Statement of Research Interests

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1 Overview

My main research interests are in analytic number theory, harmonic analysis and arithmetic geometry. More particularly, my doctoral research work is concerned with equidistribution, extremal functions in Fourier analysis, modular cusp forms, Eichler-Selberg trace formula and arithmetic of modular curves. In my Ph.D. thesis[11], I apply techniques from harmonic analysis to study the asymptotic distribution of eigenvalues of Hecke operators acting on the space of cusp forms of weight k and level N. In [6], we show how the results of this thesis form a special case of a much more general phenomenon. In the next section of this research statement, we discuss equidistribution of sequences of real numbers and a generalisation of the famous Erdös-Turán inequality. In sections 3 and 4, we study the equidistribution of eigenvalues of Hecke operators acting on spaces of modular cusp forms. Section 4 contains the main result of my Ph.D. thesis. In section 5, we discuss several applications of the theorem mentioned in section 4. Finally, in section 6, we outline topics of future research. In sections 7 and 8, I will explain two more problems in my research programme.

2 Effective equidistribution

A sequence of real numbers $\{x_n\}$ is said to be equidistributed mod 1 if for every interval $I \subset [0,1]$, we have

$$\lim_{V\to\infty} \frac{\#\{n \leq V : x_n \bmod 1 \in I\}}{V} = \mu(I),$$

where $\mu(I)$ is the usual Lebesgue measure equal to the length of the interval I. The well-known criterion of Weyl [4] states that the sequence $\{x_n\}$ is equidistributed if and only if for every integer $m \neq 0$,

$$\sum_{n\leq V} e(mx_n) = \mathrm{o}(V)$$

as $V \to \infty$, where $e(t) := e^{2\pi i t}$. In 1948, Erdös and Turán [3] proved the following inequality which can be viewed as an effective version of Weyl's criterion: there exist constants c_1, c_2 such that

$$|\#\{n \leq V : x_n \text{ mod } 1 \in I\} - V\mu(I)| \leq \frac{c_1 V}{M+1} + c_2 \sum_{m=1}^{M} \frac{1}{m} \left| \sum_{n\leq V} e(mx_n) \right|.$$

The pair of constants $c_1 = 1$ and $c_2 = 3$ is given on page 8 of Montgomery [4]. In many applications of interest, the sequence x_n may not be equidistributed with respect to the Lebesgue measure, but with respect to some other measure. With these applications in mind, it is useful to derive a variant of the Erdös-Turán inequality. To this end, suppose that the *Weyl limits*

$$c_m = \lim_{V\to\infty} \frac{1}{V} \sum_{n\leq V} e(mx_n)$$

exist for every integer m and that

$$\sum_{m=-\infty}^{\infty} |c_m| < \infty.$$

Let $\mu = F(-x)dx$, where

$$F(x) = \sum_{m=-\infty}^{\infty} c_m e(mx).$$

We also define $||\mu||$ to be the supremum of $|F(x)|$ for $x \in [0,1]$. In [6], we prove the following variant of the Erdös-Turán inequality:

Theorem 1 *Let $N_I(V) := \#\{n \leq V : x_n \in I\}$. With the c_m's defined as above, and $I = [a,b]$, set*

$$D_I(V) := |N_I(V) - V\mu(I)|.$$

Then,

$$D_I(V) \leq \frac{V||\mu||}{M+1} + \sum_{1 \leq |m| \leq M} \left(\frac{1}{M+1} + \ min \ \left(b-a, \frac{1}{\pi|m|} \right) \right) \left| \sum_{n=1}^{V} e(mx_n) - Vc_m \right|,$$

if V and M are natural numbers.

In the next two sections, we discuss the effective equidistribution of eigenvalues of Hecke operators, which can be viewed as a special example of the phenomenon explained in Theorem 1.

