hence by Rouché,

$f(z)-w$ and $f(z)-w_0$ have equal # of zeros ~~for~~ inside circle $|z-z_0|=r$.

So $f(z)-w$ has exactly $m>0$, $w$ is in the range. $\square$

Porism (= corollary of the pf.)

- Go to tutorial $\square$
- evals – take ~~to~~ 5 min. at start $\square$
- ~~last activity after we~~
- can done after end.

Mean value.

Recall Cauchy's ~~thm.~~ formula

$$f(z_0)=\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-z_0}\,d\zeta$$

$\gamma$. circle

If $z_0$ is centre of $\gamma$, then $\zeta = z_0 + re^{it}$

$0 \le t \le 2\pi$

~~$d\zeta$~~

$d\zeta = ire^{it}dt$, and

$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt.$$

f(centre of circle) = average value of f on the circle.

This mean value thm. for analytic fns

If we take Re part, $f = u + iv$,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt.$$

this is mean value thm. for harmonic functions

(can be proven ~~so~~ independently, and more is true.)

- will not see more on harmonic fn due to time const.

§3.3 Fractional- linear ~~tra~~ transformations (aka Möbius transformations)

let $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$.

$$T(z) = \frac{az+b}{cz+d}$$

is a fractional- linear ~~cont~~ transformation

$$T'(z) = \frac{ad-bc}{(cz+d)^2}, \text{ so } ad-bc = 0 \Rightarrow T \text{ is constant, not interesting.}$$

Lem. Frac-lin. trans. are injective.

Pf.

$$\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d} \iff bcz_2 + adz_1 = adz_2 + bcz_1$$

$$\iff (ad-bc)z_1 = (ad-bc)z_2.$$

$$\iff z_1 = z_2.$$

□

$T$ has pole at $-\frac{d}{c}$, $\lim_{|z|\to\infty} T(z) = \frac{a}{c}$.

Given $a, b, c, d$, ~~ad-bc~~ $\neq 0$,

consider the FLT for "~~c, a, -b, -d~~"

~~(d, -b, -c, a)~~

~~$S(z) = \frac{dz+b}{-cz-a}$.~~

~~Then~~ $(d, -b, -c, a)$

$$S(z) = \frac{dz-b}{-cz+a}$$

is inverse to $T$, i.e.

$T(S(z)) = z = S(T(z))$ $\forall z$. (book seems to have typo).

~~Fact.~~ Can view $T\left(-\frac{d}{c}\right) = \infty$,

~~then~~ and $\lim_{|z| \to \infty} T^{-1} = -\frac{d}{c}$

i.e. "$T^{-1}(\infty) = -\frac{d}{c}$".

$T$ and $T^{-1}$ are maps of the extend complex plane to itself, bijective, and holo.

Thm. Any holo. bijection $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a FLT.

Will see more next week.

143 (C)

# Aut. of unit disk

**Prop.** Every holo. fn $f: \mathbb{D} \to \mathbb{D}$ w/ holo. inverse is of the form

$$f(z) = e^{i\varphi} T_a(z) \quad , \varphi \in \mathbb{R}$$

**Pf.** Let $f$ be an aut, ~~$f(0)=$~~ $f(a) = 0$.

then $g(0) = f \circ T_{-a}(0) = 0$.

By Schwartz, $g(z) = e^{i\varphi}(z)$ for some $z$.

why? $|g(z)| \leq |z|$ by Schwartz,

and ~~$|g^{-1}(z)| \neq 1$~~

$$|g^{-1}(z)| \leq |z|$$

by Schwtz. when $z = g(w)$,

$$|w| \leq |g(w)| \ \forall w \in \mathbb{D},$$

So $|g(z)| = |z| \ \forall z \in \mathbb{D}$

fact: this implies $g(z)$ is a rotation. (Also in Schwtz).

$\square$

~~Thm (Picard). Spse f is an entire fn. s.t. f~~

**Def.** Let $f$ be an entire function. A lacunary value of $f$ is any $z \in \mathbb{C}$ s.t. $z \notin \text{image}(f)$

i.e. $\nexists\, \omega$ s.t. $f(\omega) = z$.

