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The energy of Ginzburg–Landau vortices

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We consider the Ginzburg–Landau equation in dimension two. We introduce a key notion of the vortex (interaction) energy. It is defined by minimizing the renormalized Ginzburg–Landau (free) energy functional over functions with a given set of zeros of given local indices. We find the asymptotic behaviour of the vortex energy as the inter-vortex distances grow. The leading term of the asymptotic expansion is the vortex self-energy while the next term is the classical Kirchhoff–Onsager Hamiltonian. To derive this expansion we use several novel techniques.

1 Introduction

The Ginzburg–Landau equation in various dimensions and for various internal symmetries plays a key role in condensed matter and nonlinear optics. This equation has the form

$$-\Delta\psi + g(|\psi|^2)\psi = 0, \tag{1.1}$$

where $g(|\psi|^2) = |\psi|^2 - 1$ (in fact, a particular form for g is not important, what matters is that g is monotonically increasing to ∞ and $g(0) < 0$), with the boundary condition

$$|\psi(x)| \to 1 \qquad \text{as } |x| \to \infty. \tag{1.2}$$

In this paper, we study (1.1)–(1.2) in the simplest and most important case $\psi: \mathbb{R}^2 \to \mathbb{C}$. Physically, this case is realized in nonlinear optics, superfluid thin films and high-temperature superconductors. The latter often have a layer structure with weak coupling between layers. Thus in the first approximation the layers can be considered as independent. In the case of superconductors the Ginzburg-Landau equation is coupled to a magnetic field, but in many situations the latter can be neglected, which leads to Eqns (1.1)–(1.2). Moreover, many elements of the analysis of those equations are independent of whether the magnetic field is present or not.

Solutions of equations (1.1)–(1.2) are classified by the total index (winding number) of ψ, considered as a vector field on $\mathbb{R}^2$, at ∞, i.e.

$$\deg\,\psi \;:=\; \frac{1}{2\pi}\int_{|x|=R} d(\arg\,\psi) \tag{1.3}$$

for R sufficiently large. We call this index (as opposed to local indices of ψ considered below) the *degree* (or *total vorticity*) of ψ.

It has been shown [17, 5, 10, 23] (see also Hagan [16]) that for any any n, equation (1.1) has a solution, unique modulo symmetry transformations, of the form

$$\psi^{(n)}(x) \;=\; f^{(n)}(r)e^{in\theta}, \tag{1.4}$$

where $1 > f^{(n)} \geqslant 0$ and is monotonically increasing from $f^{(n)}(0) = 0$ to 1 as r increases to ∞. Of course, deg $\psi^{(n)} = n$. For $n = 0$, $f^{(n)}(r) = 1$. These are the most symmetric solutions to (1.1), called the *n-vortices.* They were discovered by Ginzburg & Pitaevskii [14], and are similar to Abrikosov vortices [1]. Here n is the *degree* (or *vorticity*) of the vortex $\psi^{(n)}$. Of course, each solution $\psi^{(n)}$ generates a one-parameter for $n = 0$, and a three-parameter for $|n| > 0$, family of solutions of (1.1). The latter are obtained by applying symmetry transformations to $\psi^{(n)}$.

In this paper, we introduce and analyze the notion of intervortex energy, E. This notion is used in Ovchinnikov & Sigal [24] to study the dynamics of vortices. We connect properties of E with the question of existence of static multivortex solutions (this point is further pursued in Ovchinnikov & Sigal [26]). We find asymptotic behaviour of the intervortex energy at large intervortex separations. The leading term of the asymptotics is well-known in the literature as a Kirchhoff–Onsager Hamiltonian and is used to describe dynamics of vortices.

We suspect that the intervortex energy we introduce is related to the renormalized energy of Bethuel, Brezis & Hélein [2]. Now we describe the results of this paper more precisely.

The Ginzburg–Landau equation is the Euler–Lagrange equation for the renormalized Ginzburg–Landau energy functional, $\mathscr{E}_{\text{ren}}(\psi)$ (see Ovchinnikov & Sigal [23], and § 2). 'Low' energy functions $\psi : \mathbb{R}^2 \to \mathbb{C}$ are essentially determined by their vortex structure, i.e. by their zeros and their local indices. We call a collection of these data a *vortex configuration.* More precisely, consider once-differentiable functions ψ: $\mathbb{R}^2 \to \mathbb{C}$ satisfying $|\psi| \to 1$ as $|x| \to \infty$. Let $\underline{a} = (a_1, \dots, a_K)$ and $\underline{n} = (n_1, \dots, n_K)$, where $a_j \in \mathbb{R}^2$ and $n_j \in \mathbb{Z}$, $j = 1, \dots, K$. We say that ψ has the vortex configuration $\underline{c} = (\underline{a}, \underline{n})$, and write conf $\psi = \underline{c}$, if ψ has zeros (only) at $a_1, \dots, a_K$ with local indices $n_1, \dots, n_K$, respectively, i.e.

$$\int_{\gamma_j} d(\arg\,\psi) \;=\; 2\pi n_j$$

for any contour γ_j containing a_j, but not the other zeros of ψ and for $j = 1, \dots, K$. Now we define

$$E(\underline{c}) \;=\; \inf\left\{\mathscr{E}_{\text{ren}}(\psi) \mid \text{conf}\,\psi = \underline{c}\right\}. \tag{1.5}$$

(See Fröhlich & Struwe [11] for related variational problems with topological constraints.)

By property (c) of § 2, $E(\underline{c}) > -\infty$. We call $E(\underline{c})$ the *energy of the vortex configuration* $\underline{c}$. The force acting on a vortex configuration is $-\nabla_{\underline{a}} E(\underline{c})$. We suggest:

Conjecture 1.1 *Problem (1.5) has a minimizer (and consequently, equations (1.1)–(1.2) have a solution with the vortex configuration* $\underline{c}$*) if and only if* $\nabla_{\underline{a}} E(\underline{c}) = 0$.

In this paper, we prove, with some extra assumptions, the 'only if' part of this conjecture (see § 3).

In § 4–6 we find the following asymptotics:

$$E(\underline{c}) \;=\; \sum_{i=1}^{K} E_{n_i} + H(\underline{c}) + \mathrm{Rem}, \tag{1.6}$$

as $r(\underline{a}) := \min_{i \neq j} |a_{ij}| \to \infty$. Here $a_{ij} = a_i - a_j$, $\underline{c} = (\underline{a}, \underline{n})$, $E_n = \mathscr{E}_{\mathrm{ren}}(\psi^{(n)})$, the self-energy of the n-vortex,

$$H(\underline{c}) \;=\; -\pi \sum_{i \neq j} n_i n_j \ln |a_{ij}|, \tag{1.7}$$

the Kirchoff–Onsager Hamiltonian, and

$$\mathrm{Rem} \;=\; \begin{cases} O\big(r(\underline{a})^{-2}\big) & \text{if } \nabla_{\underline{a}} H(\underline{c}) = 0, \\ O\big(r(\underline{a})^{-1}\big) & \text{otherwise.} \end{cases} \tag{1.8}$$

Equation (1.6) can be tested as follows. Let a configuration $\underline{c} = (\underline{a}, \underline{n})$ correspond to distant vortices, i.e. $r(\underline{a}) := \min_{i \neq j} |a_{ij}| \gg 1$. Then we expect that the function

$$\psi_{(0)}(x) \;=\; \prod_{i=1}^{K} \psi^{(n_i)}(x - a_i),$$

describing the K 'independent' vortices, has the energy, $\mathscr{E}_{\mathrm{ren}}(\psi_{(0)})$, close to $E(\underline{c})$. That this is indeed the case follows from our analysis in § 4.

Note that the function $H(\underline{c})$, defined in (1.7), is a standard Hamiltonian of the vortex dynamics used in the literature [13, 7, 22, 8, 9, 24, 6, 20]. A similar function serves as the Hamiltonian of the vortex motion in Euler's equation (see Marchioro & Pulvirenti [21]).

We demonstrate (1.6)–(1.8) by establishing upper and lower bounds. To prove the upper bound we use that $E(\underline{c}) \leqslant \mathscr{E}_{\mathrm{ren}}(\psi)$ for any ψ with $\operatorname{conf} \psi = \underline{c}$, and show that for a certain class of ψ's (roughly, those which look like $\psi^{(n_j)}(x - a_j)$ for $|x - a_j| \ll r(\underline{a})$), $\mathscr{E}_{\mathrm{ren}}(\psi) =$ r.h.s. of (1.6). To the latter end we decompose the integral in $\mathscr{E}_{\mathrm{ren}}(\psi)$ into the integrals over the discs $D_j = \{x \in \mathbb{R}^2 \mid |x - a_j| \leqslant r_0\}$, $j = 1, \ldots, K$, and the rest

$$\mathbb{R}^2 \backslash \bigcup_{j=1}^{K} D_j$$

and estimate of each integral accordingly.

The lower bound, $E(\underline{c}) \geqslant$ r.h.s. of (1.6), is more difficult. To prove it we consider a system with 'impurities':

$$\mathscr{E}_{\underline{\lambda}}(\psi) \;=\; \mathscr{E}_{\mathrm{ren}}(\psi) + \sum_{j=1}^{K} \frac{\lambda_j}{2} \int \delta_{b_j} |\psi|^2, \tag{1.9}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_K)$, $\lambda_j > 0$, are coupling constants of impurities and $\delta_{b_j} \geqslant 0$ are their potentials which we take to be $\delta_b(x) = \frac{1}{2\pi \bar{r}} \delta(|x - b| - \bar{r})$ with $\bar{r} = O(1)$, or a smooth version of this. We place the centres, b_j, of the impurities close to the vortex centers a_j. We argue that for $\lambda_j \geqslant \mathrm{const} |\nabla_{a_j} E(\underline{c})|\ \forall j$, the energy functional $\mathscr{E}_{\underline{\lambda}}(\psi)$ has a minimizer, $\psi_{\underline{\lambda}}$, in the class of ψ's with $\operatorname{conf} \psi = \underline{c}$. Since we can insert in the right-hand side of (1.5) the condition $|\psi| \leqslant 1$ without changing the result, we have

$$E(\underline{c}) \;\geqslant\; E_{\underline{\lambda}}(\underline{c}) - \sum_{j=1}^{K} \lambda_j, \tag{1.10}$$

where $E_\lambda(\underline{c}) = \mathscr{E}_{\underline{\lambda}}(\psi_{\underline{\lambda}})$.

On the second step, using the Euler–Lagrange equation,

$$-\Delta\psi + (|\psi|^2 - 1)\psi \;=\; -\Sigma \lambda_j \delta_{b_j} \psi, \tag{1.11}$$

for $\psi_{\underline{\lambda}}$, we show that $\psi_{\underline{\lambda}}$ belongs to the class of functions used in the proof of the upper bound. Hence, $E_{\underline{\lambda}}(\underline{c}) = \mathscr{E}_{\underline{\lambda}}(\psi_{\underline{\lambda}})$ is of the form of the right-hand side of (1.6). This completes the proof of the lower bound and therefore of (1.6).