3 Modular cusp forms

In 1916, Srinivasa Ramanujan introduced a function generated by the infinite product

$$x \prod_{n=1}^{\infty} (1-x^n)^{24} = x(1-x)^{24}(1-x^2)^{24}(1-x^3)^{24} \cdots .$$

The above product can be formally written as the sum

$$\sum_{n=1}^{\infty} \tau(n)x^n = \tau(1)x + \tau(2)x^2 + \cdots ,$$

where, for every natural number n, $\tau(n)$ is the coefficient of x^n. Ramanujan obtained the values of $\tau(n)$ up to $n = 30$ and conjectured three interesting properties of these coefficients. While two of them were proved by Mordell in 1920, the third conjecture remained insurmountable for a very long time. This conjecture states that for any prime number p,

$$-2p^{11/2} \leq \tau(p) \leq 2p^{11/2}.$$

In the 1930s, Erich Hecke laid the foundations of a much deeper, underlying theory of which the Tau function is a special example, namely the theory of modular forms. Modular forms are complex-analytic functions on the upper half plane satisfying certain inner symmetries and growth conditions. Modular forms play an extremely important role in number theory and carry fundamental information about points of elliptic curves over finite fields, values of the partition function, class numbers and representations of integers by quadratic forms. We are concerned with some special modular forms, known as cusp forms. More precisely, given positive integers k and N, a cusp form of weight k and level N is an analytic function $f(z)$ on the upper half complex plane such that for all integers a, b, c, d with $ad-bc=1$ and $N|c$,

$$f\left(\frac{az+b}{cz+d}\right)=(cz+d)^k f(z).$$

In addition, such a function "vanishes at the cusps". In particular, the Delta function

$$\Delta(z):=q\prod_{n=1}^{\infty}(1-q^n)^{24},\ q=e^{2\pi iz},$$

is a classic cusp form of level 1 and weight 12. For fixed values of N and k, the set of modular cusp forms of weight k and level N is a finite dimensional vector space over $\mathbb{C}$. On such a space, Hecke defined, for each integer $n\geq 1$, some special operators T_n, which now bear his name. The eigenvalues of Hecke operators satisfy the kind of properties conjectured by Ramanujan for $\tau(n)$'s. We are now ready to describe the main results of my PhD thesis.

4 Distribution of Hecke eigenvalues

Let $S(N,k)$ be the space of cusp forms of weight k and level N and for every positive integer n, let $T_n(N,k)$ be the n-th Hecke operator acting on $S(N,k)$. Let $s(N,k)$ be the dimension of $S(N,k)$ and let $a_{p,i}$, $1\leq i\leq s(N,k)$ denote the eigenvalues of T_p, counted with multiplicity. The asymptotic distribution of eigenvalues of the Hecke operator T_p on $S(N,k)$ for a prime p is an interesting and difficult problem. By a result of Deligne, we know that the eigenvalues of T_p lie in the interval

$$[-2p^{\frac{k-1}{2}},2p^{\frac{k-1}{2}}].$$

Thus, the eigenvalues of the normalized Hecke operator $T_p'=\frac{T_p}{p^{\frac{k-1}{2}}}$ lie in the interval $[-2,2]$. If we fix N and k and vary the prime p, the distribution of these eigenvalues is predicted by the Sato-Tate conjecture (see [9]). In the generic case, the eigenvalues of a fixed Hecke eigenform are expected to be equidistributed in $[-2,2]$ with respect to the measure

$$\mu_\infty=\frac{1}{\pi}\sqrt{\left(1-\frac{x^2}{4}\right)}dx.$$

Recently, Taylor [12] has announced that this conjecture is true when $k=2$, N is squarefree and the eigenform has rational integer coefficients. Such forms correspond to elliptic curves over $\mathbb{Q}$ by a celebrated theorem of Wiles [13].