**Thm** (Picard, 1879)
Let $f$ be an entire fn. w/ two lacunary values. Then $f$ is constant.

**Def.** Write $\mathbb{D}$ = unit disk.

**Lem.** Let $a \in \mathbb{D}$. Then

$$\frac{z-a}{1-\bar{a}z} \in \mathbb{D}$$

and

$$f(z) = \frac{z-a}{1-\bar{a}z}$$

is a bijection $\mathbb{D} \longrightarrow \mathbb{D}$.

$f(a) = 0$.

**Pf.** The inverse, we'll see, is $g(z) = \dfrac{z+a}{1+\bar{a}z}$, must show

$f(z) \in \mathbb{D}$

$$|1-\bar{a}z|^2 - |z-a|^2 = 1 + |a|^2|z|^2 - |a|^2 - |z|^2$$
$$= (1-|z|^2)(1-|a|^2) > 0.$$

Thus $|1-\bar{a}z| > |z-a|$, so $|f(z)| < 1 \;\forall\, z \in \mathbb{D}$.

**Def.** For $a, b \in \mathbb{D}$, the hyperbolic distance between $a$, and $b$ is

$$d(a,b) = 2\tanh^{-1}\left(\left|\frac{a-b}{1-\bar{a}b}\right|\right) = 2\tanh^{-1}\left(|T_a(b)|\right).$$

**Fact:** This obeys $d(a,b) = 0 \iff a = b$

$$d(a,c) \leq d(a,b) + d(b,c) \quad \forall\, a,b,c \in \mathbb{D}$$

$$d(a,b) = d(b,a)$$

i.e. "behaves like a distance".

**Prop.** (Ahlfors' version of Schwarz lemma)

If $f: \mathbb{D} \to \mathbb{D}$ is holomorphic, then $f$ decreases hyperbolic distances:

$$d(f(a), f(b)) \leq d(a,b)$$

$\forall\, a, b \in \mathbb{D}$

**Pf.** If ~~f(a)~~ $a = 0$ and $f(a) = 0$, then this follows from Schwarz lemma:

$$d(f(a), f(b)) = 2\tanh^{-1}|f(b)|$$

$$d(a,b) = 2\tanh^{-1}|b| \quad \text{and} \quad |f(b)| \leq |b|.$$

If not, replace $f$ by $g = T_{f(a)} \circ f \circ (T_a)^{-1}$

**Def.** $(f^*\rho)(z) := \rho(f(z))|f'(z)|$.

Call the resulting dist the pullback distance.

**ex.** let $\rho_h(z) = \dfrac{2}{1-|z|^2}$

**Lem.** We claim $\rho_h(z-w) = d(z,w)$ $\forall\ z,w \in \mathbb{D}$.

**Pf.** $\forall$ auts. $T_a : \mathbb{D} \longrightarrow \mathbb{D}$, todo □

$$\cancel{(T_a^*\lambda)(z) = \lambda(T_a(z))}$$

$$\rho_h(T_a(z)) = \rho_h(z).$$

~~Check, we know~~

So to check the claim, suffices to prove that, for $a \in (-1,1)$,

$$\cancel{\rho_h(z) = \text{length}}$$

$$d(0,a) = 2\tanh^{-1} a = \int_0^a \frac{2}{1-t^2}\,dt = \int_{\gamma} \rho_h(t)\,dt$$

straight line

But know this from 135/136/137, etc. □

approx $\gamma$ as line segments, apply autos to make each one a segment of the real line

ex. If $\rho \equiv 1$ on $\Omega = \mathbb{C}$,
we get the same dist. as before
$$d_\rho(z,\omega) = |z-\omega|.$$

let $f: \Omega_1 \longrightarrow \Omega_2$ be noncost. holo. ~~the~~
let $\rho$ be confml wt. on $\Omega_2$, and $\gamma: [0,1] \longrightarrow \Omega_1$.