Equation (1.11) is rather subtle. We analyze it using an implicit function theorem. Denote the map $\psi \to -\Delta\psi + \big(|\psi|^2 - 1 + \sum \lambda_j \delta_{b_j}\big)\psi$ by $G_0(\psi)$. Let $\psi_0(x)$ be an approximate solution to (1.11) (e.g. see the function $\psi_{(0)}(x)$ above). Expanding $G_0(\psi)$ around ψ_0 we rewrite (1.11) as

$$L_0(\xi) \;=\; -G_0(\psi_0) - N(\xi), \tag{1.12}$$

where $\xi := \psi - \psi_0$, the operator L_0 is the linearization of $G_0(\psi)$ around ψ_0 and $N(\xi)$ is the nonlinear in ξ part of $G_0(\psi_0 + \xi)$. The next step is to invert the operator L_0, and consider the resulting equation as a fixed point equation. However, here we run into a problem. First, the continuous spectrum of the operator L_0 fills the positive semiaxis $[0, \infty)$ going all the way to 0. Secondly, L_0 has near zero modes due to the fact that the vortex solutions $\psi^{(n_j)}(x - a_j)$, $j = 1, \ldots, K$, break the translational (as well as rotational/gauge) symmetry of the original equation (1.1). These near zero modes have long-range tails, and as a result, they interact rather strongly even at large distances. A careful analysis carried out in § 6 stipulates convincingly that (1.11) has a solution of the desired form, provided the strengths, λ_j, and locations, b_j, of the impurities are adjusted in such a way that the right-hand side of the resulting equation (1.12) is orthogonal to the corresponding (near) zero translational modes. Thus, we remove small denominators and secular terms so that the perturbation theory is valid.

2 Renormalized Ginzburg–Landau energy

It is a straightforward observation that (1.1) is the equation for critical points of the following functional:

$$\mathscr{E}(\psi) \;=\; \frac{1}{2}\int \Big(|\nabla\psi|^2 + \frac{1}{2}(|\psi|^2 - 1)^2\Big). \tag{2.1}$$

Indeed, if we define the variational derivative, $\partial_\psi \mathscr{E}(\psi)$, of $\mathscr{E}$ by

$$\mathrm{Re}\int \xi \partial_\psi \mathscr{E}(\psi) \;=\; \frac{\partial}{\partial\lambda}\mathscr{E}(\psi_\lambda)\Big|_{\lambda=0} \tag{2.2}$$

for any path ψ_λ s.t. $\psi_0 = \psi$ and $\frac{\partial}{\partial\lambda}\psi_\lambda\big|_{\lambda=0} = \xi$, then the left-hand side of (1.1) is equal to $\overline{\partial_\psi \mathscr{E}(\psi)} = \partial_{\bar\psi}\mathscr{E}(\psi)$ for $\mathscr{E}(\psi)$ given in (2.1).

Equation (2.1) is the celebrated Ginzburg–Landau (free) energy. However, there is a problem with it in our context. It is shown [23] that if ψ is an arbitrary C^1 vector field on $\mathbb{R}^2$ s.t. $|\psi| \to 1$ as $|x| \to \infty$ uniformly in $\hat{x} = \frac{x}{|x|}$ and $\deg \psi \neq 0$, then $\mathscr{E}(\psi) = \infty$.

We renormalize the Ginzburg–Landau energy functional as follows (see Ovchinnikov & Sigal [23]). Let $\chi(x)$ be a smooth real function on $\mathbb{R}^2$ s.t.

$$\chi(x) \;=\; \begin{cases} 1 & \text{for } |x| \geqslant R + R^{-1}, \\ 0 & \text{for } |x| \leqslant R. \end{cases} \tag{2.3}$$

Define

$$\mathscr{E}_{\text{ren}}(\psi) \;=\; \frac{1}{2}\int \Big(|\nabla\psi|^2 - \frac{(\deg\,\psi)^2}{r^2}\chi + F(|\psi|^2) \Big) d^2x \tag{2.4}$$

where

$$F(u) \;=\; \frac{1}{2}(u-1)^2\;. \tag{2.5}$$

We list here the most important properties of $\mathscr{E}_{\text{ren}}(\psi)$ (see Ovchinnikov & Sigal [23] for the proofs):

(a) $\partial_{\bar\psi}\mathscr{E}_{\text{ren}}(\psi) = -\Delta\psi + F'(|\psi|^2)\psi$.

(b) Given n let $M_n = \Big\{ \psi = f e^{i\varphi} \mid \int\limits_{|x|\geqslant 2} \frac{1}{r^2}|1-f^2| < \infty,\ f$ is continuous and $f(0) = 0$, $\int |\nabla(\varphi - n\theta)| r^{-1} < \infty$ and $\int |\nabla(\varphi - n\theta)|^2 < \infty \Big\}$. Then $\mathscr{E}_{\text{ren}}(\psi) < \infty\ \forall \psi \in M_n$.

(c) We have the following bound from below:

$$\mathscr{E}_{\text{ren}}(\psi) \;\geqslant\; \mathscr{E}_{B(0,\bar R)}(\psi) + \frac{1}{2}\int_{|x|\geqslant \bar R} \Big(\big|\nabla|\psi|\big|^2 - \frac{1}{2}|\nabla\varphi|^4 \Big) d^2x, \tag{2.6}$$

where $\bar R = R + R^{-1}$, $\varphi = \arg\,\psi$, and for $\Omega \subset \mathbb{R}^2$,

$$\mathscr{E}_{\Omega}(\psi) \;=\; \frac{1}{2}\int_{\Omega} \Big(|\nabla\psi|^2 - \frac{(\deg\,\psi)^2}{r^2}\chi + F(|\psi|^2) \Big) d^2x. \tag{2.7}$$

3 The energy of vortex configurations

In this section we discuss the connection between $-\nabla E(\underline{c})$, the force acting on the vortex centers, and the existence of a minimizer for the variational problem (1.5). It is clear intuitively that such a minimizer exists if and only if $\nabla E(\underline{a}) = 0$. However, to establish this fact is not so easy. In what follows $\underline{n}$ is fixed and we use the notation $E(\underline{a}) = E(\underline{c})$ and $H(\underline{a}) = H(\underline{c})$ for $\underline{c} = (\underline{a}, \underline{n})$. We begin our analysis with

Proposition 3.1 *If there is a minimizer for variational problem* (1.5), *then this minimizer satisfies the Ginzburg–Landau equation* (1.1).

Proof. Let ψ be a minimizer for (1.5). Since for any differentiable function $\xi\colon \mathbb{R}^2 \to \mathbb{C}$ vanishing together with its gradient sufficiently fast at ∞ and vanishing at the points $a_1, \dots, a_m$ we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial\lambda}\mathscr{E}_{\text{ren}}(\psi + \lambda\xi)\Big|_{\lambda=0} \\ &= \text{Re}\int \bar\xi\big(-\Delta\psi + (|\psi|^2 - 1)\psi\big), \end{aligned}$$

we conclude that ψ satisfies (1.1) for $x \neq a_1, \dots, a_m$. On the other hand, since $\psi \in H_1^{\text{loc}}(\mathbb{R}^2)$, we have that $-\Delta\psi + (|\psi|^2 - 1)\psi \in H_{-1}^{\text{loc}}(\mathbb{R}^2)$. Hence $-\Delta\psi + (|\psi|^2 - 1)\psi = 0$ on $\mathbb{R}^2$. □

We assume that the function $E(\underline{a})$ is differentiable and that there are approximate minimizers $\psi_{\underline{a}}^{(\varepsilon)}$ s.t. $\nabla\mathscr{E}_{\text{ren}}(\psi_{\underline{a}}^{(\varepsilon)}) \to \nabla E(\underline{a})$ as $\varepsilon \to 0$, pointwise in $\underline{a}$. Then we have

Theorem 3.2 *Let* $\nabla E(\underline{a}) \neq 0$. *Then the variational problem* (1.5) *has no minimizer.*

Proof. Assume, on the contrary, that problem (1.5) has a minimizer, $\psi_{\underline{a}}$. By Proposition 3.1 it solves (1.1) and therefore is a critical point of the functional $\mathscr{E}_{\text{ren}}(\psi)$. Assume first that there is a path $\underline{a}(t)$, $0 \leqslant t \leqslant \varepsilon$, for some $\varepsilon > 0$, in $\mathbb{R}^{2n}$, starting at $\underline{a}$ in the direction $\underline{e}$ s.t. $\underline{e} \cdot \nabla E(\underline{a}) \neq 0$ and problem (1.5) has minimizers, $\psi_{\underline{a}(t)}$, for the points $\underline{a}(t)$. Then

$$\frac{d}{dt}\mathscr{E}_{\text{ren}}(\psi_{\underline{a}(t)})\Big|_{t=0} = \dot{\underline{a}}(0) \cdot \nabla E(\underline{a}) \neq 0,$$

which contradicts to the statement that $\psi_{\underline{a}}$ is a critical point of $\mathscr{E}_{\text{ren}}(\psi)$. If (1.5) has no minimizers for any curve $\underline{a}(t)$, $0 < t \leqslant \varepsilon$, s.t. $a(0) = \underline{a}$ and $\dot{\underline{a}}(0) \cdot \nabla E(\underline{a}) \neq 0$, then we pick approximate minimizers in accordance with the above condition and proceed as in the argument above. □

We conjecture that the assumptions formulated above are always satisfied. (Approximate minimizers which we expect satisfy it are constructed in § 5 by a method of impurities.) In any case, the proof above shows that minimizers of (1.5) can be located only on a discrete set of level sets of the function $E(\underline{a})$.

4 Asymptotics of energy of vortex configurations. Upper bound

In this section we study asymptotics of the energy, $E(\underline{a})$, of vortex configurations $(\underline{a}, \underline{n})$ as $r(\underline{a}) \to \infty$. Recall that $r(\underline{a}) = \min_{i \neq j} |a_{ij}|$. In what follows the parameter R in (2.3) is taken to be sufficiently large, and we display the R-dependence in the energies by writing $E_R(\underline{a})$ for $E(\underline{a})$ and $E_{n,R}$ for E_n. Our main result is the following relation:

$$E_R(\underline{a}) = E_R^{(0)} + \text{Rem} + O\left(\frac{1}{R^2}\right), \tag{4.1}$$

where

$$E_R^{(0)} = \sum_i E_{n_i,R} + H\left(\frac{a}{R}\right) \tag{4.2}$$

and the remainder, Rem, satisfies the estimate

$$\text{Rem} = \begin{cases} O\big(r(\underline{a})^{-2}\big) & \text{if } \nabla H(\underline{a}) = 0 \\ O\big(r(\underline{a})^{-1}\big) & \text{otherwise} \end{cases} \tag{4.3}$$

We demonstrate (4.1) by verifying that its right-hand side is an upper and lower bound for the left-hand side. The upper bound is obtained in this section, while the lower one is obtained in the next one.