In his 1997 paper [8], Serre considered a "vertical" Sato-Tate conjecture by fixing a prime p and varying N and k. He proved the following theorem:

Theorem 2 *Let N_λ, k_λ be positive integers such that k_λ is even, $N_\lambda+k_\lambda\to\infty$ and p is a prime not dividing N_λ. Then the family of eigenvalues of the normalized p-th Hecke operator*

$$T_p'(N_\lambda,\,k_\lambda)=\frac{T_p(N_\lambda,\,k_\lambda)}{p^{(k_\lambda-1)/2}}$$

is equidistributed in the interval $\Omega=[-2,2]$ with respect to the measure

$$\mu_p=\frac{p+1}{\pi}\frac{(1-x^2/4)^{1/2}}{(p^{1/2}+p^{-1/2})^2-x^2}dx.$$

Serre's theorem has many applications, and some of the most interesting ones are as follows:

For any positive integer d,

$$\#\{1 \leq i \leq r : [\mathbb{Q}(a_{p,i}) : \mathbb{Q}] \leq d\} = \mathrm{o}(s(N,k)) \text{ as } N+k \to \infty.$$

In particular,

$$\#\{1 \leq i \leq r : a_{p,i} \in \mathbb{Z}\} = \mathrm{o}(s(N,k)) \text{ as } N+k \to \infty.$$

Let $J_0(N)$ be the Jacobian of the modular curve $X_0(N)$. From the above results, one can deduce that there are only finitely many values such that $J_0(N)$ is isogenous to a product of elliptic curves. More generally, there are only finitely many values of N such that $J_0(N)$ is isogenous to a product of simple abelian varieties A_f with dimensions less than or equal to d. The drawback of Serre's theorem and its proof is that it does not give us an effective bound for these values of N. In my PhD thesis, I apply techniques from harmonic analysis to prove the following effective version of Serre's theorem:

Theorem 3 *Let N be a postitive integer and p be a prime not dividing N. For an interval $[\alpha, \beta] \subset [-2,2]$,*

$$\frac{1}{s(N,k)} \#\{1 \leq i \leq r : \frac{a_{p,i}}{2p^{\frac{k-1}{2}}} \in [\alpha, \beta]\}$$
$$= \int_{\alpha}^{\beta} \mu_p + \mathrm{O}\left(\frac{\log p}{\log kN}\right),$$

where the implied constant is effectively computable.

The above theorem is a special case of Theorem 1 as shown in [6]. We now discuss some interesting applications of Theorem 3.

5 Applications of effective equidistribution

If $\alpha = \beta$, then the integral in Theorem 3 is zero. Thus, for a fixed number α,

$$\#\{i : \frac{a_{p,i}}{p^{\frac{k-1}{2}}} = \alpha\} = \mathrm{O}\left(\frac{s(N,k)\log p}{\log(kN)}\right). \tag{1}$$

By careful estimation, in the special case $\alpha = \beta$, we also get a sharper error term. Moreover, keeping future applications in mind, we determine an explicit constant in the error term. In fact, we have the following result:

Theorem 4 *Let N be a positive integer and p be a prime not dividing N. Then, for a fixed number α lying in $[-2,2]$,*

$$\#\{i : \frac{a_{p,i}}{p^{\frac{k-1}{2}}} = \alpha\} \leq \frac{3s(N,k)\log p}{\log kN} + 63\left(kN\frac{\log p}{\log kN}\right),$$

for $N \geq e^{1024}$.

In certain cases, the estimates become substantially smaller. For example, we can obtain better bounds when N is prime or when N is squarefree. Moreover, when $k = 2$, which is a case of special interest, further improvements can be made. As a consequence of Theorem 4, we deduce the following:

Theorem 5 *Let d be a positive integer and $K_{p,i} = \mathbb{Q}(a_{p,i})$ for every $1 \leq i \leq r$. Then*

$$\#\{i : [K_{p,i} : \mathbb{Q}] = d\}$$
$$\leq d \prod_{i=1}^{d} \left(2\binom{d}{i} \left(2p^{\frac{k-1}{2}}\right)^i + 1 \right) \left(\frac{3s(N,k)\log p}{\log kN} + 63 \left(kN \frac{\log p}{\log kN} \right) \right),$$

for $N \geq e^{1024}$.

