Effective Equidistribution of Eigenvalues of Hecke Operators

Kaneenika Sinha
Queen's University
PhD Thesis
Defense Presentation
4 April 2006

Ramanujan's τ function

$$f(x) = x \prod_{n=1}^{\infty} (1 - x^n)^{24}, \quad |x| < 1.$$

$$f(x) = \sum_{n=1}^{\infty} \tau(n) x^n.$$

Ramanujan obtained the values of $\tau(n)$ for $n \leq 30$. He also made the following conjectures :

- $\tau(nm) = \tau(n)\tau(m), \quad (n,m) = 1$
- $\tau(p^n)\tau(p) = p^{11}\tau(p^{n-1}) + \tau(p^{n+1})$
- $|\tau(p)| \leq 2p^{\frac{11}{2}}$

The first two conjectures were proved by Mordell in 1917, but not the third.

Consider

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}.$$

$$\mathfrak{H} = \{x + iy | x, y \in \mathbb{R},\ y > 0\}$$

is the upper-half plane. For $z \in \mathfrak{H}$, let

$$\gamma z := \frac{az + b}{cz + d}.$$

Consider the set of all holomorphic functions $f : \mathfrak{H} \to \mathbb{C}$, such that

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z)$$

for all $\gamma \in SL_2(\mathbb{Z})$. Since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}),$$

$$f(z + 1) = f(z), \quad z \in \mathfrak{H}.$$

Thus, $f(z)$ has a **Fourier expansion at infinity**

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(f) q^n, \quad q = e^{2\pi i z}.$$

We say that f **is holomorphic at infinity** if $c_n(f) = 0$ for all $n < 0$.

A modular form of weight k on $SL_2(\mathbb{Z})$ is a map $f : \mathfrak{H} \to \mathbb{C}$ such that

(1) f is holomorphic on $\mathfrak{H}$,

(2) $f(\gamma z) = (cz + d)^k f(z)$ for all $\gamma \in SL_2(\mathbb{Z})$.

(3) f is holomorphic at infinity.

If in addition, $c_0(f) = 0$, then f is said to **vanish at infinity**. We say that $f(z)$ is a **cusp form** of weight k.

$$\Delta(z) := \sum_{n=1}^{\infty} \tau(n) q^n$$

is a cusp form of weight 12. For $k \geq 4$, k even, the **Eisenstein series**

$$E_k(z) = \frac{1}{2\zeta(k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}$$

is a modular form of weight k.

M_k and S_k are both finite dimensional vector spaces.

Hecke Operators

For

$$f(z) = \sum_{n=1}^{\infty} c_n(f) q^n, \quad q = e^{2\pi i z},$$

define, for every $n \geq 1$,

$$T_n(f(z)) := \sum_{m=1}^{\infty} \left(\sum_{d|(m,n)} d^{k-1} c_{\frac{mn}{d^2}} \right) q^m.$$

- $T_m T_n = T_{mn}, \quad (m,n) = 1.$

- $T_{p^n} T_p = p^{k-1} T_{p^{n-1}} + T_{p^{n+1}}$

For example, S_{12} is one-dimensional and generated by $\Delta(z)$.

$$T_p(\Delta(z)) = \tau(p)\Delta(z).$$

Consider the space S_{24}. The functions

$\Delta E_6^2 = q - 1032q^2 + 245196q^3 + 10965568q^4 \cdots$

and

$$\Delta^2 = q^2 - 48q^3 + 1080q^4 + \cdots$$

form a basis of S_{24}. The matrix of T_2 is

$$\begin{pmatrix} -1032 & 1 \\ 18289152 & 2112 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\lambda^2 - 1080\lambda - 20468736.$$

The eigenvalues $a_{2,1}$ and $a_{2,2}$ are

$$540 + 12\sqrt{144169} \text{ and } 540 - 12\sqrt{144169}.$$

$$|a_{2,i}| < 2 \cdot 2^{\frac{23}{2}}.$$

Let $a_{p,i}$ denote the eigenvalues of T_p on S_k. Ramanujan- Petersson conjecture states that

$$|a_{p,i}| \leq 2p^{\frac{k-1}{2}}.$$

This was proved by Deligne in 1974. Define the **normalized Hecke operator**

$$T_p' = \frac{T_p}{p^{(k-1)/2}}.$$

The eigenvalues of T_p' lie in $[-2,2]$. Serre fixes a prime p and considers the sequence of spaces $S_{12}, S_{14}, S_{16}, \cdots$.

How are the eigenvalues of T_p' distributed in $[-2,2]$ as the spaces vary?