Theorem 4.1 [Upper bound.] *In the notation above,*

$$E_R(\underline{a}) \;\leqslant\; E_R^{(0)} + \mathrm{Rem} + O(R^{-2})\,.$$

A proof of this theorem follows from the variational inequality

$$E_R(\underline{a}) \;\leqslant\; \mathscr{E}_{\mathrm{ren}}(\psi)\,, \tag{4.4}$$

for any function ψ having the given vortex configuration, and Proposition 4.2 below, showing that for an appropriate ψ, $\mathscr{E}_{\mathrm{ren}}(\psi)$ has the asymptotics given by the right-hand side of (4.1).

Define a class of functions ψ on which we test (4.4) by the following relations:

$$\psi \;=\; fe^{i\varphi_0}, \quad \text{where} \quad \varphi_0 \;=\; \sum_j \varphi_j, \quad \text{with} \quad \varphi_i(x) \;=\; n_i\theta(x-a_i), \tag{4.5}$$

$$f = f_i + O\Big(\frac{1}{r_i \cdot r(\underline{a})^n}\Big) \quad \text{and} \quad \int_0^{2\pi} \mathrm{Re}(f-f_i)d\theta_i = O\Big(\frac{1}{r(\underline{a})^{n+1}}\Big), \quad \text{if } r_i \ll r(\underline{a}), \tag{4.6}$$

where r_i and θ_i are the polar coordinates of $x - a_i$, $\forall\, i$, and

$$f = 1 + O\Big(\frac{1}{d(x,\underline{a})^2}\Big) \qquad \text{if } d(x,\underline{a}) \gg 1\,, \tag{4.7}$$

with the corresponding estimates of their first derivatives, where $n = 2$ if $\nabla H(\underline{a}) = 0$ and $n = 1$ otherwise and where $\psi_i(x) = \psi^{(n_i)}(x - a_i)$, $f_i = |\psi_i|$, and

$$d(x,\underline{a}) \;=\; \min_j |x - a_j|.$$

An example of such a function is $\psi_0 = f_0 e^{i\varphi_0}$, where φ_0 is as above and $f_0 = \sum_1^K f_j\chi_j$, where $\{\chi_j\}_1^K$ is a partition of unity, $\sum_1^K \chi_j = 1$, having the following properties $\forall\ j$:

$$B\Big(a_j, \frac{1}{3}r(\underline{a})\Big) \subset \operatorname{supp}\chi_j \quad \text{and} \quad \nabla^n\chi_j = O\big(r(\underline{a})^{-n}\big), \quad n = 0,1,2.$$

In what follows we need the following notation:

$$\varphi_{(i)} \;=\; \sum_{j, j\neq i} \varphi_j.$$

Proposition 4.2 *Assume ψ satisfies* (4.5)–(4.7). *Then*

$$\mathscr{E}_{\mathrm{ren}}(\psi) \;=\; E_R^{(0)} + \mathrm{Rem} + O\Big(\frac{1}{R^2}\Big), \tag{4.8}$$

where, we recall, $E_R^{(0)}$ is given by (4.2) *and*

$$\mathrm{Rem} \;= \begin{cases} O\big(r(\underline{a})^{-2}\big) & \text{if } \nabla H(\underline{a}) = 0, \\ O\big(r(\underline{a})^{-2}\ln r(\underline{a})\big) & \text{otherwise.} \end{cases} \tag{4.9}$$

Remark 4.1. Of course, to prove the upper bound in Theorem 4.1 it suffices to estimate $\mathscr{E}_{\text{ren}}(\psi)$ for one function only, so we can take, for example, $f = f_j$ for $|x - a_j| \ll r(\underline{a})\ \forall\ j$. However, Proposition 4.2 is also used below (see § 5) to obtain a lower bound on $E_R(\underline{a})$.

Proof. Let $D_j = D(a_j, r_0)$, the disc with the centre at a_j and of the radius r_0. We specify r_0 as $r_0 < \frac{1}{2}r(\underline{a})$ and $r_0 = O(r(\underline{a}))$. We decompose the energy functional as

$$\mathscr{E}_{\text{ren}}(\psi) \;=\; \sum_j \int_{D_j} e(\psi) + \int_{(\cup D_j)^c} e(\psi), \tag{4.10}$$

where $D^c := \mathbb{R}^2 \backslash D$ and $e(\psi)$ is the energy density,

$$e(\psi) \;=\; \frac{1}{2}|\nabla\psi|^2 + \frac{1}{4}(|\psi|^2 - 1)^2.$$

Let $e_1(\varphi) = \frac{1}{2}|\nabla\varphi|^2$ and $\langle f(\psi)\rangle = f(\psi) - \sum_k f(\psi_k)$. Eqn (4.7) implies

$$\int_{(\cup D_k)^c} e(\psi) \;=\; \int_{(\cup D_k)^c} e_1(\varphi_0) + \int_{(\cup D_k)^c} O\big(d(x,\underline{a})^{-4}\big). \tag{4.11}$$

Next, the estimates

$$|\psi_i| \;=\; 1 + O(r_i^{-2}) \tag{4.12}$$

and

$$\nabla|\psi_i| \;=\; O(r_i^{-3}) \tag{4.13}$$

give

$$\int_{(\cup D_k)^c} e_1(\varphi_i) \;=\; \int_{(\cup D_k)} e(\psi_i) + O(r_0^{-2}). \tag{4.14}$$

This together with (4.11) yields

$$\int_{(\cup D_k)^c} \langle e(\psi)\rangle \;=\; \frac{1}{2}\sum_{i \neq j} \int_{(\cup D_k)^c} \nabla\varphi_i \nabla\varphi_j + O(r_0^{-2}). \tag{4.15}$$

Next, we write ψ in the region D_i as $\psi = e^{i\varphi_0}(f_i + \xi)$, where $f_i \equiv |\psi_i|$. As a single-valued harmonic function in D_i, $\varphi_{(i)}$ has the following expansion

$$\varphi_{(i)} \;=\; \sum_{m=0}^{\infty} c_m r_i^m \cos m(\theta_i - \beta_i^{(m)}),$$

where, we recall, r_i and θ_i are the polar coordinates of $x - a_i$ and c_m and $\beta_i^{(m)}$ are some constants. This implies that

$$\int_{D_i} \nabla\varphi_i \cdot \nabla\varphi_{(i)} \;=\; 0.$$

Using this relation and that

$$\int_{D_j} f_j \nabla\varphi_j \cdot \nabla \operatorname{Im}\xi \;=\; n_j \int_{D_j} f_j \frac{\partial}{\partial\theta} \operatorname{Im}\xi \;=\; 0,$$

we obtain

$$\int_{D_i} e(\psi) \;=\; \int_{D_i} e(\psi_i) + \int_{D_i} e_1(\varphi_{(i)}) + R_1 + R_2,$$

where

$$R_1 = \int_{D_i} (f_i^2 - 1)\Big(\nabla\varphi_i \cdot \nabla\varphi_{(i)} + \frac{1}{2}|\nabla\varphi_{(i)}|^2\Big),$$

and

$$\begin{aligned} R_2 = \int_{D_i} \Big\{ & (|\nabla\varphi_0|^2 + f_i^2 - 1) f_i \operatorname{Re}\xi + f_i^2 (\operatorname{Re}\xi)^2 \\ & + \frac{1}{2}|\nabla\varphi_0|^2|\xi|^2 + \frac{1}{2}|\nabla\xi|^2 + 2\nabla f_i \cdot \nabla \operatorname{Re}\xi + f_i \nabla\varphi_{(i)} \cdot \nabla \operatorname{Im}\xi \\ & + \operatorname{Im}(\xi\nabla\varphi_0 \cdot \nabla\xi) + \frac{1}{2}(f_i^2 - 1 + 2 f_i \operatorname{Re}\xi)|\xi|^2 + \frac{1}{4}|\xi|^4 \Big\}. \end{aligned}$$

Using that $|\nabla\varphi_{(i)}(x)|^2 = O\big(d(x,\underline{a})^{-2}\big)$ and $\nabla\varphi_i(x) = O(r_i^{-1})$, expanding

$$\nabla\varphi_{(i)}(x) \;=\; \nabla\varphi_{(i)}(a_i) + O\Big(\frac{r_i}{r(\underline{a})^2}\Big)$$

and using that $\int_{D_i}(1 - f_i^2)\nabla\varphi_i = 0$, we obtain

$$R_1 \;=\; O\Big(\frac{\ln r_0}{r(\underline{a})^2}\Big).$$

Using that, due to (4.6), $\xi = O\Big(\frac{1}{r \cdot r(\underline{a})}\Big)$ and $\int_0^{2\pi} \operatorname{Re}\xi \, d\theta = O\Big(\frac{1}{r(\underline{a})^2}\Big)$, and using that $|\nabla\varphi_i|^2 + f_i^2 - 1 = O(r_i^{-4})$, we find

$$R_2 \;=\; O\Big(\frac{\ln r_0}{r(\underline{a})^2}\Big).$$

Finally, we observe that due to (4.14),

$$\begin{aligned} \frac{1}{2}\int_{D_k} |\nabla\varphi_{(k)}|^2 &= \sum_{j \neq k} \int_{D_k} \big(e_1(\psi_j) + I\big) \\ &= \sum_{j \neq k} \int (e(\psi_j) + I) + O(r_0^{-2}), \end{aligned}$$

where $I := \frac{1}{2}\sum_{i \neq j} \nabla\varphi_i \nabla\varphi_j$. Collecting the estimates above, we arrive at

$$\int_{D_k} \big(\langle e(\psi)\rangle - I\big) \;= O\Big(\frac{\ln r_0}{r(\underline{a})^2}\Big) + O\Big(\frac{1}{r_0^2}\Big), \tag{4.16}$$

which together with (4.10) and (4.15) yields

$$\mathscr{E}_{\mathrm{ren}}(\psi) \;=\; E + O\Big(\frac{1}{r_0^2}\Big), \tag{4.17}$$

where $E = \int \Big(g - \frac{n^2}{r^2}\chi\Big)$ with $g = \sum_j e(\psi_j) + I$ and $n = \deg \psi$.