Theorem 5 gives us important arithmetic information about $J_0(N)$, the Jacobian of the modular curve $X_0(N)$. For which values of N is $J_0(N)$ isogenous to a product of $\mathbb{Q}$-simple abelian varieties of dimension less than or equal to d? Serre's theory (see Theorem 7 of [8]) shows that there are only finitely many for a given value of d but does not give an effective bound. Theorem 5 in the special case $k = 2$ gives us such a bound. We have the following result:

Theorem 6 *For every positive integer $d \geq 1$, there is an effectively computable constant B_d such that if $J_0(N)$ is isogenous to a product of $\mathbb{Q}$-simple abelian varieties A_f with* $\dim A_f \leq d$, *then $N \leq B_d$.*

6 Future research

From the standpoint of applications, it would be interesting to improve the constants in our estimates. This would have important consequences for numerical computation, especially in the context of factorisation of Jacobians of modular curves into $\mathbb{Q}$-simple abelian varieties of bounded dimension. For example, Henri Cohen [1] has proved that an odd N such that $J_0(N)$ is isogenous to a product of elliptic curves is bounded above by 48800. Then, with the help of numerical calculations he further establishes that the only odd values for N such that $J_0(N)$ is isogenous to a product of elliptic curves are $N \leq 21$ and $N =$ 25, 27, 33, 37, 45, 49, 57, 75, 99 and 121. In [14], Takuya Yamauchi determined all positive integers N(odd and even) for which $J_0(N)$ is isogenous to a product of elliptic curves. However, the effective determination of all values of N for which $J_0(N)$ is isogenous to a product of simple abelian varieties each of dimension ≤ 2 has not been carried out. Our theorem gives an effective bound for N. However, this is a crude bound and finer estimates can be obtained with more care. Also, after finding a bound for N, it is another matter to actually determine the finite values of N for which this holds. I hope to address these issues in future research.

Serre's results and our effective versions are restricted to the case when p is a prime not dividing N. In the future, I would also like to investigate the distribution of eigenvalues of T_p acting on $S(N,k)$ in the case when p is a prime dividing N.

7 Multiple Hurwitz zeta functions

The study of special values of the Riemann zeta function has motivated the study of multiple zeta values (MZVs) or the multiple zeta functions defined as:

$$\zeta(s_1, ..., s_r) = \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}.$$

In the paper [5], we have studied the multiple Hurwitz zeta function:

$$\zeta(s_1, ..., s_r; x_1, ..., x_r) := \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{(n_1 + x_1)^{s_1} \cdots (n_r + x_r)^{s_r}},$$

as well as the cognate multiple L-functions of Goncharov [2]:

$$L(s_1, ..., s_r; \chi_1, ..., \chi_r) := \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{\chi_1(n_1)\chi_2(n_2)\cdots\chi_r(n_r)}{n_1^{s_1} n_2^{s_r} \cdots n_r^{s_r}},$$

where $\chi_1, \chi_2, ..., \chi_r$ are Dirichlet characters (of necessarily the same modulus). We show that the meromorphic continuation of the multiple Hurwitz zeta function $\zeta(s_1, ..., s_r; x_1, ..., x_r)$ is a simple consequence of the meremorphic continuation of the multiple zeta function. This also allows us to deduce the meromorphic continuation of multiple L-functions to $\mathbb{C}^r$. The precise location of the singularities of the multiple L-functions is an open problem worthy of future research. This I plan to investigate in the coming years. I would also like to study special values of the multiple Hurwitz zeta function.