Define
$$\text{length}_{f^*\rho}(\gamma) = \text{length}_\rho(f\circ\gamma).$$

Get a notion of distance on $\Omega_1$:

$$d_{f^*\rho}(z,\omega) = \min_{\substack{\gamma:[0,1]\to\Omega_1 \\ \text{fr } z \text{ to } \omega}} \{\text{lgth}_\rho(f\circ\gamma)\}.$$

By the chain rule,
$$\text{lngth}_{f^*\rho}(\gamma) = \int_0^{2\pi} \rho((f\circ\gamma)(t))\,|(f\circ\gamma)'(t)|\,dt$$
$$= \int_0^{2\pi} \rho((f\circ\gamma)(t))\,|f'(\gamma(t))\,\gamma'(t)|\,dt$$

So $(\rho\circ f)(z)\,|f'(z)|$ is a conformal wt.
for this distance on $\Omega_1$. Note $(\rho\circ f)$ and $f'$ have isolated zeros on $\Omega_1$, so so does their product.
(why?)

Want to try to prove Picard the same way,

w/ e.g. $\mathbb{C}\setminus\{0,1\}$ instead of $\mathbb{D}$.

Original pf. was more conceptual, but needs ~3 weeks setup at our pace.

<u>Def.</u> $\Omega \subseteq \mathbb{C}$ open. Cts. fn

$$\rho: \Omega \longrightarrow \mathbb{R}$$

is a <u>conformal weight</u> if

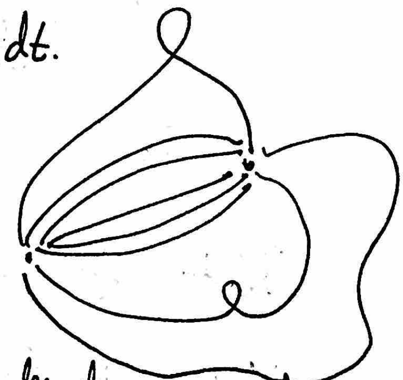
$\exists$ set $N \subset \Omega$ of isolated pts. s.t.

- $\rho(n) = 0 \ \forall n \in N$
- On $\Omega \setminus N$, all partial derivs of $\rho$ exist & are cts
  - $\rho > 0$.

Given $\rho$, <u>the metric corresp to $\rho$</u> is defined as follows.

$\forall \ \gamma: [0,1] \longrightarrow \Omega$ paths,

$$\text{length}_\rho(\gamma) = \int_0^1 \rho(\gamma(t)) \, |\gamma'(t)| \, dt.$$



$\forall \ x, y \in \Omega$,

$$\text{dist}_\rho(x,y) = \min_{\substack{\gamma \text{ s.t.} \\ \gamma(0)=x \\ \gamma(1)=1}} \text{lgth}_\rho(\gamma).$$

depending on $\rho$, distinct $\gamma$ could be "shortest"

Now this is holo and maps 0 to 0.

~~By~~ thus

$$|g(T_a(b))| \leq |\underbrace{T_a(b)}_{z}|$$

So

$$2\tanh^{-1}|g(T_a(b))| \leq 2\tanh^{-1}|g(T_a(b))|$$
$$\| \qquad\qquad\qquad \|$$
$$d(f(a), f(b)) \qquad\qquad d(a, b).$$

$\square$

<u>Warmup</u> to <u>proving</u> little <u>Picard</u>:

<u>Thm</u> (Liouville, again) A bounded entire fn. is const.

<u>Pf.</u> Rescale and translate $f$ so that $f: \mathbb{C} \to \mathbb{D}$

let $r > 0$ and let

$$B_r = \{z \in \mathbb{D} \mid d(0, z) < r\}.$$

~~By~~ let

$$D_r = \{z \in \mathbb{D} \mid |z| < \tanh(\tfrac{1}{2} r)\}.$$

By Ahlfors:

$$d(f(z)) \leq d(0, z) = 2\tanh^{-1}(\tanh \tfrac{1}{2} r) = r$$

So

$$f(D_r) \subseteq B_r.$$

But the same holds for $f_n(z) = f(nz)$ $\forall n = 1, 2, 3 \ldots$

So $im(f) \subseteq B_r$. $r$ is arb. $\Rightarrow$ $f$ is const.

$\square$

**Def.** The curvature of $d_\rho$ is

$$\kappa_\rho = \frac{-1}{\rho^2}\Delta(\log \rho) : \Omega \to \cancel{\mathbb{R}}\,[0,\infty)$$

(Not defined on $N \subset \Omega$).

**Def.** $d_\rho$ is ultrahyperbolic if $\kappa_\rho \leq -1$ on $\Omega \setminus N$.

ex. If $d_\rho$ = hyperbolic metric, i.e.