Now, by the definition of the cut-off function χ ($\chi \geqslant 0$, $\chi = 1$ for $|x| \geqslant R$) we have

$$E \leqslant \int_{D_R} g + \int_{D_R^c} \Big(g - \frac{n^2}{2r^2}\Big), \tag{4.18}$$

where D_R is the disc around the origin of radius R. Now, by the definition ($a_i \ll R$)

$$\int_{D_R} e(\psi_i) \;=\; \int_{D_R + a_i} e(\psi^{(n_i)}) \;=\; E_{n_i,R} + O\Big(\frac{1}{R^2}\Big). \tag{4.19}$$

Next, we show that

$$\frac{1}{2}\int_{D_R}\nabla\varphi_i\nabla\varphi_j \;=\; -\pi n_i n_j \ln\Big(\frac{|a_{ij}|}{R}\Big). \tag{4.20}$$

We compute

$$\int_{D_R}\nabla\varphi_i\nabla\varphi_j \;=\; n_i n_j\int_0^{2\pi}\int_0^R \frac{r-a\cos\theta}{r^2+a^2-2ar\cos\theta}dr\,d\theta,$$

where $a=|a_{ij}|$. Furthermore,

$$\int_0^{2\pi}\frac{r-a\cos\theta}{r^2+a^2-2ar\cos\theta}d\theta \;=\; \frac{2\pi}{r}\begin{cases}1 & \text{if } r>a,\\ 0 & \text{if } r<a.\end{cases}$$

The last two equations yield (4.20). Observe also that up to a multiplicative constant expression (4.20) can be found from symmetry considerations: the invariance of the integral on the left-hand side under translations ($a_i\to a_i+h$ and $a_j\to a_j+h$ $\forall h\in\mathbb{R}^2$) and rotations ($a_i\to ga_i$ and $a_j\to ga_j$ $\forall g\in O(2)$) imply that it depends only upon $|a_{ij}|$. Its scaling properties under the dilations ($a_i\to\lambda a_i$ and $a_j\to\lambda a_j$ $\forall\lambda\in\mathbb{R}$) imply that it is a multiple of $\ln\Big(\frac{|a_{ij}|}{R}\Big)$.

Equations (4.19) and (4.20) imply

$$\int_{D_R} g = \sum E_{n_i,R} + H(\underline{a}/R) + O(1/R^2). \tag{4.21}$$

Next we estimate the second integral on the r.h.s. of (4.18). By Eqns (4.13) and (4.14) we have

$$g = \frac{1}{2}|\nabla\varphi_0|^2 + O\big(d(x,\underline{a})^{-4}\big).$$

Furthermore, expanding the terms $\nabla\theta(x-a_j)$ in $\nabla\varphi_0(x)=\sum n_j\nabla\theta(x-a_j)$ around the point x we obtain

$$\nabla\varphi_0(x) = n\nabla\theta(x) - \theta''(x)\sum n_j a_j + O\Big(\frac{\sum n_j a_j^2}{d(x,\underline{a})^3}\Big), \tag{4.22}$$

where $\theta''(x)$ is the Hessian of $\theta(x)$. Choosing the origin so that $\sum n_j a_j = 0$ eliminates the second term on the right-hand side. (Otherwise we could have used that by an explicit computation we have

$$\theta''(x)\nabla\theta(x) = -\frac{x}{r^4},$$

the integral of which over the exterior of the ball $B(0,R)$ vanishes.) Hence

$$\int_{D_R^c}\Big(g-\frac{n^2}{2r^2}\Big) = \int_{D_R^c} O\Big(\frac{\sum n_j a_j^2}{d(x,\underline{a})^4}\Big) \tag{4.23}$$

$$= O\Big(\frac{\sum n_j a_j^2}{R^2}\Big). \tag{4.24}$$

Equations (4.17)–(4.21) with $r_0 = O\big(r(\underline{a})\big)$ and Eqn (4.23) imply (4.8) with Rem $=$ $O\Big(\frac{\ln r(\underline{a})}{r(\underline{a})^2}\Big)$. Similarly, one obtains (4.8)–(4.9) in the forceless case. □

Remark 4.2. The estimate (4.9) can be considerably improved in the force-free case, if we use instead of $e_1(\varphi) = \frac{1}{2}|\nabla\varphi|^2$ the density

$$e_2(\varphi) \;=\; \frac{1}{2}|\nabla\varphi|^2 - \frac{1}{2}|\nabla\varphi|^4,$$

which is a better approximation to the density $e(\psi)$, and instead of (4.12) and (4.13) we use

$$|\psi_i| \;=\; 1 - \frac{1}{2}|\nabla\varphi_i|^2 + O(r_i^{-4}) \tag{4.25}$$

and

$$\nabla|\psi_i| \;=\; -\frac{1}{2}\nabla|\nabla\varphi_i|^2 + O(r_i^{-5}), \tag{4.26}$$

respectively. Indeed, proceeding as above, we find in the force-free case that

$$\mathscr{E}(\psi) \;=\; E_R^{(0)} + K + O\left(\frac{\ln r(\underline{a})}{r(\underline{a})^4}\right) + O\left(\frac{1}{R^2}\right), \tag{4.27}$$

where

$$K \;=\; -\frac{1}{2}\int_{D_R}\left(|\nabla\varphi_0|^4 - \sum_j |\nabla\varphi_j|^4\right). \tag{4.28}$$

This result is used in Ovchinnikov & Sigal [26].

5 Lower bound on energy of vortex configurations. Pinning effect

Lower bounds are notoriously difficult. An additional problem which faces us is that unless the condition $\nabla E_R(\underline{a}) = 0$ is satisfied minimization problem (1.5) has no minimizer. To circumvent the latter difficulty we introduce defects into the system, and use the fact that sufficiently strong defects bind the vortices (the effect of pinning). More precisely, we introduce the new energy functional

$$\mathscr{E}_{\underline{\lambda}}(\psi) \;=\; \mathscr{E}_R(\psi) + \Sigma\frac{1}{2}\lambda_j\int\delta_{b_j}|\psi|^2, \tag{5.1}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_K)$, $\lambda_j > 0$, are coupling constants of the defects and $\delta_{b_j} \geqslant 0$ are their potentials, centered at points $b_j \in \mathbb{R}^2$ depending on $\underline{a}$ and very close to the a_j's. The λ_j's and b_j's will be determined later. We take δ_b to be either

$$\delta_b = \frac{1}{2\pi\bar{r}}\,\delta(|x-b| - \bar{r}), \tag{5.2}$$

where $\bar{r} = O(1)$, or a smooth version of this, i.e. δ_b is a smooth function supported in the annulus

$$\{x \in \mathbb{R}^2 \mid \bar{r} \leqslant |x-b| \leqslant \bar{r} + \delta\} \tag{5.3}$$

for some sufficiently small δ and satisfying

$$\int\delta_b \;=\; 1. \tag{5.4}$$

Remark 5.1. Sometimes it is convenient to modify the definition of δ_b in such a way that $\forall j$, $\delta_{b_j} f_j$ does not contain harmonics in θ with $|m| \geqslant 2$, where (r,θ) are the polar coordinates of $y = x - a_j$ (see the harmonic analysis of (6.8) in the next section). To this end, we replace (5.2) by

$$\delta_b \;=\; \frac{1}{2\pi\bar{r}}\frac{\partial\gamma}{\partial r}(x-b)\delta\big(\gamma(x-b)\big),$$

where $\gamma(x)$ is a slight deformation (modulation) of the function $|x| - \bar{r}$, or by a smooth version of the latter function, so that

$$\int_0^{2\pi} \delta_{b_j} f_j e^{im\theta} d\theta \;=\; 0 \qquad \text{for } |m| \geqslant 2.$$

With the potential δ_b defined as above it is argued below that $\mathscr{E}_{\underline{\lambda}}(\psi)$ has a minimizer among functions with the given vortex configuration $(\underline{a}, \underline{n})$, provided

$$\lambda_j \;\geqslant\; C|\nabla_{a_j} E(\underline{a})| \tag{5.5}$$

for an appropriate constant C.

We argue as follows. Clearly, a minimizer, if it exists, has near a_j the form of the jth vortex, $\psi_j, \forall_j$. The relevant contribution of the second term on the right-hand side of (5.1) near a_j is $\frac{1}{2}\lambda_j \int \delta_{b_j}|\psi|^2$. If the centre of the vortex is at the centre of the ring $\operatorname{supp}\delta_{b_j}$, i.e. $a_j = b_j$, then the contribution of this term is approximately $\frac{1}{2}\lambda_j\alpha_{n_j}^2\varepsilon^{2n_j}$, where $\varepsilon = \bar{r}$ is the radius of the interior boundary of the support of δ_{b_j}, provided $\varepsilon \ll 1$ and α_{n_j} is defined by the expansion

$$|\psi^{(n)}(x)| \;=\; \alpha_n|x|^n + O(|x|^{n+2}) \tag{5.6}$$

for $|x| \ll 1$ (remember, $\psi_j(x) = \psi^{(n_j)}(x - a_j)$). On the other hand, if the centre of the vortex is in $\operatorname{supp}\delta_{b_j}$ (i.e. $a_j \in \operatorname{supp}\delta_{b_j}$), then the corresponding contribution is approximately

$$\begin{aligned}\frac{1}{2}\lambda_j \int \delta_{b_j} f_j^2 &= \frac{1}{2}\lambda_j\alpha_{n_j}^2 \frac{\varepsilon^{2n_j}}{2\pi}\int_0^{2\pi}\Big(2\cos\frac{\theta}{2}\Big)^{2n_j} d\theta \\ &= \frac{1}{2}\lambda_j\alpha_{n_j}^2\varepsilon^{2n_j}\binom{2n_j}{n_j}.\end{aligned}$$

Since $\binom{2n_j}{n_j} \geqslant 1$, this shows that it is more energetically advantageous for the vortex to be inside the ring, supp δ_{b_j}, than in its middle. Moreover, the force needed to remove the vortex from the inside of the ring is approximately

$$\frac{1}{2}\lambda_j \int \delta_{b_j}\frac{\partial}{\partial r} f_j \;=\; -\frac{2}{\pi}\lambda_j n_j \alpha_{n_j}^2 \varepsilon^{2n_j-1}\sum_{m=0}^{2n_j-1}\frac{(-1)^{n_j-m}\binom{2n_j-1}{m}}{2(n_j-m)-1}, \tag{5.7}$$

where we have used that

$$\int_0^{\pi}\Big(2\cos\frac{\theta}{2}\Big)^{2n_j-1} d\theta \;=\; -2\sum_{m=0}^{2n_j-1}\frac{(-1)^{n_j-m}\binom{2n_j-1}{m}}{2(n_j-m)-1}.$$

On the other hand, the force with which the remaining vortices act on the jth vortex is $-\nabla_{a_j}E(\underline{a})$. This shows that for a fixed ε, to keep the j-th vortex inside the ring $\operatorname{supp}\delta_{a_j}$ we need $\lambda_j = O\big(|\nabla_{a_j}E(\underline{a})|\big)$, hence condition (5.5) for the existence of minimizer for energy functional (5.1).

Remark 5.2. In fact, the force $-\nabla_{a_j}\cdot\frac{1}{2}\lambda_j\int\delta_{b_j}f_j^2\approx\frac{1}{2}\lambda_j\cdot\nabla f_j^2(|b_i-a_j|)$ exerted on the vortex j by the defects is also present when the vortex is outside of the defect ring and it takes its greatest value at the distance r_0 defined by

$$\frac{\partial^2 f_j^2}{\partial r^2}\Big|_{r=r_0}, \;=\; 0, \tag{5.8}$$

provided $\varepsilon=\bar{r}\ll 1$. This greatest value is

$$F_{\max} \;=\; \lambda_j\,\frac{\partial f_j^2}{\partial r}\Big|_{r=r_0}. \tag{5.9}$$

This implies, in particular, that the range of the potential created by the defect is $O(1)$.