More generally, we need to consider

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : N|c \right\},$$

where N is a positive integer.

A cusp form of weight k and level N is a map $f : \mathfrak{H} \to \mathbb{C}$ such that

(1) f is holomorphic on $\mathfrak{H}$,

(2) $f(\gamma z) = (cz + d)^k f(z)$ for all $\gamma \in \Gamma_0(N)$.

(3) f "vanishes at the cusps."

For example, if $f(z) \in S_k$, then $f(Nz) \in S(N, k)$.

Let p be a prime not dividing N. Then, the eigenvalues of $T_p(N, k)$ lie in the interval

$$[-2p^{\frac{k-1}{2}}, 2p^{\frac{k-1}{2}}].$$

Thus, the family of eigenvalues of the normalized p-th Hecke operator

$$T_p'(N,\, k) = \frac{T_p(N,\, k)}{p^{(k-1)/2}}$$

lie in the interval $[-2, 2]$. How are the eigenvalues of $T_p'(N, k)$ distributed as we vary N and k? For any closed interval $[\alpha, \beta]$ contained in $[-2, 2]$,

$D(p, [\alpha, \beta]) = \#\{$ Eigenvalues of T_p' lying in $[\alpha, \beta].\}$.

Serre's Equidistribution Theorem

Let N_λ, k_λ be positive integers such that k_λ is even, $N_\lambda + k_\lambda \to \infty$ and p is a prime not dividing N_λ. We denote the dimension of $S(N_\lambda, k_\lambda)$ as r_λ.

Then, as $N_\lambda + k_\lambda \to \infty$, for every $[\alpha, \beta] \subseteq [-2, 2]$,

$$\frac{D(p, [\alpha, \beta])}{r_\lambda} \sim \int_\alpha^\beta \mu_p$$

where

$$\mu_p = \frac{2(p+1)}{\pi} \frac{\left(1 - \frac{x^2}{4}\right)^{1/2}}{(p^{1/2} + p^{-1/2})^2 - x^2} dx.$$

$$\text{dim } S(N,k) \sim \psi(N)\frac{k-1}{12}$$

as $N + k \to \infty$ where

$$\psi(N) = N \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

Theorem 1 :

For any interval $[\alpha, \beta] \subseteq [-2, 2]$,

$$D(p, [\alpha, \beta]) = \psi(N)\frac{k-1}{12}\int_{\alpha}^{\beta} \mu_p$$

$$+ \mathrm{O}\left(\frac{\psi(N) k \log p}{\log(kN)}\right).$$

If we choose $\alpha = \beta$, then

$$\left(\int_{\alpha}^{\beta} \mu_p\right) = 0.$$

For a fixed a lying in $[-2, 2]$, let $D(p, a)$ denote the number of eigenvalues of $T_p'(N, k)$ equal to a.

Corollary

For $a \in [-2, 2]$,

$$D(p, a) = \mathrm{O}\left(\frac{k\psi(N)\log p}{\log(kN)}\right).$$

Let us denote the dimension of $S(N,k)$ as r. For $1 \leq i \leq r$, let $a_{p,i}$ denote the i-th eigenvalue of $T_p(N,k)$. Then, for a positive integer N and a fixed prime p not dividing N,

Theorem 2 :

$$\#\{i :\ 1 \leq i \leq r,\ \frac{a_{p,i}}{p^{\frac{k-1}{2}}} = \alpha\}$$

$$\leq 289 \left((k-1)\psi(N) \frac{\log p}{\log(kN)} \right).$$

Theorem 3

$$\#\{i : 1 \leq i \leq r,\ a_{p,i} \in \mathbb{Z}\}$$

$$\leq 289 \left(4p^{\frac{k-1}{2}} + 1 \right) \left((k-1)\psi(N) \frac{\log p}{\log(kN)} \right).$$

Theorem 4

Let d *be a positive integer and* $K_{p,i} = \mathbb{Q}(a_{p,i})$ *for every* $1 \le i \le r$. *Then*

$$\#\{i : 1 \le i \le r,\ [K_{p,i} : \mathbb{Q}] = d\}$$

$$\le 289 C_d \left((k-1)\psi(N) \frac{\log p}{\log(kN)} \right),$$

where

$$C_d = d \prod_{i=1}^{d} \left(2\binom{d}{i} \left(2p^{\frac{k-1}{2}} \right)^i + 1 \right).$$

Theorem 5

For odd N, there is an effectively computable constant B_d such that if $J_0(N)$ is isogenous to a product of simple abelian varieties with dimensions less than or equal to d, then $N \leq B_d$.