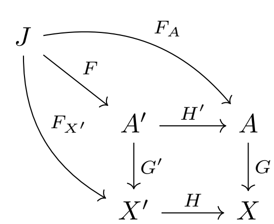
   Of course, the usual free object provides a solution to the solution set condition, but the utility of the theorem is that we can do something easier, by using sub(rings, groups etc.) where things like associativity hold automatically. Given $f\colon x \to UR$ in **Set**, then the set of all rings S of cardinality at most card(x) is small, and so is the set $\{x \to S\}_S$ of functions $x \to S$. For any R, we can consider the subring $S \subset R$ generated by the image of $f\colon x \to R$. This satisfies the solution set criterion. The existence of the free rings $\mathbb{Z}\langle x\rangle$ follows.

   For the functor $A\colon$ **Rng** $\to$ **Ab** sending a ring to its underlying abelian group, we can perform exactly the same construction above, yielding subrings $S \subset R$ with maps $A(\iota)\colon A(S) \to A(R)$ where $\iota\colon S \to R$ is inclusion in **Rng**.

   For $U\colon$ **Cat** $\to$ **Grph** we must check that U preserves products and equalizers, and then the solution set condition. By construction (see excise 1 of §2.5) U preserves products (note that the product of graphs therein defined is a categorical product). The equalizer of parallel morphisms of graphs $G \to G'$ is the subgraph of G with vertex set the equalizer of the vertex functions and arrow the set-theoretic equalizer of the arrow function. Recalling the construction of equalizers in **Cat** as pullbacks (see 3 (a) of §3.5, we see that this graph is the underlying graph of the pullback in **Cat**.

   We use subcategories for the solution set condition. Namely the set of all small categories S whose object sets have cardinality at most that of the vertex set of a given graph x is a small set. Then for any morphism of graphs $f\colon x \to UC$ for a small category C, consider the full subcategory S of C such that $\operatorname{Obj} S = \operatorname{im}(f)_O$. Then f factors through a morphism $g\colon x \to US$ via the inclusion functor $S \to C$. It follows that free categories exist (c.f. §2.7).

3. We claim the functor H' as defined the statement creates limits. Let $C =$ **Cat**. Let J be any category and consider the pullback diagram



   in C. Say that $\lim F_A$ exists. Then as G preserves limits and H creates them, we have

$$G \lim H'F = \lim GH'F = \lim HG'F = H \lim G'F.$$

   Using this equation and the description of A' in C as a certain full subcategory of $(H \downarrow G)$, one can finish the exercise by hand from here.

Alternatively, recall that limit in functor categories are calculated pointwise by theorem 2 of §5.3. Note that F induces a morphism in $C^{\to\cdot\leftarrow}$ from ΔJ to the above span, which has a limit. Recalling the adjunction from exercise 6 of §4.2, the functor $\mathbb{V}$ preserves limits, and so $F\colon J \to A'$ has a limit. Because H' is a component of the counit of the adjunction, it follows that that this limiting in A is sent to $\lim F_A$. This says that H' creates limits.

check last sentence

4. We give an alternative and superior proof of 1. It is easy to check that we have a pullback diagram

$$\begin{array}{ccc} (x \downarrow G) & \xrightarrow{\text{proj}} & A \\ \big\downarrow{\scriptstyle \text{incl}} & & \big\downarrow{\scriptstyle G} \\ (x \downarrow X) & \xrightarrow{\text{proj}} & X. \end{array}$$

By hypothesis G is continuous and the lower projection has been shown to create limits, and so the upper projection creates limits by exercise 3.