The minimizer, $\psi_{\underline{\lambda}}$, of $\mathscr{E}_{\underline{\lambda}}(\psi)$ satisfies the Euler-Lagrange equation

$$-\Delta\psi+(|\psi|^2-1)\psi \;=\; -\Sigma\lambda_j\delta_{b_j}\psi. \tag{5.10}$$

An analysis of this equation conducted in the next section shows that this minimizer satisfies conditions (4.5)–(4.7). Then Proposition 4.2 implies that the energy

$$E_{\underline{\lambda}}(\underline{a}) \;:=\; \inf\{E_{\underline{\lambda}}(\psi)\mid \operatorname{conf}\psi=\underline{c}\} \tag{5.11}$$

which is equal to $\mathscr{E}_{\underline{\lambda}}(\psi_{\underline{\lambda}})$, satisfies

$$E_{\underline{\lambda}}(\underline{a}) \;=\; E_R^{(0)}+\sum\frac{1}{2}\lambda_j\int\delta_{b_j}|\psi_{\underline{\lambda}}|^2+\mathrm{Rem}+O\Big(\frac{1}{R^2}\Big), \tag{5.12}$$

where Rem is given in (5.9). On the other hand, since the infimum can be taken over ψ's with $|\psi|\leqslant 1$, we have that

$$E_R(\underline{a}) \;\geqslant\; E_{\underline{\lambda}}(\underline{a})-\Sigma\lambda_j. \tag{5.13}$$

Due to (5.5), $\Sigma\lambda_j$ can be taken to be of the same order as Rem. Hence, we conclude that

$$E_R(\underline{a}) \;\geqslant\; E_R^{(0)}+\mathrm{Rem}+O\Big(\frac{1}{R^2}\Big) \tag{5.14}$$

with Rem given in (4.3).

6 Equation (5.10): method of geometric solvability

In this section we show that (5.10) has a solution satisfying (4.5)–(4.7), provided condition (5.5) (or (6.11)) holds. This solution is the minimizer of variational problem (5.11). This result was used in § 5 to obtain estimate (5.12).

We explain the main ideas of our method. We rewrite (5.10) as $G(f)=0$, where $f=e^{-i\varphi_0}\psi$, and the map G is defined by

$$\begin{aligned}G(f):&=e^{-i\varphi_0}\big(-\nabla(e^{i\varphi_0}f)+(|e^{i\varphi_0}f|^2-1+\sum\lambda_j\delta_{b_j})e^{i\varphi_0}f\big)\\&=-\Delta_{\nabla\varphi_0}f+(f^2-1+\sum\lambda_j\delta_{b_j})f,\end{aligned}$$

with $\Delta_A:=\nabla_A^2$, $\nabla_A:=\nabla+iA$. Let $\psi_0=f_0e^{i\varphi_0}$ be an approximate solution to (5.10), i.e. f_0 is an approximate solution to $G(f)=0$. We look for a solution of the latter equation in the form

$$f=f_0+\xi \tag{6.1}$$

where ξ is a small fluctuation of the order $O(\frac{1}{r(\underline{a})})$. We expand

$$G(f_0 + \xi) = G(f_0) + L(\xi) - R(\xi),$$

where L is the linearized operator for the map $f \to G(f)$ around the function f_0:

$$L(\xi) := [-\Delta_{\nabla\varphi_0} + f_0^2 - 1 + \sum \lambda_j \delta_{b_j}]\xi + 2f_0 \mathrm{Re}(\bar{f}_0 \xi)$$

and the term $R(\xi)$ is the nonlinear in ξ part of $G(f_0 + \xi)$:

$$R(\xi) = -2\xi \mathrm{Re}(\bar{f}_0 \xi) - |\xi|^2 \xi.$$

Note that the operator L is self-adjoint in the inner product $\langle \xi, y \rangle := \mathrm{Re} \int \bar{\xi} y$. Now the equation $G(f_0 + \xi) = 0$ can be rewritten as

$$L(\xi) = -G(f_0) + R(\xi). \tag{6.2}$$

The first task now is to show that this equation can be solved for ξ. We demonstrate this nonrigorously by showing that for the choice of the parameters as mentioned above, the right-hand side – in the leading order – is orthogonal to the almost zero modes of the adjoint operator L^* $(= L)$. The latter modes are just the zero modes of the operator $e^{-i\varphi_j} \cdot L_{\psi_j} \cdot e^{i\varphi_j} =: L_j$, where L_{ψ_j} are the linearizations of the original equation (1.1), i.e. of the map $\psi \to \Delta\psi + (|\psi|^2 - 1)\psi$, around the shifted vortex solutions ψ_j. They are due to the fact that the vortex solutions, ψ_j, brake the translational (and rotatonal/gauge) symmetry of the original equation (1.1).

Finally, we specify the approximate solution, f_0, mentioned above. We define f_0 so that

$$f_0 = f_j \text{ in } D_j \ , \ 1 \leqslant j \leqslant k \ , \text{ and } f_0 = 1 + O\left(\frac{1}{r(\underline{a})^2}\right) \text{ in } D_0,$$

with the corresponding estimates on the derivatives of the remainder in the last equation. Such a function can be constructed with the help of an appropriate partition of unity (see the paragraph after equation (4.7) and the end of this section).

Now we proceed to the analysis of (6.2). We study (6.2) in each of the domains $D_j = \{x \in \mathbb{R}^2 \mid |x - a_j| \leqslant r_0\}$, $j = 1, \dots, K$, and $D_0 = \{x \in \mathbb{R}^2 \mid |x - a_j| \geqslant r_1 \forall j\}$, where $r_0 \ll r(\underline{a})$ and $r_1 \gg 1$, separately.

The disc D_j, $1 \leqslant j \leqslant k$

We fix j and set $r = r_j$. In D_j we have

$$L = L_j + O(\frac{1}{r(\underline{a})})$$

where the operator L_j was defined above and can be explicitly written out as

$$L_j(\xi) \ = \ (-\Delta + |\nabla\varphi_j|^2 + 2f_j^2 - 1)\xi - 2i\nabla\varphi_j \cdot \nabla\xi + f_j^2 \bar{\xi},$$

and

$$-G(f_0) = F_j + |\nabla\varphi_{(j)}|^2 f_j,$$

where

$$F_j = \lambda_j \delta_{b_j} f_j + 2i\nabla\varphi_{(j)} \cdot (\nabla f_j + i\nabla\varphi_j f_j). \tag{6.3}$$

Here we used that $|\nabla\varphi_0|^2 - |\nabla\varphi_j|^2 = 2\nabla\varphi_{(j)}\cdot\nabla\varphi_j + |\nabla\varphi_{(j)}|^2$. Observe that

$$e^{i\varphi_j}(\nabla f_j + i\nabla\varphi_j f_j) = \nabla\psi;$$

is the translational zero mode of the operator L_{ψ_j} and $\nabla f_j + i\nabla\varphi_j f_j$ is the zero mode of the operator L_j.

Thus, the equation (6.2) can be written as

$$L_j(\xi) \;=\; F_j + R_j(\xi), \tag{6.4}$$

where F_j is the leading part of a free term defined in (6.3) and $R_j(\xi) = F' + R'(\xi) + R''(\xi)$ with

$$F' = -|\nabla\varphi_{(j)}|^2 f_j\ , \tag{6.5}$$

$$R'(\xi) = -\lambda_j\delta_{b_j}\xi - (|\nabla\varphi_0|^2 - |\nabla\varphi_j|^2)\xi + 2i\nabla\varphi_{(j)}\cdot\nabla\xi, \tag{6.6}$$

and

$$R''(\xi) = -f_j|\xi|^2 - 2f_j(\operatorname{Re}\xi)\xi - |\xi|^2\xi. \tag{6.7}$$

Observe that the term $\sum_{i\neq j}\lambda_i\delta_{b_i}\psi$ is absent, since it is zero in the region $|x-a_j| \ll r(\underline{a})$.

Assuming $\xi = O\big(\frac{1}{r(\underline{a})}\big)$ and dropping the term $R_j(\xi)$, which is of the order $O\big(\frac{1}{r(\underline{a})^2}\big)$, from the right-hand side of (6.7), we arrive at the equation

$$L_j(\xi) = F_j. \tag{6.8}$$

As mentioned above, the operator L_j is related to the operator L_{ψ_j}, obtained by linearizing (1.1) around the solution ψ_j (see Ovchinnikov & Sigal [23]), as follows:

$$L_j(\xi) = e^{-i\varphi_j}L_{\psi_j}(e^{i\varphi_j}\xi). \tag{6.9}$$

Observe that L_j is self-adjoint, $L_j^* = L_j$, in the scalar product

$$\langle\eta,\xi\rangle = \operatorname{Re}\int\bar{\eta}\xi.$$

The only zero modes of the operator $L_j^* = L_j$, which decay at ∞, are those related to the translation symmetry of the equation, namely

$$\eta_k = e^{-i\varphi_j}\partial_{x_k}\psi_j, \qquad k = 1,2. \tag{6.10}$$

Hence, (6.8) is solvable only if

$$\operatorname{Re}\int\bar{\eta}_k F_j = 0 \quad \text{for } k = 1,2. \tag{6.11}$$

Below we will find conditions on λ_j and b_j for (6.11) to hold. For the moment we assume (6.11) and push on with our analysis.

