Effective
Poisson Summation Formula

If $|f(x)| \le K g(x)$ for some constant K, then we say

$$f(x) = \mathrm{O}(g(x))$$

or

$$f(x) \ll g(x).$$

$$\sum_{0<|m|\le M} \frac{1}{2\pi i m} = 0.$$

Denote

$$e(x) = e^{2\pi i x}.$$

Let f be piecewise continuous, bounded and periodic with period 1.

The m-th Fourier Coefficient of f is

$$c_m(f) = \int_0^1 f(x) e(-mx) dx.$$

A Fundamental Question is :

$$f(x) = \sum_{m \in \mathbb{Z}} c_m(f) e(mx)?$$

We will prove that for some special functions f,

$$\sum_{m \in \mathbb{Z}} c_m(f) = \frac{f(0) + f(1)}{2}.$$

First, let us consider the following function :

$$\psi(x) = \begin{cases} x - [x] - 1/2 & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases} \tag{1}$$

It can be shown that $c_0 = 0$ and $c_m = -\frac{1}{2\pi im}$ if $m \neq 0$. The associated fourier series of $\psi(x)$ does not converge to $\psi(x)$. We would like to know how closely this series can approximate $\psi(x)$.

We will prove that

Theorem 1.

$$\left|\psi(x) - \sum_{|m|\leq M} c_m e(mx)\right| \leq \frac{1}{2\pi M||x||},$$

where $||x||$ denotes the distance from x to its nearest integer.

Using Theorem 1, we will prove the following theorem, known as the Effective Poisson Summation Formula .

Theorem 2.

Let $f(x)$ *be a differentiable function on* $[0, 1]$ *such that* $|f'(x)| \le K$. *Then,*

$$\sum_{m=-\infty}^{\infty} c_m(f) = \frac{f(0) + f(1)}{2}.$$

More precisely,

$$\left| \sum_{|m| \le M} c_m(f) - \frac{f(0) + f(1)}{2} \right| \ll \frac{K \log M}{M}.$$

Coming back to the saw-tooth function, we have to prove the following :

$$\left|\psi(x)+\sum_{0<|m|\leq M}\frac{e(mx)}{2\pi im}\right|\leq\frac{1}{2\pi M||x||}.$$

If $x\in\mathbb{Z}$, the result is clear since

$$\sum_{0<|m|\leq M}\frac{1}{2\pi im}=0=\psi(x).$$

If $x\notin\mathbb{Z}$, WLOG assume $0<x<1$. First, suppose $0<x\leq 1/2$. Observe that

$$\int_{\frac{1}{2}}^{x}e(mt)dt=\frac{e(mx)}{2\pi im}-\frac{(-1)^m}{2\pi im.}$$

Step 1.

$$\sum_{0<|m|\le M} \frac{e(mx)}{2\pi i m} = \sum_{0<|m|\le M} \int_{1/2}^{x} e(mt)dt.$$

Step 2. Adding $\psi(x) = x - 1/2 = \int_{1/2}^{x} e(0)dt$ to both sides, we get

$$\int_{1/2}^{x} \sum_{0\le|m|\le M} e(mt)dt = x - \frac{1}{2} + \sum_{0<|m|\le M} \frac{e(mx)}{2\pi i m}.$$

Step 3.

$$\psi(x) + \sum_{0<|m|\le M} \frac{e(mx)}{2\pi i m} = \int_{1/2}^{x} \sum_{-M}^{M} e(mt)dt$$

$$= \int_{1/2}^{x} \frac{\sin((2M+1)\pi t)}{\sin(\pi t)} dt.$$

Step 4.

$$\left| \psi(x) + \sum_{0<|m|\le M} \frac{e(mx)}{2\pi i m} \right| \le \frac{1}{2M\pi x}.$$

$1/2 < x < 1$, Similar steps.

Finally,

$$\left|\psi(x)+\sum_{0<|m|\le M}\frac{e(mx)}{2\pi i m}\right|\le\frac{1}{2\pi M||x||}.$$

So,

$$-\sum_{0<|m|\le M}\frac{e(mx)}{2\pi i m}=\psi(x)+\mathrm{O}\left(\frac{1}{M||x||}\right).$$

Now, let $f(x)$ be any differentiable function on $[0,1]$ satisfying $|f'(x)| \leq K$. For $m \neq 0$,

$$\int_0^1 f(x)e(mx)dx = \frac{f(1)-f(0)}{2\pi im} - \int_0^1 \frac{f'(x)e(mx)}{2\pi im}dx.$$