$\rho(z) = \dfrac{2}{1-|z|^2}$, then

$$\begin{aligned}
\kappa_\rho &= -\frac{1}{\rho^2}\Delta(\log \rho) \\
&= \frac{-1}{\rho^2}\Delta\left(-\log(2(1-x^2-y^2))\right) \\
&= \frac{-1}{\rho^2}\left[\frac{\partial}{\partial x}\left(\frac{1}{1-x^2-y^2}(-2x)\right) + \frac{\partial}{\partial y}\left(\frac{1}{1-x^2-y^2}(-2y)\right)\right) \\
&= -\frac{1}{\rho^2}\left(2\,\frac{(1-x^2-y^2)-2x(-2x)+2(1-x^2-y^2)-2y(-2y)}{(1-x^2-y^2)^2}\right) \\
&= -\frac{1}{\rho^2}\left(\frac{4}{(1-x^2-y^2)^2}\right) = -\frac{1}{4}\cdot 4 = -1.
\end{aligned}$$

so $d_\rho$ is hyperbolic.

Now can prove Picard    Go to 151

Let $\Omega$ be any region w/ an ultrahyperbolic metric. $\rho_\Omega$

then $\forall x, y \in \mathbb{D}$,

$$\rho_\Omega(f(x), f(y)) \le \rho_h(x,y).$$ ~~152~~ 152

Pf. ETS   1). Pullback of ultrahyperbolic is ultrahyperbolic

2). $\forall$ ultrahyp. metrics $\lambda$ on $\mathbb{D}$, $\lambda \le \rho_h$.

Pf. of 1).   ETS   $K_{f^*\rho_\Omega}(z) = K_{\rho_\Omega}(f(z))$.

$$f^*\rho_\Omega = \rho_\Omega(f(z))|f'(z)|.$$

By def

$$K_{f^*\rho_\Omega}(z) = \frac{-1}{|f'(z)|^2 \rho_\Omega(f(z))^2} \Delta\Big(\log \rho_\Omega(f(z)) + \underbrace{\log|f'(z)|}_{\Delta(\text{this}) = 0}\Big)$$

by lemma: direct computation.

$$\overset{\text{Lemma}}{=} \frac{\Delta(\log(\rho_\Omega))(f(z))|f'(z)|^2}{\rho_\Omega(f(z))^2|f'(z)|^2}$$

$$= K_{\rho_\Omega}(f(z)) \le -1.$$

Proves first claim.

Lem. Let $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ be the Laplacian on $\mathbb{R}^2$.

1). In polar coords: $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

2). If $f$ is holo, then $\forall r : \mathbb{C} \to \mathbb{R}$,

$$\Delta(r \circ f) = (\Delta r)(f(z))\,|f'(z)|.$$

Pf. 1). See 235 chain rule ¦ C-R eqns (long computation).

$u_x = v_y$

$u_y = -v_x$

2). ~~By the chain rule:~~

~~write $f = u + iv$~~

$\cancel{\mathbb{R} \xrightarrow{u} \mathbb{R}^2 \xrightarrow{r} \mathbb{R}}$

$\mathbb{R}^2 \xrightarrow{u(x,y),\, v(x,y)} \mathbb{R}^2 \xrightarrow{r} \mathbb{R}$

$$\frac{\partial r}{\partial x} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial r}{\partial y} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial y}$$

$$\Delta(r \circ f) = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial r}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial y}\right)$$

$$= \partial$$

$$\cancel{(\Delta r)(f) = \frac{\partial^2(r \circ f)}{\partial x^2} + \frac{\partial^2(r \circ f)}{\partial y^2}}$$

$$\cancel{= \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial u}\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial r}{\partial\right.}}$$

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Pf of 2)

Let $\lambda$ be ultrahyp. on $\mathbb{D}$.

Let $r<1$.