Expand ξ in (6.8) in the Fourier series

$$\xi(x) \;=\; \sum_{m=-\infty}^{\infty}\xi^{(m)}(r)e^{im\theta}, \tag{6.12}$$

and define

$$\hat{\xi} \;=\; \bigoplus_{m\geqslant 0}\begin{pmatrix}\xi^{(m)}\\ \bar{\xi}^{(-m)}\end{pmatrix}. \tag{6.13}$$

Then, obviously ξ and $\hat{\xi}$ are in one-to-one correspondence, which we denote by $\xi \leftrightarrow \hat{\xi}$. Observe now that if $\xi \leftrightarrow \hat{\xi}$, then

$$L_j\xi \leftrightarrow \hat{L}\hat{\xi} := \bigoplus_{m\geqslant 0} L^{(m)}\begin{pmatrix}\xi^{(m)}\\ \bar{\xi}^{(-m)}\end{pmatrix}, \tag{6.14}$$

where

$$L^{(m)} = \begin{pmatrix} -\Delta_r + \frac{(n_j+m)^2}{r^2} + 2f_j^2 - 1 & f_j^2 \\ f_j^2 & -\Delta_r + \frac{(n_j-m)^2}{r^2} + 2f_j^2 - 1 \end{pmatrix}.$$

Here Δ_r stands for the radial Laplacian, $\Delta_r f = r^{-1}\partial_r(r\partial_r f)$ and we have used that $\varphi_j(x) = n_j\theta(x-a_j)$. Eqn (6.14) implies that (6.8) can be rewritten as

$$\hat{L}\hat{\xi} = \hat{F}, \tag{6.15}$$

where $\hat{F} = \bigoplus_{m\geqslant 0}\binom{F^{(m)}}{\bar{F}^{(-m)}}$ with

$$F^{(m)}(r) = (2\pi)^{-1}\int_0^{2\pi} F_j e^{-im\theta}d\theta. \tag{6.16}$$

Finally, observe that the translational zero modes (6.10) in the new representation become

$$\hat{\eta}_1 = i\hat{\eta}_2 = \bigoplus_{m\geqslant 0}\frac{1}{2}\begin{pmatrix} f_j' - \frac{n_j}{r_j}f_j \\ f_j' + \frac{n_j}{r_j}f_j\end{pmatrix}\delta_{m,1}. \tag{6.17}$$

This formula implies that (6.11) is equivalent to the relation

$$\int_0^\infty \begin{pmatrix} f_j' - \frac{n_j}{r_j}f_j \\ f_j' + \frac{n_j}{r_j}f_j\end{pmatrix}\cdot\begin{pmatrix}F^{(1)}\\ \bar{F}^{(-1)}\end{pmatrix} r dr = 0. \tag{6.18}$$

We analyze the operators $L^{(m)}$, $m \geqslant 0$. The operator-matrix $L^{(0)}$ can easily be diagonalized. A Perron–Frobenius argument given in Ovchinnikov & Sigal [23] shows that $L^{(0)} \geqslant 0$ and 0 is not an eigenvalue of $L^{(0)}$. Next, a similar (but more subtle) argument shows that $L^{(1)} \geqslant 0$ and 0 is a non-degenerate eigenvalue of $L^{(1)}$ (with the eigenfunction $\left(f_j' - \frac{n_j}{r_j}f_j, f_j' + \frac{n_j}{r_j}f_j\right)$ corresponding to the breaking of the translational symmetry of the Ginzburg–Landau equation by ψ_j (see Ovchinnikov & Sigal [23] for details). Here f_j' stands for the derivative of f_j w.r.t. to $r_j = |x - a_j|$. Finally,

$$L^{(m)} - L^{(1)} = \frac{m-1}{r_j^2}\begin{pmatrix} 2n_j + m + 1 & 0 \\ 0 & -2n_j + m + 1\end{pmatrix} \geqslant 0$$

and $\not\equiv 0$ for $m \geqslant 2n_j - 1$. Hence $L^{(m)} \geqslant 0$ and 0 is not an eigenvalue of $L^{(m)}$ for $m \geqslant 2n_j - 1$. For $2 \leqslant m < 2n_j - 1$, $L^{(m)}$ have negative eigenvalues, but still do not have an eigenvalue at zero (note that in general such eigenvalues, unless related to symmetries, are unstable and can be easily removed by small perturbations). We leave this fact without a proof since we can choose δ_{b_j} so that $F^{(m)} = 0$ for $|m| \geqslant 2$ (see Remark 5.1), so that we can solve (6.22) without using properties of the operators $L^{(m)}$, $m \geqslant 2$.

Due to condition (6.11), (6.18) has a unique solution which we write in the form

$$\hat{\xi} = (\hat{L})^{-1}\hat{F} = \bigoplus_{m\geqslant 0} G^{(m)}\begin{pmatrix} F^{(m)} \\ \bar{F}^{(-m)} \end{pmatrix}, \tag{6.19}$$

where $G^{(m)}$ is the (left, if $m = 1$) inverse of $L^{(m)}$. Observe now that $L^{(m)}$ are (matrix) ordinary differential operators, their (regularized) Green's functions can be found in terms of some special solutions to the homogeneous equations. This is done in Ovchinnikov & Sigal [27]. (It is convenient for technical reasons to include a part of $R'(\xi)$ into $L_j(\xi)$, namely, to replace L_j in (6.8) by $L_j + \lambda_j\delta_{b_j}$.) Results of [27] imply that $\hat{\xi}$ is of the same order as $\hat{F}$, i.e. as will be shown below, $O\left(\frac{1}{r(\underline{a})^2}\right)$ in the forceless case and $O\left(\frac{1}{r(\underline{a})}\right)$, otherwise. This, due to (6.1) and (6.12), implies (4.5) and the first part of (4.6).

Region D_0

In this region (5.10) coincides with Ginzburg–Landau equation (1.1), i.e. the right-hand side of (5.10) vanishes. In this region

$$L = L_0 + O(\frac{1}{r(\underline{a})^2}),$$

where the operator L_0 is related to the linearization of the map $\psi \to \Delta\psi + (|\psi|^2 - 1)\psi$ around $e^{i\varphi_0}$,

$$L_0(\xi) := \big(-\Delta - 2i\nabla\varphi_0\cdot\nabla + |\nabla\varphi_0|^2\big)\xi + 2\mathrm{Re}\xi, \tag{6.20}$$

and

$$G(f_0) = G(1) = |\nabla\varphi_0|^2 =: -F_0.$$

Assuming that $\nabla^n\xi = O\big(r(\underline{a})^{-n-2}\big)$ in the region D_0 and dropping terms of the order $O\big(r(\underline{a})^{-5}\big)$ we arrive at the equation

$$L_0(\xi) \;=\; F_0. \tag{6.21}$$

Taking the real and imaginary parts of this equation, we obtain (to leading order in $\frac{1}{|x|}$)

$$\mathrm{Re}\,\xi_0 \;=\; -\frac{1}{2}|\nabla\varphi_0|^2 \tag{6.22}$$

and

$$-\Delta\mathrm{Im}\,\xi_0 \;=\; -\nabla\varphi_0\cdot\nabla|\nabla\varphi_0|^2. \tag{6.23}$$

The last two equations show that

$$|\xi_0| \;=\; O\Big(\frac{1}{d(x,\underline{a})^2}\Big)$$

so that property (4.7) holds for the solution ψ. Moreover, these equations imply that ψ is of the form

$$\psi \;=\; e^{i(\varphi_0+\mathrm{Im}\,\xi_0)}\Big(1 - \frac{1}{2}|\nabla\varphi_0|^2 + O\Big(\frac{1}{d(x,\underline{a})^4}\Big)\Big), \tag{6.24}$$

where we remember $\mathrm{Im}\,\xi_0$ solves (6.23).

To solve (6.23) we have to take into account the boundary conditions on ∂D_0. Instead of this, we use the solutions of (6.8) as sources. Namely, we proceed as follows. Writing

$$\varphi = \varphi_0 + \operatorname{Im}\xi$$

and using that λ_j are real, we derive from (5.10)

$$-\Delta \operatorname{Im}\xi = \nabla(\varphi_0 + \operatorname{Im}\xi)\cdot\nabla\ln f^2, \tag{6.25}$$

where, we recall, $f = |\psi|$. Observe that while φ_0 is a multivalued function, $\operatorname{Im}\xi$ is a regular function on $\mathbb{R}^2$ vanishing at ∞. Thus, (6.25) can be written as

$$\operatorname{Im}\xi(x) = \frac{1}{2\pi}\int \ln|x-y|\nabla\big(\varphi_0(y) + \operatorname{Im}\xi(y)\big)\cdot\nabla\ln f^2(y)dy. \tag{6.26}$$

Let ξ_j be the solution of equation (6.8) for $j = 1,\dots,K$, and of equations (6.22)–(6.23) for $j = 0$, let $f_j = |\psi_j|$ for $j = 1,\dots,K$, and $= 1 - \frac{1}{2}|\nabla\varphi_0|^2$ for $j = 0$. In the right-hand side of (6.26) we take $\xi = \xi_j$ in D_j, $j = 0,\dots,K$, where $D_0 = \mathbb{R}^2\setminus\bigcup_j D_j$. Plugging this into the right-hand side of (6.26), we obtain the following equation for $\operatorname{Im}\xi_0$:

$$\begin{aligned}\operatorname{Im}\xi_0 = \frac{1}{2\pi}\sum_{j=0}^{K}\int_{D_j}\ln|x-y|\Big\{&\nabla\varphi_0(y)\cdot\nabla\ln\big(f_j(y) + \operatorname{Re}\xi_j\big)^2\\ &+\nabla\operatorname{Im}\xi_j(y)\cdot\nabla\ln f_j(y)^2\Big\}d^2y,\end{aligned} \tag{6.27}$$

where $\operatorname{Im}\xi_j$, $j = 1,\dots,k$, are given as above. We iterate this equation. On the first step, we drop $\operatorname{Im}\xi_0$ from the right-hand side. The resulting expression for $\operatorname{Im}\xi_0$ suffices for us.

The free term F_j (see (6.3))

In the rest of this section, we keep j fixed and let $y = x - a_j$, and let r and θ be the polar coordinates of the vector y. We consider the cases $\nabla_{a_j}H(\underline{a}) \neq 0$ and $\nabla_{a_j}H(\underline{a}) = 0$ separately.

(a) $\nabla_{a_j}H(\underline{a}) \neq 0$. The definition of $H(\underline{a})$, (1.7) (recall that $H(\underline{a}) = H(\underline{c})$), implies that

$$J\nabla_{a_j}H(\underline{a}) = -2\pi n_j\nabla\varphi_{(j)}(a_j), \tag{6.28}$$

where, we recall, $J = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}$. Using this equation and the fact that $J^* = -J$, we obtain that, modulo $O\Big(\frac{1}{r(\underline{a})^2}\Big)$,

$$\begin{aligned}&-2ie^{-i\varphi_j}\nabla\psi_j\cdot\nabla\varphi_{(j)}\\ &= -\frac{1}{\pi n_j r}\Big(if_j'J\hat{y} + \frac{1}{r}f_j\hat{y}\Big)\cdot\nabla_{a_j}H(\underline{a}).\end{aligned} \tag{6.29}$$

Then, taking into account Remark 5.1, we obtain for the Fourier coefficients, $F^{(m)}$, of F_j (see (6.16)) that

$$F^{(\pm m)} = F^{(\pm 1)}\delta_{m,\pm 1}. \tag{6.30}$$

Let $k_j = |k_j|e^{-i\alpha_j}$ be the complex number corresponding to the vector $-\frac{1}{2\pi n_j}J\nabla_{a_j}H(\underline{a})$.