Summing both sides over $0 < |m| \leq M$ gives

$$\sum_{0<|m|\leq M} f(x)e(mx)dx =$$

$$\sum_{0<|m|\leq M} \frac{(f(1)-f(0))}{2\pi im} - \int_0^1 \sum_{0<|m|\leq M} \frac{f'(x)e(mx)}{2\pi im}dx$$

$$= \int_0^1 f'(x)\left(\psi(x) + O\left(\frac{1}{M||x||}\right)\right)dx$$

$$= \int_0^1 f'(x)\left(x - \frac{1}{2}\right)dx + O\left(\frac{K \log M}{M}\right)$$

$$= \frac{f(1)+f(0)}{2} - \int_0^1 f(x)dx + O\left(\frac{K \log M}{M}\right).$$

So,

$$\sum_{m=-M}^{M} f(x)e(mx) = \frac{f(1)+f(0)}{2} + \mathrm{O}\left(\frac{K\log M}{M}\right).$$

As a result, as $M \to \infty$,

$$\sum_{m\in\mathbb{Z}} c_{-m} = \sum_{m\in\mathbb{Z}} c_m = \frac{f(0)+f(1)}{2}.$$

This formula has many applications to number theory. Applying it to the functions $\cos\left(\frac{2\pi x^2}{N}\right)$ and $\sin\left(\frac{2\pi x^2}{N}\right)$, we can evaluate the Gauss sum

$$\sum_{n=0}^{N-1} e\left(\frac{n^2}{N}\right).$$

This gives a new and insightful proof of the famous Quadratic Reciprocity Law.

Yet, another application is to evaluate the Riemann Zeta Function at positive even integers.

Take $f(x) = x^2$. For $m \neq 0$,

$$\int_0^1 f(x)e(mx)dx = \frac{1}{2\pi im} + \frac{1}{2\pi^2 m^2}.$$

For $m = 0$, $\int_0^1 x^2 dx = \frac{1}{3}$.

Applying the above formula, we get

$$\frac{1}{3} + \sum_{0<|m|\leq M} \left(\frac{1}{2\pi im} + \frac{1}{2\pi^2 m^2}\right) = \frac{1}{2} + O\left(\frac{\log M}{M}\right).$$

So,

$$\frac{1}{3} + 2\sum_{m=1}^{M} \frac{1}{2\pi^2 m^2} = \frac{1}{2} + O\left(\frac{\log M}{M}\right).$$

As $M \to \infty$,

$$\zeta(2) = \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

The same idea can be extended to evaluate $\zeta(2n)$.

Let $F(x)$ be a continuous function on R such that

$$\int_{-\infty}^{\infty} |F(x)| dx < \infty.$$

We define its m-th Fourier Transform as follows :

$$\widehat{F}(m) = \int_{-\infty}^{\infty} F(x) e(-mx).$$

Suppose that the series

$$G(v) = \sum_{m \in \mathbb{Z}} F(m+v)$$

converges absolutely and uniformly in v. Thus, $G(v)$ is a continuous function of v of period 1. Let us further suppose that

$$\sum_{m \in \mathbb{Z}} |\widehat{F}(m)| < \infty.$$

The Poisson Summation Formula states that

$$\sum_{m \in \mathbb{Z}} F(m+v) = \sum_{m \in \mathbb{Z}} \widehat{F}(m) e^{2\pi i m v}.$$

Putting $v = 0$ gives us

$$\sum_{m\in\mathbb{Z}} F(m) = \sum_{m\in\mathbb{Z}} \widehat{F}(m).$$

If we change the conditions a bit, we get a weaker version of the Poisson Summation Formula, by applying Theorem 2 .

Let us assume that G is differentiable and that $G'(v)$ is bounded.

The Fourier coefficients of G are given by

$$\begin{aligned}
c_m &= \int_0^1 G(v)e^{-2\pi imv}dv \\
&= \int_0^1 \left(\sum_{n\in\mathbb{Z}} F(n+v)\right) e^{-2\pi imv}dv \\
&= \sum_{n\in\mathbb{Z}} \int_0^1 F(n+v)e^{-2\pi imv}dv \\
&= \sum_{n\in\mathbb{Z}} \int_n^{n+1} F(x)e^{-2\pi imx}dx \\
&= \int_{-\infty}^{\infty} F(x)e^{-2\pi imx}dx \\
&= \widehat{F}(m).
\end{aligned}$$

Since $G(0) = G(1)$, we can apply the “Effective Poisson Summation “ Formula to get

$$\sum_{m\in\mathbb{Z}} \widehat{F}(m) = \sum_{m\in\mathbb{Z}} c_m$$

$$= \frac{G(0)+G(1)}{2} = G(0) = \sum_{m\in\mathbb{Z}} F(m).$$

Applications to Number Theory

Analytic continuation of the Riemann Zeta function and it's functional equation.

The classical large sieve.

Uniform distribution of the sequence (n^{α}), α not an integer.