6 Chapter 6

7 Chapter 7

8 Abelian Categories

8.1 Additive Categories

1. Let A be an additive category, and let $\kappa\colon a_1 \coprod \cdots \coprod a_n \to a_1 \times \cdots \times a_n$ be the canonical map corresponding to the identity matrix. Because the right-hand side is also a coproduct, we have arrows $\iota_i'\colon a_i \to \prod a_j$ as well as $\iota_i\colon a_i \to \coprod a_j$, and because the left-hand side is also a product, we have projections $p_i'\colon \coprod a_j \to a_i$. Therefore we can define a map $\eta\colon \prod a_j \to \coprod a_j$ by $p_i'\eta\iota_j' = \delta_{ij}$. We see then that
$$p_k'\eta\kappa\iota_j = p_k'\eta(\iota_1 p_1 + \iota_2' p_2)\kappa\iota_j = p_k'\eta(\iota_j' p_j \kappa\iota_j) = p_k'\eta\iota_j' = \delta_{jk}$$
when considering only two factors a_1 and a_2. The general claim follows by duality, then induction on n.

2. The map κ is defined by the same equations as before, but it need not be an isomorphism. For example, in **Ab**, infinite products are different than infinite direct products; the former can have infinitely many nonzero components.

3. That the biproduct is associative and commutative up to isomorphism follows from the fact that the product (or coproduct) is, and then theorem 2.

4.

5.

6.

8.2 Abelian Categories

1. We claim an additive functor $T\colon A \to B$ between abelian categories is exact iff it preserves short exact sequences. If T preserves all finite limits and finite colimits, then given

$$0 \longrightarrow a \xrightarrow{f} b \xrightarrow{g} c \longrightarrow 0$$

we have

$$\ker(Tg) = T(\ker g) = T(\operatorname{im}(f)) = \operatorname{im}(()Tf),$$

because $\operatorname{im}(()f) = \ker(\operatorname{coker}(f))$. The same happens at a and c. Conversely, an equalizer of $f, g\colon a \to b$ is the kernel of $f - g$, and so

$$0 \longrightarrow \ker(f-g) \longrightarrow a \xrightarrow{f-g} \operatorname{coim}(f-g) \longrightarrow 0$$

is sent to the exact sequence

$$0 \longrightarrow T\ker(f-g) \longrightarrow Ta \xrightarrow{T(f-g)} T\operatorname{coim}(f-g) \longrightarrow 0.$$

Therefore $\ker(T(f-g)) = \ker(Tf - Tg)$ and T preserves equalizers. Additivity implies T in particular preserves products, and so T preserves all finite limits. Showing T preserves all finite coequalizers and coproducts is similar.

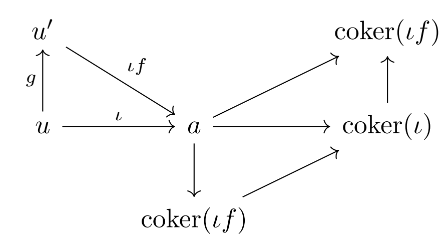
2. Trivial.

3. The category of all free abelian groups is not abelian. Suppose otherwise. Then the map $x \mapsto 2x$ from $\mathbb{Z} \to \mathbb{Z}$ has 0 kernel and 0 cokernel in the category of free abelian groups, and is therefore an epimorphism and a monomorphism, hence an isomorphism. But this map has no inverse.

4. The category of finite abelian groups is abelian. Existence of a zero object, binary biproducts, kernels and cokernels are obvious. In this category monomorphisms are exactly injective homomorphisms, and epimorphisms are exactly surjective homomorphisms, and so every monomorphism $N \hookrightarrow G$ is the kernel of $G \twoheadrightarrow G/N$, and every epimorphism $\phi\colon G \twoheadrightarrow H$ is the cokernel of $\ker\phi \hookrightarrow G$.

5. Let R be left-Noetherian, then we claim the category of finitely-generated left R-modules is abelian. Clearly it is an additive category with a zero object and binary biproducts. Again, monomorphisms are precisely injective R-linear maps, and epimorphisms, surjective R-linear maps. Therefore cokernels are quotient maps and kernels are inclusions of submodules. There is no problem with cokernels, and kernels are finitely-generated by the Noetherian hypothesis.