We compute

$$F^{(\pm 1)} = i|k_j|e^{\mp i\alpha_j}\Big(f_j' \mp \frac{n_j}{r}f_j\Big) - \frac{1}{2\pi}\int_0^{2\pi}\lambda_j\delta_{b_j}f_je^{\mp i\theta}d\theta \tag{6.31}$$

(a) $\nabla_{a_j}H(\underline{a}) = 0$. First, we observe that since $\varphi_{(j)} = \sum_{k\neq j}\varphi_k$ is a single-valued function in the region $r = |x-a_j| < r(\underline{a})$ and is harmonic in $\mathbb{R}^2$, it has in this region the following Fourier series expansion:

$$\varphi_{(j)} = \sum_{m=0}^{\infty} c_m r^m \cos m(\theta - \beta_j^{(m)}) \tag{6.32}$$

for some amplitudes c_m and phases $\beta_j^{(m)}$ (with $c_m = O\big(r(\underline{a})^{-m}\big)$). Moreover, in the force-free configuration $c_1 = 0$. Using this we find that in the force-free case, the expression for (6.16) is

$$F^{(\pm m)} = \pm 2i\alpha re^{\mp 2i\beta_j^{(2)}}\Big(\pm f_j' - \frac{n_j}{r}f_j\Big)\delta_{m,2} - \frac{1}{2\pi}\int_0^{2\pi}\lambda_j\delta_{b_j}f_je^{\mp i\theta}\delta_{m,1}, \tag{6.33}$$

where the coefficient α is $O\big(r(\underline{a})^{-2}\big)$ (see (6.32)).

Conditions on λ_j and b_j

Now we derive the restrictions on the parameters λ_j and b_j implied by solvability conditions (6.11) or (6.18). We consider separately two cases.

(a) $\nabla_{a_j}H(\underline{a}) \neq 0$. Let $\vec{\eta} = \binom{\eta_1}{\eta_2}$, where η_j are given in (6.10). It is shown in Appendix A that

$$\mathrm{Re}\int \bar{\vec{\eta}}F_j = \nabla_{a_j}H(\underline{a}) - \frac{1}{2}\lambda_j\int\delta_{b_j}\nabla f_j^2, \tag{6.34}$$

where $\Delta a_j = b_j - a_j$. Equations (6.11) and (6.28) imply then that

$$\frac{1}{2}\lambda_j\int\delta_{b_j}\nabla f_j^2 = \nabla_{a_j}H(\underline{a}). \tag{6.35}$$

This fixes the direction in which the jth vortex centre, a_j, must be shifted relative to the center, b_j, of the circle $\operatorname{supp}\delta_{a_j}$. Indeed, it is shown in Appendix A that

$$\int\delta_{b_j}\nabla f_j^2 \approx \frac{1}{2}\Delta a_j f_j^{2\prime}(r_1).$$

This equation together with (6.21) implies that

$$\lambda_j = |\nabla_{a_j}H(\underline{a})|/|\Delta a_j|f_j^{2\prime}(r_1) \qquad \text{if } \nabla_{a_j}H(\underline{a}) \neq 0 \tag{6.36}$$

and that the direction $\widehat{\Delta a_j} = \Delta a_j/|\Delta a_j|$ should satisfy

$$\widehat{\Delta a_j} = \nabla_{a_j}H(\underline{a})/|\nabla_{a_j}H(\underline{a})| \qquad \text{if } \nabla_{a_j}H(\underline{a}) \neq 0. \tag{6.37}$$

(b) $\nabla_{a_j}H(\underline{a}) = 0$. Recall that the operator $L^{(2)}$ does not have a bounded zero mode. Hence, due to (6.33), we can set $\lambda_j = 0$ when solving (6.15) to the order of $O\big(r(\underline{a})^{-2}\big)$. Hence, (6.15) has a unique solution $O\big(r(\underline{a})^{-2}\big)$. The need for λ_j arises only in the next step of the perturbation theory in the small term $R(\xi)$, neglected previously, i.e. at $O\big(r(\underline{a})^{-3}\big)$. Thus, in this case we can take $\lambda_j = O\big(r(\underline{a})^{-3}\big)$ and so on.

Iteration scheme

Now we derive an equation allowing us to go beyond the first order perturbation theory. To this end, we use the method of geometric parametrices of Sigal [30]. Let $\{\chi_j\}_0^K$ be the partition of unity, i.e. $\sum_{j=0}^{K} \chi_j = 1$, s.t. $\operatorname{supp}\chi_j \subset D_j$. Let G_j be the left inverse (or the (regularized) Green's function) for the operator $M_j = L_j - 2i\nabla\varphi_{(j)} \cdot \nabla + \lambda_j \delta_{b_j}$, $j = 0, \dots, K$ (for $j = 0$ the last two terms on the right-hand side are absent). Assume the unknown function ξ is orthogonal to the translational zero modes of the operators $L_1, \dots, L_K$. Applying the operator $\sum_{j=0}^{K} \chi_j G_j$ to (5.10), and using (6.4) and the equations $G_j M_j(\xi) = \xi$ for $j = 0, \dots, K$, we obtain

$$\xi \;=\; \sum_{j=0}^{K} \chi_j G_j F_j + \sum_{j=0}^{K} \chi_j G_j \overline{R}_j(\xi), \tag{6.38}$$

where $\overline{R}_j(\xi) = R_j(\xi) - 2i\nabla\varphi_{(j)} \cdot \nabla\xi + \lambda_j \delta_{b_j} \xi$, and where we used that $M_j \xi = F_j + \overline{R}_j(\xi)$. Equation (6.38) is a fixed point problem. One can try to show that this problem has a solution in an appropriate Banach space. This goal is outside the scope of this paper. Here this equation is used to find the function ξ iteratively to an arbitrary order in $\frac{1}{r(\underline{a})}$. After that one recovers the solution ψ of (5.10) as

$$\psi \;=\; e^{i\varphi_0}(f_0 + \xi). \tag{6.39}$$

This analysis shows that in the leading order in perturbation theory (5.10) has a solution satisfying (4.5)–(4.7) with $n = 1$, provided the λ_j's and b_j's are s.t. (6.11) holds.

7 Conclusion

In this paper, we have introduced and analyzed the intervortex energy for the Ginzburg–Landau equation. We have described its key role in finding (non-minimizing) multivortex solutions, i.e. solutions 'composed' of several single vertices (in a separate paper [24] we use the intervortex energy to describe dynamics of vortices). We also found its asymptotic behaviour as the intervortex distances increase. A part of the latter result (upper bound – easy part) is rigorous, while the other part (lower bound – hard part) is justified by detailed analysis. This analysis uses an auxiliary energy functional – pinning energy functional – which differs from the original one by extra potentials. The rôle of these potentials is to hold down the vortices from moving as a result of mutual interactions. The infimum of the pinning functional yields a lower bound on the original Ginzburg–Landau functional. It is argued that the new functional has a local minimizer (the point important in its own right as it relates to an important phenomenon of pinning) corresponding to the vortex configuration of interest. (We venture that there should be a mountain pass-type argument showing this.) Using the corresponding Euler–Lagrange (or modified Ginzburg–Landau) equation we estimate this local minimizer and its energy. The latter energy gives a desired lower bound on the Ginzburg–Landau energy under consideration.

Appendix A

In this appendix we perform some computations required in § 6. In what follows we set $r = |x - a_i|$.

First prove (6.34). We claim that

$$\mathrm{Re} \int \bar{\vec{\eta}}(F_j + \lambda_j \delta_{b_j} f_j) \;=\; \nabla_{a_j} H(\underline{a}). \tag{A 1}$$

Indeed, (6.17) shows that η_k, $k = 1, 2$, have only the $m = \pm 1$ harmonics. Using this equation together with (6.3), or (6.34), we obtain

$$\mathrm{Re} \int \bar{\vec{\eta}}(F_j + \lambda_j \delta_{b_j} f_j) \;=\; -\mathrm{Re}\, \pi i k_j \int_0^\infty \eta \cdot J\eta r dr \begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{A 2}$$

where $\eta = \binom{f_j' - \frac{n_j}{r} f_j}{f_j' + \frac{n_j}{r} f_j}$. The right-hand side can be computed explicitly:

$$\begin{aligned} \text{R.H.S. (A 2)} &= -\mathrm{Re}\, \pi k_j \int_0^\infty \Big(-4 \frac{n_j}{r} f_j f_j' \Big) r dr \begin{pmatrix} i \\ 1 \end{pmatrix} \\ &= 2\pi n_j \mathrm{Re} k_j \begin{pmatrix} i \\ 1 \end{pmatrix}. \end{aligned}$$

Since k_j is the complex version of $-\frac{1}{2\pi n_j} J \nabla_{a_j} H(\underline{a})$, (A 1) follows.

Now we compute the term $\mathrm{Re} \int \bar{\xi}_k \lambda_j \delta_{b_j} f_j$ under the assumption that $|\varDelta a_j| \ll \bar{r}$, where, recall, $\varDelta a_j = b_j - a_j$. To simplify the computations we let δ_{b_j} be the true δ-function, not a smeared one. Under the first assumption, the equation $|x - b_j| = \bar{r}$ for the circle ($= \operatorname{supp} \delta_{b_j}$) can be written in the leading approximation in $\varDelta a_j$ as

$$r \approx \bar{r} + \varDelta a_j \cdot \hat{y}, \tag{A 3}$$

where, remember, r and θ are the polar coordinates of $y = x - a_j$. Thus, $\delta_{b_j} = \delta(|x - b_j| - \bar{r})$ can be replaced by $\delta(r - \bar{r} - \varDelta a_j \cdot \hat{y})$. This yields

$$\begin{aligned} \mathrm{Re} \int \bar{\vec{\eta}} \delta_{b_j} f_j &\approx \int f_j f_j' \hat{y} \delta(r - \bar{r} - \varDelta a_j \cdot \hat{y}) \\ &\approx f_j(\bar{r}) f_j'(\bar{r}) \int_0^{2\pi} \hat{y}(\bar{r} + \varDelta a_j \cdot \hat{y}) d\theta. \end{aligned}$$

Hence

$$\mathrm{Re} \int \vec{\eta} \delta_{b_j} f_j \;\approx\; \pi f_j(\bar{r}) f_j'(\bar{r}) \varDelta a_j. \tag{A 4}$$

Appendix B Region $D_j \cap D_0$

Now we investigate the behaviour of the solution $\hat{\xi}$ in the regions $D_j \cap D_0$. We require that asymptotics in $D_j \cap D_0$ of the solutions found in D_j and D_0 match. We fix j and let r and θ denote the polar coordinates of $y = x - a_j$. As before we consider two cases.

$$\nabla_{a_j} H(\underline{a}) \neq 0$$

Now we find the leading asymptotic of $\hat{\xi}$ in the region $1 \ll r \ll r(\underline{a})$. Since only the operator $L^{(1)}$ has a zero mode, the leading term for $r \gg 1$ comes from the $m = 1$ sector. Thus, we consider (6.15) in this sector:

$$L^{(1)}\begin{pmatrix}\xi^{(1)}\\ \bar{\xi}^{(-1)}\end{pmatrix} = \begin{pmatrix}F^{(1)}\\ \bar{F}^{(-1)}\end{pmatrix}. \tag{B 1}$$

We write the solution of this equation in the form

$$\begin{pmatrix}\xi^{(1)}\\ \bar{\xi}^{(-1)}\end{pmatrix} = \begin{pmatrix}\xi_0^{(1)}\\ \bar{\xi}_0^{(-1)}\end{pmatrix} + c\begin{pmatrix}f_j' - \frac{n_j}{r}f_j\\ f_j' + \frac{n_j}{r}f_j\end{pmatrix}, \tag{B 2}$$

where $\left(\begin{smallmatrix}\xi_0^{(1)}\\ \bar{\xi}_0^{(-1)}\end{smallmatrix}\right)$ is a special solution, c is a constant and the vector which multiplies c is, recall, the translational zero mode. The value of c is fixed from original nonlinear equation (6.4) (say, by the perturbation theory). We show [27] that $\xi_0^{(\pm 1)}$ are bounded.