$\rho_r = \dfrac{2}{r^2-|z|^2}$ ~~with~~ hyperbolic on $D(0,r)\subseteq \mathbb{D}$

$$t(z)=\frac{\lambda(z)}{\rho_r(z)}$$

is cts. on $\overline{D(0,r)}$, closed disk, $\because$

as $|z|\to r$, $t(z)\to 0$:

$\rho_r(z)\longrightarrow\infty$ as $|z|\to r$

So $\dfrac{1}{\rho_r(z)}\longrightarrow 0$ and $\lambda(z)$ ~~$\geq 0$~~ ~~doesn't go to $\infty$, etc~~ is cts.

So $t$ attains its max at some $a\in D(0,r)^{\circ}$

must show $t(a)\leq 1$. WLOG $a\notin N$ (why? $\rho(a)=0$)

$$K_\lambda(a)\leq -1 = K_{\rho_h}(a).$$

$\log(t)$ has max at $a$, so

$$0\geq (\Delta\log t)(a) = \big((\Delta\log\lambda)(a)-(\Delta\log\rho_h)(a)\big)$$

$$= -K_\lambda\lambda(a)^2 - K_\lambda\rho_h(a)^2 \geq \lambda(a)^2-\rho_h(a)^2$$

So $\nexists$ So $\rho_h(a)\geq\lambda(a)\Rightarrow t(a)\leq 1$. This is max, so $\rho_h(z)\geq\lambda(z)\ \forall z$. Now let $r\to 1$. $\square$

**lem.** Let $\Omega$ have some ultrahyp. metric. Then any entire fn. $f: \mathbb{C} \to \Omega$ is constant. 15 $\underline{\underline{\#4}}$

Pf.

To prove l.P., enough to prove

**lem.** $\forall\, a, b \in \mathbb{C}$, it is possible to define an ult. metric on $\Omega = \mathbb{C} \setminus \{a, b\}$. ~~has an ultrahy~~

Rk. In his original paper, Picard ~~ad~~ says what this metric is.

Pf of lem. WLOG (hard-ware) $a = 0, b = 1$.

Let $\varphi(r) = \dfrac{(1 + r^{\alpha})^{\beta}}{r^{\gamma}}$, $r \in \mathbb{R}$, $\alpha, \beta, \gamma \in \mathbb{R}$.

Will look at metrics of the form

$$\rho(z) = \varphi(|z|)\,\varphi(|z-1|)$$

and adjust $\alpha, \beta, \gamma$ to get the curvature we want.

By Laplace lemma (both parts)

$$\Delta(\log \varphi)(r) = \beta\, \Delta(\log(1 + r^{\alpha}))$$

$$= \frac{\beta \alpha^2 r^{\alpha - 2}}{(1 + r^{\alpha})^2}$$

Thus

$$K_\rho = -\beta\alpha^2\left(\frac{|z|^{\alpha-2-2\gamma}\,|z-1|^{2\gamma}}{(1+|z|^\alpha)^{2+2\beta}(1+|z-1|^\alpha)^{2\beta}} + \frac{|z-1|^{\alpha-2-2\gamma}\,|z|^{2\gamma}}{(1+|z-1|^\alpha)^{2+2\beta}(1+|z|^\alpha)^{2\beta}}\right) \le 0.$$

Let $\alpha-2-2\gamma=0$ (can always have this), $K_\rho = -\beta\alpha^2\left(\frac{|z-1|^{2\gamma}}{(1+|z|^\alpha)^{2+2\beta}(1+|z-1|^\alpha)^{2\beta}} + \frac{|z|^{2\gamma}}{(1+|z-1|^\alpha)^{2+2\beta}(1+|z|^\alpha)^{2\beta}}\right)$

then as $z\to 0$ or $z\to 1$,

~~$K_\rho$ has a finite limit.~~

$$K_\rho \longrightarrow -\beta\alpha^2 2^{-2\beta}.$$

Now enforce $\gamma=\alpha(1+2\beta)$.

Then as $z\to\infty$, $K_\rho \longrightarrow -2\beta\alpha^2$

So the curvature is bounded:

$$K_\rho < C < 0$$

for some $C$. Rescaling $\rho$ gives that $K_\rho<1$. $\square$

This proves little Picard thm.

We are actually slightly close to proving big Picard:

<u>Thm</u> (Picard) Let $z$ be an ess. sing of $f: D\to\mathbb{C}$. In any nbhd of $z$, $f$ assumes every value in $\mathbb{C}$ inf. often, w/ at most one exception.