6. For simplicity we define quotients of an object a by a subobject $u \to a$. Let $a/u := \operatorname{coker}(\iota\colon u \to a)$. We must show this is well-defined. Suppose that $f\colon u' \to u$ and $g\colon u \to u'$ are monics as in the definition of the equivalence relation defining subobjects. Consider the diagram



Each of the maps between cokernels is unique, and the identity $\operatorname{coker}(\iota f) \to \operatorname{coker}(\iota f)$ makes the diagram commute as well. Therefore the map $\operatorname{coker}(\iota f) \to \operatorname{coker}(\iota)$ is a monomorphism, and is also clearly an epimorphism. In an abelian category this forces $\operatorname{coker}(\iota f) \simeq \operatorname{coker}(\iota)$ and quotient objects are well-defined.

8.3 Diagram Lemmas

1. Following the given proof of the five lemma, we see that we need only f_4 and f_2 monic, and f_1 epi. These hypotheses are minimal. Indeed, consider the following diagrams in **Ab**:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i_1} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{p_2} & \mathbb{Z} & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow{\scriptstyle \Delta} & & \big\downarrow{\scriptstyle \Delta\circ p_1} & & \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \end{array},$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i_1} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{p_2} & \mathbb{Z} & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle p_2} & & \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array},$$

   and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}.$$

2. Let the diagram for five lemma be labelled as in the chapter. Then let x be a member of b_3. If $x_1 \equiv h_3(x)$, then there is $y_1 \in_m a_4$ such that $g_3(y_2) \equiv y_3$ for some member y_2. Then $h_3(f_3y_2 - x) \equiv 0$ and so $f_3y_2 - x \equiv h_2z \equiv f_2h_2z'$ for some $z' \in_m a_2$. Now $f_3g_2z' \equiv h_2f_2z' \equiv f_3y_2 - x$ and so $f_3(-g_2z' + y_2) \equiv x$. This says that f_3 is an epimorphism.

3. Exactness of the connecting morphism δ is straightforward.

4. Consider the functors $F, G\colon \mathrm{Ses}(A)^{\mathbf{2}} \to A$ given on objects by $F(z) = \ker f_3$ and $G(z) = \mathrm{coker}(f_1)$ if

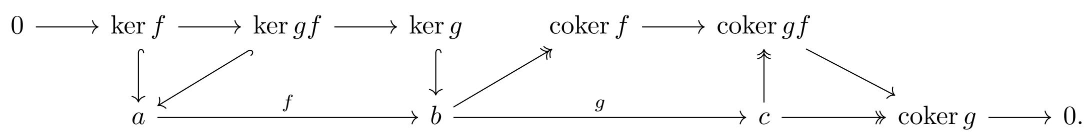
$$z = \begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_2} & & \big\downarrow{\scriptstyle f_3} \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}.$$

   A moment of diagram chasing using the fact that δ does not depend on any of the choices made during its construction then shows that δ is a natural transformation from F to G.

5. (a) Diagram chase using members and the rules of theorem 3.
   (b) Applying the snake lemma to the bottom two rows gives an exact top row parallel to the existing row. We claim the morphisms in the row provided by the snake lemma and the morphisms in the existing row are equal. Indeed, both rows make the top two square commute, and the vertical morphisms in these squares are monic. Therefore the top row is exact.
   (c) Suppose the top and bottom rows are exact, and all columns are exact. We claim the middle row is exact **provided that the middle row composes to the zero morphism**. **Otherwise, the exercise is incorrect as stated.** To see that the first and last morphisms are monic and epi, respectively, use two applications of the short five lemma "vertically." Finally, use exactness of the bottom row and the outer columns to see that the last part of rule (v) from theorem 3 is satisfied.

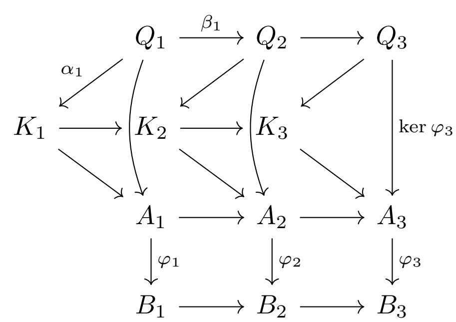
       For a counter-example if the middle row is not exact, see *Lectures on Homological Algebra* by Weizhe Zheng, Remark 1.6.38.