Dropping in (B 1) the derivatives of ξ which are of a higher order in r^{-1} as well as the terms $f_j^2 - 1$ and $\frac{(n_j \pm 1)^2}{r^2}$ in $L^{(1)}$, and dropping the terms coming from $\lambda_j \delta_{b_j}$ and the terms containing f_j' from $F_0^{(1)}$ and $\bar{F}_0^{(-1)}$, we arrive at the asymptotic equation

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\xi^{(1)}\\ \bar{\xi}^{(-1)}\end{pmatrix} = -ik_j\frac{n_j}{r}\begin{pmatrix}1\\ 1\end{pmatrix}. \tag{B 3}$$

A particular solution of this equation, $\frac{n_j k_j}{2ir}\left(\begin{smallmatrix}1\\ 1\end{smallmatrix}\right)$, and the asymptotics, $\frac{n_j}{r}\left(\begin{smallmatrix}-1\\ 1\end{smallmatrix}\right)$, of the translational zero mode for $r \gg 1$ lead to the asymptotics of the general solution to (6.15):

$$\begin{pmatrix}\xi^{(1)}\\ \bar{\xi}^{(-1)}\end{pmatrix} = \frac{n_j k_j}{2ir}\begin{pmatrix}1\\ 1\end{pmatrix} + ive^{-i\alpha_j}\begin{pmatrix}-1\\ 1\end{pmatrix} \tag{B 4}$$

plus higher order terms in r^{-1} (remember, $k_j = |k_j|e^{-i\alpha_j}$). Here v is a function of r (incorporating $c\frac{n_j}{r}$); it cannot be found from (B 3) and an explicit expression for it will be given below. This together with (6.12) yields the asymptotic expression for the solution to (6.8):

$$\xi_j = \frac{n_j|k_j|}{r}\sin(\theta - \alpha_j) - 2iv \cdot \cos(\theta - \alpha_j) \tag{B 5}$$

plus higher order terms in r^{-1}. The second term on the right-hand side yields, in the leading order, a correction to the phase of ψ (due to a small translation of the center of the vortex), while the first term, to $|\psi|$:

$$\varphi = \varphi_j + 2v\cos(\theta - \alpha_j) \tag{B 6}$$

and

$$|\psi| = 1 - \frac{n_j^2}{2r^2} + \frac{n_j|k_j|}{r}\sin(\theta - \alpha_j).$$

Consider now equation (6.23) in the region $1 \ll r = |x - a_j| \ll r(\underline{a})$. We obtain

$$-\Delta \mathrm{Im}\xi_0 = \frac{4n_j^2|k_j|}{r^3}\cos(\theta - \alpha_j).$$

The solution to this equation decreasing at infinity is

$$\mathrm{Im}\,\xi_0 = 2|k_j|n_j^2\cos(\theta - \alpha_j)\frac{1}{r}\ln\left(\frac{r}{r_*}\right), \tag{B 7}$$

where r_* is a constant, which can be found from solving (B 1) at $r = O(1)$. Comparing (B 5) and (B 7) and using that $\xi = \xi_0$, modulo higher order terms, we find

$$v \;=\; -|k_j| n_j^2 \frac{1}{r} \ln\left(\frac{r}{r_*}\right). \tag{B 8}$$

Observe now that in the region $1 \ll r \ll r(\underline{a})$ (which is a part of $d(x, \underline{a}) \gg 1$)

$$\nabla\varphi_0 \;=\; \frac{n_j}{r}(-\sin\theta, \cos\theta) + k_j, \tag{B 9}$$

which implies that ψ is, modulo $O\big(r(\underline{a})^{-2}\big)$, of the form (6.39) (remember that $k_j = O\big(|r(\underline{a})|^{-1}\big)$). Thus, the obtained solutions in the regions D_j, $j = 1, \ldots, K$, and D_0 match (modulo higher order terms) in the common domain.

Thus, we have shown that solutions of (5.10) in the regions D_j, $j = 1, \ldots, K$, have asymptotics in $D_j \cap D_0$ given by (B 6), while the solution in D_0 has asymptotics in the same domains $D_j \cap D_0$ given by (6.22) and (B 7). Thus, in the overlapping regions the obtained local solutions match.

The case $\nabla_{a_j} H(\underline{a}) = 0$

Equation (6.33) with $\lambda_j = 0$ shows that in this approximation $\xi^{(m)} = 0$ for $m \neq 2$. Consider the sector $m = 2$. In the region $1 \ll |x - a_j| \ll r(\underline{a})$, (6.16) in the sector $m = 2$ leads to

$$\xi^{(\pm 2)} \;=\; \pm i n_j \alpha e^{\mp 2i\beta_j^{(2)}} \Big[-1 - \frac{4n_j^2 - 2}{r^2} \pm \frac{3n_j}{2} \Big]. \tag{B 10}$$

Combining this with (6.1) and (6.12), we obtain the following expression for the correction to the phase, φ_0, in this region

$$\delta\varphi \;=\; 3\alpha n_j^2 \cos\big(2(\theta - \beta_j^{(2)})\big). \tag{B 11}$$

Such a correction leads to the correction to the energy of the order $O\Big(\frac{\ln r(\underline{a})}{r(\underline{a})^4}\Big)$. The rest of the analysis of the general case can be carried over into the force-free case without a change. This shows that in the case $\nabla H(\underline{a}) = 0$, in the leading order in perturbation theory, the solutions of (5.10) in the regions D_j, $j = 0, \ldots, K$, match in the overlaps $D_0 \cap D_j$ of these regions.

The analysis above can be carried out to an arbitrary order of perturbation theory with the conclusions not changed. This concludes our argument that (5.10) has a unique solution satisfying (4.5) and (4.6).

Appendix C Supplement

In this supplement we show in the first order of perturbation theory, that the solution of (5.10) found in § 6 has in fact the vortex configuration $\underline{c}$. Below j is fixed and $r = |x - a_j|$. We assume $n_j > 0$. We consider two cases.

The case $\nabla_{a_j} H(\underline{a}) \neq 0$

Consider (B 1) for $r \ll 1$. We have in the leading order

$$\begin{pmatrix} -\Delta_r + \frac{(n_j+1)^2}{r^2} & \alpha_{n_j}^2 r^{2n_j} \\ \alpha_{n_j}^2 r^{2n_j} & -\Delta_r + \frac{(n_j-1)^2}{r^2} \end{pmatrix} \begin{pmatrix} \xi^{(1)} \\ \xi^{(-1)} \end{pmatrix}$$
$$= -2in_j k_j \alpha_{n_j} r^{n_j-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{C 1}$$

where, recall, α_{n_j} is determined by (5.6). The general solution regular at $r \to 0$ is

$$\frac{i\alpha_{n_j} k_j}{2} r^{n_j+1} \begin{pmatrix} \frac{\alpha_{n_j}^2}{8(n_j+1)^2} r^{2(n_j+1)} \\ 1 \end{pmatrix}$$
$$+c_1 \begin{pmatrix} \frac{\alpha_{n_j}^2}{4n_j(2n_j+1)} r^{3n_j+1} \\ r^{n_j-1} \end{pmatrix} + c_2 \begin{pmatrix} r^{n_j+1} \\ \frac{\alpha_{n_j}^2}{4(n_j+2)(2n_j+5)} r^{3(n_j+1)} \end{pmatrix}, \tag{C 2}$$

where c_1 and c_2 are some constants. Note that the translational mode $\begin{pmatrix} f_j' - \frac{n_j}{r} f_j \\ f_j' + \frac{n_j}{r} f_j \end{pmatrix} = \begin{pmatrix} 0 \\ 2\alpha_{n_j} n_j r^{n_j-1} \end{pmatrix} + O(r^{n_j+1})$ as $r \to 0$. Hence, (C 2) can be written as

$$\frac{1}{2} i \ \alpha_{n_j} k_j r^{n_j+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3 2\alpha_{n_j} n_j r^{n_j-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$+c_2 \begin{pmatrix} r^{n_j+1} \\ 0 \end{pmatrix} + O(r^{n_j+1}), \tag{C 3}$$

where $c_3 = c_1 / 2\alpha_{n_j} n_j$. Thus, in this approximation the n_j-vortex, shifted to some distance from the initial position, does not split.

The case $\nabla_{a_j} H(\underline{a}) = 0$

On the small distances, $r \ll 1$, we obtain from (6.15) in the sector $m = 2$ that

$$\xi^{(\pm 2)} \ = \ \pm i\alpha\alpha_{n_j} e^{\mp 2i\beta_j^{(2)}} [r^{n_j+2} C_{\pm 1} + r^{|n_j-2|} C_{\pm 2}], \tag{C 4}$$

where $C_{-1} = 1$, $C_2 = 0$ and C_1 and C_{-2} are some real constants of the order $O(1)$, while the numbers α_{n_j} are defined from (5.6).

If $n_j = 1$, then it follows from (C 4) that the j-th vortex is slightly deformed. In the case $n_j \geqslant 2$, if we know from, say, symmetry considerations that the jth vortex does not split, then we should add to the function ψ a shift solution (the $m = 1$ sector) with such a coefficient, c, that the term given by (C 4) and the new term added produce only a shift z_0 of the jth zero of ψ, i.e.

$$z^{n_j} + z^{n_j-2} \alpha e^{2i(\beta_j^{(2)} - \pi/4)} c_2 + \frac{cz^{n_j-1}}{\alpha_{n_j}}$$
$$= \ z^{n_j} - n_j z_0 z^{n_j-1} + z_0^2 n_j (n_j - 1) \frac{z^{n_j-2}}{2}. \tag{C 5}$$

Using this equation, we obtain

$$
\begin{aligned}
z_0 &= \pm e^{i(\beta_j^{(2)}-\pi/4)}\Big(\frac{2\alpha C_2}{n_j(n_j-1)}\Big)^{1/2} \\
&= O\big(r(\underline{a})^{-1}\big).
\end{aligned} \tag{C 6}
$$

Such a shift of the function ψ contributes $O\big(r(\underline{a})^{-4}\big)$ to the energy (see Remark 5.2). Choosing the centers b_j of our potentials δ_{b_j} appropriately, we move the zeros of ψ to the old positions.

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