8 Averages of exponents in factoring integers

A natural number $n \geq 2$ can be written uniquely as a product of prime numbers,

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l},\ \alpha_i \geq 1.$$

We define

$$M(n) = \mathrm{Max}\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$$

and

$$m(n) = \min\{\alpha_1, \alpha_2, \cdots, \alpha_l\}.$$

We will take $M(1) = m(1) = 1$. The sums $\sum_{n \leq x} M(n)$ and $\sum_{n \leq x} m(n)$ have been well investigated in the literature. In 1969, Ivan Niven [7] proved that

$$\sum_{n \leq x} M(n) \sim Bx, \ \text{ as } x \to \infty \tag{2}$$

where

$$B = 1 + \sum_{k=2}^{\infty} \left(1 - \frac{1}{\zeta(k)}\right).$$

B can be evaluated to be 1.705211 approximately. He also showed that

$$\sum_{n \leq x} m(n) = x + \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)} x^{\frac{1}{2}} + \mathrm{o}(x^{\frac{1}{2}}). \tag{3}$$

The order of the error term $\sum_{n \leq x} M(n) - Bx$ is extremely interesting because it is related to the largest real part of the zeroes of the Riemann zeta function. Thus, it has a clear connection with the Riemann hypothesis. Niven's estimate in equation (3) for $\sum_{n \leq x} m(n)$ can also be further sharpened by assuming the Riemann hypothesis. In [10], we prove the following theorem:

Theorem 7 *Assuming the Riemann hypothesis,*

$$\sum_{n \leq x} M(n) = Bx + \mathrm{O}(x^{\frac{17}{54} + \epsilon}). \tag{4}$$

Moreover, unconditionally, we have

$$\sum_{n \leq x} M(n) = Bx + \Omega(x^{\frac{1}{4}}). \tag{5}$$

In order to describe our next result which sharpens equation (3), we note that for $k \geq 2$, and $K = \frac{1}{2}(3k^2 + k - 2)$, there are constants $a_{r,k}$ $(2k + 2 < r \leq K)$ such that

$$\left(1 + \frac{v^k}{1-v}\right)(1 - v^k)(1 - v^{k+1}) \cdots (1 - v^{2k-1}) = 1 - v^{2k+2} + \sum_{r=2k+3}^{K} a_{r,k} v^r.$$

We define

$$F_k(s) = \zeta(ks)\zeta((k+1)s) \cdots \zeta((2k-1)s) \prod_p (1 - p^{-(2k+2)s} + \sum_{r=2k+3}^{K} a_{r,k} p^{-rs}),$$

where the infinite product runs over all primes p. We also define, for $0 \leq i \leq k - 1$,

$$\gamma_{i,k} = \text{Res}_{\ s=1/(k+i)} \frac{F_k(s)}{s}.$$

The following theorem has been proved in [10]:

Theorem 8 *Assuming the Riemann hypothesis, we have*

$$\sum_{n \leq x} m(n) = x + \sum_{i=2}^{7} A(i) x^{\frac{1}{i}} + \mathrm{O}(x^{\frac{12}{85}+\epsilon}), \tag{6}$$

where

$$A(2) = \gamma_{0,2},\ A(3) = \gamma_{1,2} + \gamma_{0,3},\ A(4) = \gamma_{1,3} + \gamma_{0,4},$$
$$A(5) = \gamma_{2,3} + \gamma_{1,4} + \gamma_{0,5},\ A(6) = \gamma_{2,4} + \gamma_{1,5} + \gamma_{0,6},$$

and

$$A(7) = \gamma_{3,4} + \gamma_{2,5} + \gamma_{1,6} + \gamma_{0,7}.$$

Also, unconditionally, we have

$$\sum_{n \leq x} m(n) = x + \sum_{i=2}^{7} A(i) x^{1/i} + \Omega(x^{\frac{1}{10}}). \tag{7}$$

In the future, I would like to investigate if the estimates in the above two theorems can be further improved under the assumption of the Riemann hypothesis or even some quasi-Riemann hypothesis. Could one also obtain optimal error terms for the sums $\sum_{n \leq x} M(n)$ and $\sum_{n \leq x} m(n)$ conditionally or unconditionally?

The techniques used in proving the above theorems can also be generalised to estimate averages of exponents in factoring integers which lie in a certain arithmetic progression or integers which are of the form $p - 1$, p being a prime number. This is work in preparation.

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