6. We construct the given short exact sequence. Consider the diagram



All the maps except f and g are induced from the universal properties of (co)kernels or are part of the data of (co)kernels. Exactness at each point can be checked using fact (v) of theorem 3 of this section, and proposition 1 from §VIII.3.

7. We claim the category $\mathbf{Ses}(A)$ is not abelian in a moderately specific way, namely by showing that it will not in general have kernels. Suppose otherwise and first note that a monomorphism in $\mathbf{Ses}(A)$ has monomorphisms for all its components. Then consider the following diagram



where Q is the kernel in $\mathbf{Ses}(A)$ of $\varphi\colon A \to B$ and $K_i = \ker \varphi_i$ in A. The diagonal maps exist by the universal property of kernels in A, and the maps $K_i \to K_{i+1}$ are induced using the universal property of K_{i+1} and, crucially, are the morphisms in the snake lemma. By the opening remark, the α_i are all monomorphisms, because the bent morphisms are monomorphisms. By considering short the short exact sequences

$$0 \longrightarrow K_1 \xrightarrow{\text{id}} K_1 \longrightarrow 0 \longrightarrow 0, \qquad 0 \longrightarrow 0 \longrightarrow K_2 \xrightarrow{\text{id}} K_2 \longrightarrow 0$$

and

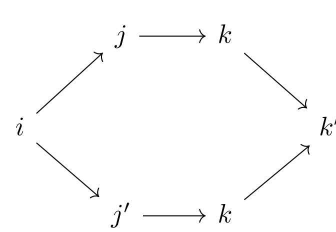
$$0 \longrightarrow 0 \longrightarrow K_3 \xrightarrow{\text{id}} K_3 \longrightarrow 0$$

one obtains inverses to the α_i. It follows that $\ker \varphi$ in $\mathbf{Ses}(A)$ has $Q_i = K_i$ and morphisms β_i as in the snake lemma. But this sequence is not in general exact.

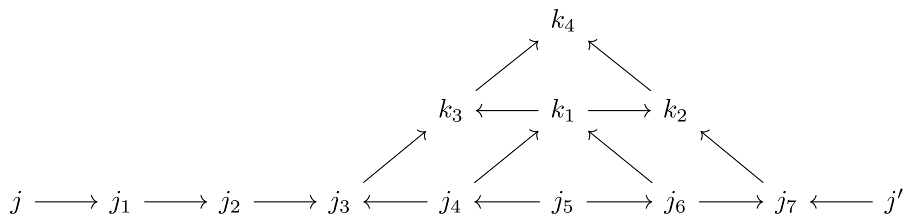
9 Special Limits

9.1 Interchange of Limits

1. We claim that a category J is filtered iff it is connected and pseudofiltered. If J is filtered, it is obviously connected by condition (a), and condition (b) obviously holds. By conditions (a) and (b), we have a diagram



showing that J is pseudofiltered. Let J be connected and pseudofiltered and let $j, j' \in \operatorname{Obj} J$. We need to check condition (a) for filtration. The proof is by induction on the number of times arrows change direction in the path of arrows joining j to j'. Instead of writing this carefully we present an example diagram:



Next we claim a category is pseudofiltered if its connected components are filtered. By the above connected components are filtered in a pseudofiltered category. Conversely, condition (b) refers only to a single connected component, and so holds in any category whose connected components are filtered. The same holds for condition (a)', and so a category whose connected components are filtered is pseudofiltered.

2. We claim that the coproduct commutes with pullback in **Set**. Let P be the finite category $\bullet \longrightarrow \bullet \longleftarrow \bullet$ and let $J = \mathbf{Set}$. We will show that pullback commutes with any coproduct indexed by a small set. Note that **Set** has a terminal object, so is filtered.